Discussiones Mathematicae
Graph Theory 43 (2023) 999–1017
https://doi.org/10.7151/dmgt.2415

A CHARACTERIZATION OF UNIQUELY REPRESENTABLE GRAPHS[1]

Péter G.N. Szabó

Department of Computer Science and Information Theory
Budapest University of Technology and Economics
Budapest, H-1111, Hungary

e-mail: szape@cs.bme.hu

Abstract

The betweenness structure of a finite metric space $M = (X, d)$ is a pair $\mathcal{B}(M) = (X, \beta_M)$ where β_M is the so-called betweenness relation of M that consists of point triplets (x, y, z) such that $d(x, z) = d(x, y) + d(y, z)$. The underlying graph of a betweenness structure $\mathcal{B} = (X, \beta)$ is the simple graph $G(\mathcal{B}) = (X, E)$ where the edges are pairs of distinct points with no third point between them. A connected graph G is uniquely representable if there exists a unique metric betweenness structure with underlying graph G. It was implied by previous works that trees are uniquely representable. In this paper, we give a characterization of uniquely representable graphs by showing that they are exactly the block graphs. Further, we prove that two related classes of graphs coincide with the class of block graphs and the class of distance-hereditary graphs, respectively. We show that our results hold not only for metric but also for almost-metric betweenness structures.

Keywords: finite metric space, metric betweenness, block graph, distance-hereditary graph, graph representation.

2020 Mathematics Subject Classification: 05C12, 54E35.

1. Introduction

Metric space is a universal concept with various applications in physics, molecular biology and phylogenetics as well as in different branches of mathematics including geometry, topology, graph theory, computer science and algebra. Although finite metric spaces are trivial objects from a topological point of view, they have

[1]This work was supported by the National Research, Development and Innovation Office — NKFIH, No. 108947.

rich combinatorial properties which were investigated from different angles over the last century. The concept of metric betweenness appears to play a central role in the related literature which ranges from combinatorial geometry to metric graph theory.

An important line of study in metric graph theory was concerned with the axiomatization of different graph classes in terms of geodesic betweenness and more generally, in terms of different path transit functions. Further, graph classes satisfying certain transit axioms were characterized in different ways, for example, by means of forbidden induced subgraphs. Among the earliest results are Sholander's axiomatic characterization of trees, lattices and partially ordered sets in terms of segments, medians and betweenness [45].

The (geodesic) interval function of a connected graph is one of the most fundamental concepts in metric graph theory. It was first extensively studied by Mulder [35], who introduced the five classical axioms of the interval function and used the interval function to study different classes of graphs. In a series of papers starting from 1994, Nebeský *et al.* presented a characterization of the interval function of a connected graph in terms of first order transit axioms, each time improving the proof [36, 38, 39, 40, 41, 42]. Their approach was to extend the set of the five classical axioms with additional postulates.

Besides the interval function, two other path transit functions were extensively studied on connected graphs: the induced path function and the all-paths function. In [37], Mulder introduced the general notion of transit function to unify the three concepts and presented a list of prototype problems that can be uniformly applied to different kinds of transit functions.

In [12], Changat *et al.* characterized the all-paths function of a connected graph by using only first-order transit axioms. Interestingly, Nebeský established in [43] that such a characterization of the induced path function J of an arbitrary connected graph is impossible. Nevertheless, Changat *et al.* succeeded in axiomatizing J for certain classes of graphs [15, 16]. The authors also investigated different axiomatic restrictions on J such as the Peano axiom, the Pasch axiom and monotonicity [14, 16, 19]. Several well-known graph classes—including trees, block graphs, bipartite graphs, chordal graphs and Ptolemaic graphs—were characterized by axiomatizing their interval function or induced path function [4, 13, 18]. For a thorough survey on geodesic and induced path betweenness, see [17].

Beside the above mentioned results, there are two other lines of research that are related to the topic of this paper. Based on the pioneering work of Isbell [30] and Buneman [10], Dress *et al.* developed T-theory, a combinatorial theory of discrete metric spaces, and extensively studied algorithmic and combinatorial aspects of phylogenetic trees [5, 26]. These results have a wide range of applications in evolutionary biology. Another notable problem that is related to metric betweenness and gained a lot of attention lately is the generalization of the de Bruijn-Erdős theorem—known from geometry—to finite metric spaces, originally conjectured by Chen and Chvátal in 2008 [21]. The conjecture is still open but has already been proved for several important classes of metric spaces and graph metrics [1, 3, 7, 8, 22, 31, 44], among others, for metric spaces induced by distance-hereditary graphs [2].

The aim of our research in the most general sense is to better understand the combinatorial aspects of the metric betweenness relation. We believe that finite metric spaces are interesting objects on their own right, therefore, there are some important differences between our approach and that of metric graph theory. First, instead of studying metric properties of graphs to draw graph-theoretic conclusions, we focus on the combinatorial aspects of finite metric spaces with the help of graph-theoretic tools and arguments. Second, we formulate our results in the language of the metric betweenness relation. It is essentially equivalent to the language of transit functions but better suited for potentially non-graphic problems. Third, we do not restrict our attention to betweenness structures obtained in some way from simple graphs but only require them to be metric or almost-metric (see Definition 1). Results obtained in this framework are general enough to allow for a broader range of applications in different areas of mathematics and natural sciences.

2. Definitions

A *finite metric space* is a pair $M = (X, d)$ where X is a finite nonempty set and d is a *metric* on X, i.e., an $X \times X \to \mathbb{R}$ function which satisfies the following conditions for all $x, y, z \in X$.

1. $d(x, y) = 0 \Leftrightarrow x = y$ (*identity of indiscernibles*);
2. $d(x, y) = d(y, x)$ (*symmetry*);
3. $d(x, z) \leq d(x, y) + d(y, z)$ (*triangle-inequality*).

The non-negativity of metric follows from the definition, as for all $x, y \in X$, $0 = d(x, x) \leq d(x, y) + d(y, x) = 2d(x, y)$. We will refer to the ground set and the metric of a metric space M by $X(M)$ and d_M, respectively. All metric spaces in this paper will be assumed to be *finite* ($|X(M)| < \infty$).

In order to capture the relevant combinatorial properties of metric spaces, we introduce the following abstraction. A *betweenness structure* is a pair $\mathcal{B} = (X, \beta)$ where X is a nonempty finite set and $\beta \subseteq X^3$ is a ternary relation, called the *betweenness relation* of $\mathcal{B}$. The statement $(x, y, z) \in \beta$ will be denoted by $(x\ y\ z)_{\mathcal{B}}$ or simply by $(x\ y\ z)$ if $\mathcal{B}$ is clear from the context, and we say that y is *between* x and z.

We say that a betweenness structure $\mathcal{C} = (X, \gamma)$ is an *extension* of $\mathcal{B}$ (in notation, $\mathcal{B} \preccurlyeq \mathcal{C}$) if $\beta \subseteq \gamma$. The *substructure* of $\mathcal{B}$ induced by a nonempty subset $Y \subseteq X$ is the betweenness structure $\mathcal{B}|_Y = (Y, \beta \cap Y^3)$.

We believe that stating and proving our results on the abstraction level of betweenness structures makes our arguments clearer and easier to understand, as they are essentially about the combinatorial properties of the betweenness relation of metric spaces.

There is a natural way to associate a betweenness structure with a metric space. The *betweenness structure induced by a metric space* $M = (X, d)$ is $\mathcal{B}(M) = (X, \beta_M)$ where

$$\beta_M = \{(x, y, z) \in X^3 : d(x, z) = d(x, y) + d(y, z)\}$$

is the *betweenness relation* of M. To simplify notations, we will write $(x\ y\ z)_M$ for $(x\ y\ z)_{\mathcal{B}(M)}$.

The betweenness structure $\mathcal{B}$ is *metric* if it is induced by some metric space $M = (X, d)$. A metric betweenness relation satisfies the following elementary properties for any three points $x, y, z \in X$.

(P1) $(x\ x\ z)$;

(P2) $(x\ y\ z) \Rightarrow (z\ y\ x)$;

(P3) $(x\ y\ z) \wedge (y\ x\ z) \Rightarrow x = y$.

The *trichotomy* of betweenness follows straight from properties (P1)–(P3): for any three distinct points $x, y, z \in X$, at most one of the relations $(x\ y\ z)$, $(y\ z\ x)$, $(z\ x\ y)$ can hold.

Additionally, for any four points $x, y, z, w \in X$, we have

(P4) $(x\ y\ z) \wedge (x\ w\ y) \Rightarrow (x\ w\ z) \wedge (w\ y\ z)$.

This property, which we call the *four relations property*, is the simplest non-trivial property of metric betweenness. Notice that properties (P1)–(P4) are not sufficient to guarantee that the betweenness structure is metric (think, for example, about the Fano plane, which is proved to be non-metric [20, 23]).

Definition 1. A betweenness structure is *almost-metric* if it satisfies properties (P1)–(P4).

Different notions of betweenness were developed by Sholander [45], Morgana [34] and Burigana [11]. In this context, our concept of almost-metric betweenness can be best viewed as a stronger version of Morgana's betweenness. There, trichotomy and the condition $(x\ y\ z) \wedge (x\ w\ y) \Rightarrow (x\ w\ z)$ is assumed in place of properties (P3) and (P4). To see that our version is indeed a strengthening, consider the set $X = \{w, x, y, z\}$ of four points with betweenness relation

$\beta = \{(w,y,z),(z,y,w),(w,x,y),(y,x,w),(w,x,z),(z,x,w)\} \cup \{(u,u,v) : u,v \in X\} \cup \{(u,v,v) : u,v \in X\}$. The betweenness structure $\mathcal{B} = (X,\beta)$ is not almost-metric as it violates property (P4), but β is a betweenness in the sense of Morgana.

Quite interestingly, several properties of finite metric spaces can be generalized to almost-metric betweenness structures without modification. The reader will see that our main results fall in this category as well.

The *underlying graph* (or *adjacency graph*) of an almost-metric betweenness structure $\mathcal{B} = (X,\beta)$ is the graph $G(\mathcal{B}) = (X, E(\mathcal{B}))$ where the edges are such pairs of distinct points for which no third point lies between them (see Figure 1). More formally,

$$E(\mathcal{B}) = \left\{ xz \in \binom{X}{2} : \forall\, y \in X,\ (x\ y\ z)_{\mathcal{B}} \Rightarrow\ y = x \vee y = z \right\}.$$

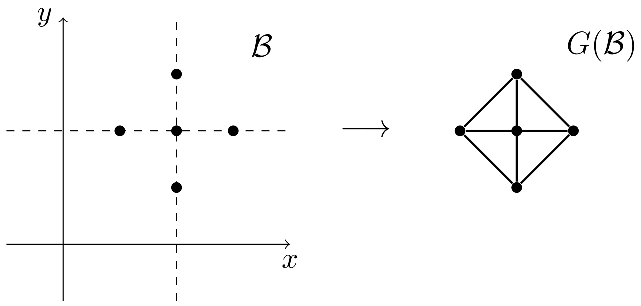


Figure 1. The underlying graph of a metric betweenness structure induced by five points in the Euclidean plane.

These edges are also called primitive pairs by some authors. The *underlying graph of a metric space M* is $G(M) = G(\mathcal{B}(M))$. The underlying graph is an important graph-theoretic tool in studying betweenness structures, therefore, it is highly desirable to gain a better understanding of its properties.

The fact that the underlying graph of a metric betweenness structure is connected is part of the mathematical folklore. Changat *et al.* generalized this observation to the path transit function equivalent of almost-metric betweenness structures in [16].

Proposition 2 [16]**.** *The underlying graph of an almost-metric betweenness structure is connected.*

Next, we introduce some definitions for simple and weighted graphs. A *weighted graph* is a triple $W = (V, E, \omega)$ where $G = (V, E)$ is a simple connected graph and ω is a positive real-valued function on the set of edges, also called the *edge-weighting* of W. We will also denote V, E, G and ω by $V(W)$,

$E(W)$, $G(W)$ and ω_W, respectively. The connectivity of G and the positivity of ω guarantee that a proper weighted graph metric can be defined on W (see below). We note that any simple connected graph $G = (V, E)$ can be regarded as a weighted graph $W(G)$, with edge-weighting $\omega_G : E \to \{1\}$, thus, any definitions for weighted graphs can be naturally applied to simple connected graphs as well.

Let $W = (V, E, \omega)$ be a weighted graph and $P = v_0 e_1 v_1 e_2 \cdots e_\ell v_\ell$ be a path in W. The *length* of P is $|P| = \ell$ and the *weight* of P is $\omega(P) = \sum_{i=1}^{\ell} \omega(e_i)$. We want to emphasize that the length and the weight of a path are usually different, even though the two coincide in case of simple graphs. The *metric space induced by the weighted graph* W is $M(W) = (V, d_W)$ where $d_W(u, v)$ is the minimum weight of a u-v path in W for any vertices $u, v \in V$. Such a path is called a u-v *geodesic* of W. Note that subpaths of a geodesic are also geodesics. Because of our definition of weighted graphs, d_W is a metric that we call the *weighted graph metric* of W. For a simple connected graph G, d_G is the usual graph metric induced by the shortest paths of the graph. We remark that every (finite) metric space $M = (X, d)$ is induced by some weighted graph W. For example, take d as the edge-weighting of a complete graph on vertex set X.

The *betweenness structure induced by* W is the betweenness structure induced by $M(W)$, denoted by $\mathcal{B}(W)$. For simplicity, we will write $(x\ y\ z)_W$ for $(x\ y\ z)_{\mathcal{B}(W)}$. Let us remark that the betweenness relation of a weighted graph W can be described in terms of its geodesics: $(x\ y\ z)_W$ holds if and only if y is on an x-z geodesic of W.

The weighted graph Z is a *weighted subgraph* of W (in notation, $Z \leq W$) if $G(Z) \leq G(W)$ and ω_Z is the restriction of ω_W to $E(Z)$. Let U be a set of vertices that induces a connected subgraph of $G(W)$. The weighted subgraph of W induced by U is the (uniquely determined) weighted subgraph $W[U] \leq W$ which satisfies $G(W[U]) = G(W)[U]$. We say that a weighted subgraph $Z \leq W$ is *isometric* if $M(Z)$ is a metric subspace of $M(W)$, i.e., for every $x, y \in V(Z)$, $d_Z(x, y) = d_W(x, y)$. We remark that $\mathcal{B}(Z) \leq \mathcal{B}(W)$ is also true in that case.

By graph, we will always mean a finite simple connected graph in the rest of the paper. A betweenness structure/metric space is

- *graphic* if it is induced by a graph;
- *ordered* if it is induced by a path.

The ordered betweenness structure induced by the path $P = x_1 x_2 \cdots x_n$ will be denoted by $[x_1, x_2, \ldots, x_n]$. We also say that Y is an ordered subset of $\mathcal{B}$ if it induces an ordered substructure of $\mathcal{B}$. Remark that almost-metric betweenness structures are typically not graphic. However, there is an important connection between the underlying graph and the graphicity of an almost-metric betweenness structure.

Observation 3. *For every graph G, $G(\mathcal{B}(G))) = G$. Further, for every almost-metric betweenness structure $\mathcal{B}$, $\mathcal{B}(G(\mathcal{B})) = \mathcal{B}$ if and only if $\mathcal{B}$ is graphic.*

Next, we recall some useful results about block graphs and distance-hereditary graphs. Graph G is a *block graph* if every block (2-connected component) of G is a clique, or equivalently, if every cycle of G induces a complete subgraph (see Figure 2). Block graphs can be seen as natural generalizations of trees. Graph G is a *distance-hereditary graph* if all of its connected induced subgraphs are isometric. It can be easily seen that all block graphs are distance-hereditary.

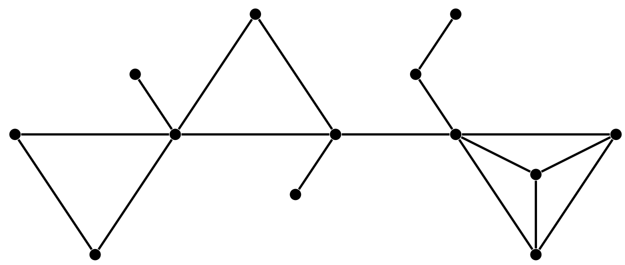


Figure 2. A block graph.

Both block graphs and distance-hereditary graphs are well-understood graph classes and have been characterized in many different ways. Block graphs were first studied by Harary in [27]. They were characterized in terms of various metric properties (such as the four point condition, the Ptolemaic property and the weakly geodetic property) [28, 32], forbidden induced subgraphs [6], bulge [28], k-Steiner intervals [9], vertex induced partitions [25] and betweenness axioms [4]. Further, Le and Tuy gave a characterization and a linear time recognition algorithm for squares of block graphs [33].

Distance-hereditary graphs were characterized in terms of conditions on paths and cycles by Howorka [29], who also showed that distance-hereditary graphs are perfect. Bandelt and Mulder gave a good summary on the most important characterizations of distance-hereditary graphs including recursive, metric and forbidden subgraph characterizations [6], while Morgana and Mulder provided a characterization in terms of axioms imposed on the induced path function [34].

Below, we list some of the results on which the proof of Theorem 13 and Theorem 14 will rely on. First, observe the following property of block graphs, which is a straightforward consequence of the definition.

Observation 4. *In a block graph, any two vertices are connected by a unique induced path.*

A *chord* of a cycle is an edge that connects two non-consecutive vertices of the cycle. A graph is *chordal* if it does not contain any induced cycle of length at least four, or equivalently, if it does not contain a chordless cycle. A *diamond* is a 4-cycle with exactly one chord (see Figure 3(b)). Next, we recall a characterization of block graphs by Bandelt and Mulder, and a characterization of distance-hereditary graphs by Howorka.

Proposition 5 [6]**.** *Block graphs are exactly the diamond-free chordal graphs.*

Proposition 6 [29]**.** *A graph is distance-hereditary if and only if every induced path in it is a geodesic.*

The following statement is an easy consequence of Proposition 6.

Observation 7. *A distance-hereditary graph does not have any induced cycle of length at least* 5*.*

3. Main Results

Below, we introduce and characterize uniquely representable graphs, as well as two related classes of graphs that bound their representations from below and from above (Theorem 13 and Theorem 14). These results can be regarded as a new metric characterization of block graphs and distance-hereditary graphs and—as we point out later—generalize an interesting observation of Dress about trees (see Observation 15).

We say that an almost-metric betweenness structure $\mathcal{B}$ is a *representation* of a graph G if G is the underlying graph of $\mathcal{B}$. It follows from Observation 3 that $\mathcal{B}(G)$ is always a metric representation of G but there may be other representations. Below, we investigate the question whether it is possible to infer a betweenness structure from its underlying graph or at least obtain a subset or superset of its betweenness relation.

Definition 8. A graph G is *uniquely representable* in the metric/almost-metric sense (ur-m/ur-am) if $\mathcal{B}(G)$ is the only metric/almost-metric representation of G.

We will see later that not all graphs are uniquely representable, moreover, one may have a representation $\mathcal{B}$ of a graph G such that neither $\mathcal{B} \preccurlyeq \mathcal{B}(G)$ nor $\mathcal{B} \succcurlyeq \mathcal{B}(G)$ holds (Lemma 20). This motivates the following generalizations of Definition 8.

Definition 9. A graph G

- *bounds its representations from below* in the metric/almost-metric sense (brb-m/brb-am) if $\mathcal{B}(G) \preccurlyeq \mathcal{B}$ holds for all metric/almost-metric representations $\mathcal{B}$ of G;

- *bounds its representations from above* in the metric/almost-metric sense (bra-m/bra-am) if $\mathcal{B}(G) \succcurlyeq \mathcal{B}$ holds for all metric/almost-metric representations $\mathcal{B}$ of G.

We denote the corresponding classes of graphs by $\mathcal{C}^{m}_{ur}, \mathcal{C}^{am}_{ur}, \mathcal{C}^{m}_{bra}, \mathcal{C}^{am}_{bra}, \mathcal{C}^{m}_{brb}$ and $\mathcal{C}^{am}_{brb}$, respectively. It is easy to see that the following relations hold between these classes.

Observation 10. 1. $\mathcal{C}^{m}_{ur} = \mathcal{C}^{m}_{bra} \cap \mathcal{C}^{m}_{brb}$ *and* $\mathcal{C}^{am}_{ur} = \mathcal{C}^{am}_{bra} \cap \mathcal{C}^{am}_{brb}$*;*

2. $\mathcal{C}^{am}_{ur} \subseteq \mathcal{C}^{m}_{ur}$*,* $\mathcal{C}^{am}_{bra} \subseteq \mathcal{C}^{m}_{bra}$ *and* $\mathcal{C}^{am}_{brb} \subseteq \mathcal{C}^{m}_{brb}$*.*

Below, we give an example for a metric representation $\mathcal{B}$ of a graph G such that $\mathcal{B}(G) \succnsim \mathcal{B}$.

Example 11. Let G be a 4-cycle with consecutive vertices x_1, x_2, x_3, x_4, and let W denote the weighted graph obtained from G by increasing the weight of the edge $e = x_4x_1$ by 1 (so that $\omega(e) = 2$). Then,

- the non-trivial betweennesses of $\mathcal{B}(G)$ (up to symmetry) are $(x_1\ x_2\ x_3)$, $(x_2\ x_3\ x_4)$, $(x_3\ x_4\ x_1)$ and $(x_4\ x_1\ x_2)$;
- the non-trivial betweennesses of $\mathcal{B}(W)$ (up to symmetry) are $(x_1\ x_2\ x_3)$ and $(x_2\ x_3\ x_4)$.

For example,

$$
\begin{aligned}
d_W(x_3, x_1) = 2 < d_W(x_3, x_4) + d_W(x_4, x_1) \\
d_W(x_4, x_2) = 2 < d_W(x_4, x_1) + d_W(x_1, x_2).
\end{aligned}
$$

Now, it is obvious that $\mathcal{B} = \mathcal{B}(W)$ is a representation of G and $\mathcal{B}(G) \succnsim \mathcal{B}$. It can also be shown that G bounds its representations from above.

Remark 12. Repeating the construction in Example 11 with an *odd* cycle leads to a completely different outcome, as one would get a metric representation $\mathcal{B}$ of the graph $G \simeq C_{2k+1}$ for which $\mathcal{B}(G) \not\succsim \mathcal{B}$. This shows that the relation between a graph and its representations is sensitive to the parity of the cycles in the graph.

The next two theorems are the main results of this paper. We characterize uniquely representable graphs, as well as graphs that bound their representations from below and from above in both the metric and the almost-metric sense. The proofs can be found in Section 4.

Theorem 13. *The following statements are equivalent.*

1. $G \in \mathcal{C}^{m}_{bra}$;
2. $G \in \mathcal{C}^{am}_{bra}$;
3. *G is a distance-hereditary graph.*

Theorem 14. *The following statements are equivalent.*

1. $G \in \mathcal{C}_{ur}^{m}$;
2. $G \in \mathcal{C}_{ur}^{am}$;
3. $G \in \mathcal{C}_{brb}^{m}$;
4. $G \in \mathcal{C}_{brb}^{am}$;
5. *G is a block graph.*

Our motivation for characterizing uniquely representable graphs was twofolded. First, we wanted to have a better understanding on the relationship of betweenness structures and their underlying graphs. We have observed that under certain conditions, a metric betweenness structure can be fully reconstructed from its underlying graph, which is exactly what unique representability means. For example, in [24], Remark R3, Dress made an interesting observation which implies that trees are uniquely representable in the metric sense.

Observation 15 [24]**.** *Let $\mathcal{B}$ be a metric betweenness structure such that $T = G(\mathcal{B})$ is a tree. Then $\mathcal{B}$ is induced by T.*

Our second motivation is to find useful generalizations of Observation 15 that can be applied to a larger set of problems. Graphs that bound their representations from below and from above arise as natural generalizations of uniquely representable graphs. As Theorems 13 and 14 show, these broader classes of graphs have nice characterizations, too.

We would like to highlight two interesting aspects the main results. First, there is no difference between the metric and almost-metric cases: $\mathcal{C}_{ur}^{m} = \mathcal{C}_{ur}^{am}$, $\mathcal{C}_{bra}^{m} = \mathcal{C}_{bra}^{am}$ and $\mathcal{C}_{brb}^{m} = \mathcal{C}_{brb}^{am}$. Second, the results show a significant asymmetry: while $\mathcal{C}_{ur}^{m}$ is a proper subclass of $\mathcal{C}_{bra}^{m}$, it coincides with $\mathcal{C}_{brb}^{m}$ (and the same holds in the almost-metric case).

Finally, we note that in [37], Prototype problem 3, Mulder discussed a question involving a kind of unique representability but it is applied to transit functions and convexities instead of betweenness structures and underlying graphs. Although the concept is quite different from the one discussed here, this example shows that unique representability is a general concept, and investigating related problems for different kinds of combinatorial structures may give rise to interesting new ideas and results.

4. Proof of the Main Results

We prepare for the proof of Theorems 13 and 14 by introducing tight weighted graphs and extending the concept of geodesics to almost-metric betweenness structures.

We say that an edge $e = xy$ of a weighted graph W is *tight* if e is the unique x-y geodesic in W, in other words, every x-y geodesic in W is of length 1. A weighted graph is *tight* if all of its edges are tight. Note that all simple graphs are tight. Further, every geodesic in a tight weighted graph W is an induced path in $G(W)$.

Proposition 16. *If W is a tight weighted graph, then $G(W) = G(\mathcal{B}(W))$.*

Proof. Let $W = (V, E, \omega)$ be a tight weighted graph. Observe that $G(\mathcal{B}(W)) \leq G(W)$ is true for any weighted graph W, since every vertex y on an x-z geodesic of W satisfies $(x\ y\ z)_W$. Thus, it suffices to show that $G(W) \leq G(\mathcal{B}(W))$. Let $e = xz$ be an edge of $G(W)$. If $e \notin E(\mathcal{B}(W))$, then there exists a vertex $y \in V\backslash\{x, z\}$ such that $(x\ y\ z)_W$ holds, i.e., $d_W(x, z) = d_W(x, y) + d_W(y, z)$. But then, we would get an x-z geodesic of length at least 2 by concatenating an x-y and a y-z geodesic, in contradiction with the tightness of e. ■

Remark 17. Proposition 16 can be reversed: if $G(W) = G(\mathcal{B}(W))$, then W is a tight weighted graph.

Next, we generalize the notion of geodesics to almost-metric betweenness structures and summarize their most important properties. Let $\mathcal{B}$ be an almost-metric betweenness structure on ground set X, and let $x, z \in X$ be two points of $\mathcal{B}$. An x-z *geodesic in* $\mathcal{B}$ is an induced x-z path P in $G(\mathcal{B})$ such that $\mathcal{B}|_{V(P)}$ is an ordered substructure of $\mathcal{B}$.

Proposition 18. *Let $\mathcal{B}$ be an almost-metric betweenness structure on ground set X. Then*

1. *for every geodesic P in $\mathcal{B}$, $\mathcal{B}|_{V(P)} = \mathcal{B}(P)$;*
2. *for every maximal ordered set Y in $\mathcal{B}$, $G(\mathcal{B})[Y]$ is a geodesic in $\mathcal{B}$;*
3. *for every $x, y \in X$, there exists an x-y geodesic in $\mathcal{B}$;*
4. *if $\mathcal{B}$ is induced by a tight weighted graph W, then the geodesics of $\mathcal{B}$ coincide with the geodesics of W;*
5. *for every $x, y, z \in X$, $(x\ y\ z)_{\mathcal{B}}$ holds if and only if there exists an x-z geodesic in $\mathcal{B}$ that contains y;*
6. *for every almost-metric betweenness structure $\mathcal{C}$ for which $G(\mathcal{C}) = G(\mathcal{B})$ holds, $\mathcal{B} \preccurlyeq \mathcal{C}$ if and only if all geodesics of $\mathcal{B}$ are geodesics of $\mathcal{C}$.*

Proof. Part 1 is obvious from the definition of geodesics, while Part 2 follows from the four relations property: if $\mathcal{B}|_Y = [y_1, y_2, \ldots, y_k]$ and $G(\mathcal{B})[Y]$ is not an induced path, then there exists an index $1 \leq i < k$ and a point $x \in X\backslash Y$ such that $(y_i\ x\ y_{i+1})_{\mathcal{B}}$, which implies that $Y + x$ is ordered by property (P4), in contradiction with the maximality of Y.

Part 3 is a simple consequence of Part 2: take the geodesic induced by a maximal ordered set containing x and y and then take its subpath between x and y.

As for Part 4, let $W = (V, E, \omega)$ and $G = G(W)$. Since W is tight, $G(\mathcal{B}) = G$ by Proposition 16, hence, it is enough to show that an induced path $P = y_1 y_2 \cdots y_\ell$ in G is a geodesic in W if and only if it is a geodesic in $\mathcal{B}$ (here, we used the fact that all geodesics in W and $\mathcal{B}$ are induced paths).

Observation 19 (Polygon-equality)**.** *Let $\mathcal{B}$ be a metric betweenness structure induced by a metric space $M = (X, d)$ and $Y = \{y_1, y_2, \ldots, y_\ell\} \subseteq X$ be a nonempty set of points in $\mathcal{B}$. Then $d(y_1, y_\ell) = \sum_{i=1}^{\ell-1} d(y_i, y_{i+1})$ if and only if $\mathcal{B}|_Y = [y_1, y_2, \ldots, y_\ell]$.*

Now, the above statement follows from Part 1 and Observation 19: P is a geodesic in W if and only if $d_W(y_1, y_\ell) = \sum_{i=1}^{\ell-1} d_W(y_i, y_{i+1})$ if and only if $\mathcal{B}|_{V(P)} = [y_1, y_2, \ldots, y_\ell]$ if and only if P is a geodesic in $\mathcal{B}$.

In order to prove Part 5, suppose first that $(x\ y\ z)_{\mathcal{B}}$ holds, and let Y be a maximal ordered set in $\mathcal{B}$ that contains x, y and z. By Part 2, $P = G(\mathcal{B})[Y]$ is a geodesic in $\mathcal{B}$, and Part 1 implies $\mathcal{B}|_Y = \mathcal{B}(P)$ from which $(x\ y\ z)_P$ follows. Second, suppose that P is an x-z geodesic in $\mathcal{B}$ and $y \in V(P)$, i.e., $(x\ y\ z)_P$ holds. Because of Part 1, $\mathcal{B}|_{V(P)} = \mathcal{B}(P)$, hence, $(x\ y\ z)_{\mathcal{B}}$ holds as well.

Finally, Part 6 is a straightforward consequence of Part 5. ■

We conclude preparations with a lemma that plays a key role in the proof of the main results.

Lemma 20. *Let G be a graph that contains an induced path which is not a geodesic in G. Then $G \notin \mathcal{C}^m_{bra}$ and $G \notin \mathcal{C}^m_{brb}$.*

Proof. We construct a metric representation $\mathcal{B}$ of G such that $\mathcal{B} \not\preceq \mathcal{B}(G)$ and $\mathcal{B} \not\succeq \mathcal{B}(G)$. Let P be an induced x-y path of length ℓ in G which is not a geodesic. Further, let us define the weighted graph W for which $G(W) = G$ and $\omega_W = \mathbf{1}_{E(G) \setminus E(P)} + \varepsilon \mathbf{1}_{E(P)}$ where $0 < \varepsilon < 1/\ell$ and $\mathbf{1}_A$ denotes the indicator function of the set $A \subseteq E(G)$.

First, we show that P is an x-y geodesic in W and it is the only one. Let Q be an x-y geodesic in W. The weight of Q is $\omega(Q) \le \omega(P) = \varepsilon\ell < 1$, hence, Q cannot have edges from outside of $E(P)$. Therefore, Q must be a subpath of P that connects x and y, so, $Q = P$.

Next, we prove that W is tight. The edges of P are obviously tight because they are the lightest edges of W. Now, let $e = uv \in E(G) \setminus E(P)$. If e is not tight, then there exists a u-v geodesic Q' of length at least 2 such that $\omega(Q') \le \omega(e) = 1$. Again, this can only be true if Q' is a subpath of P. But then, e would connect two non-adjacent vertices of P, which contradicts the choice of P as an induced path.

Now, let $\mathcal{B} = \mathcal{B}(W)$ and let Q be an x-y geodesic in G. Note that $Q \neq P$ because P was not a geodesic in G. On the other hand, we have shown that P is a unique geodesic in W, thus, Q cannot be a geodesic in W. Finally, we obtain from Proposition 16 and Observation 3 that $G(\mathcal{B}) = G(W) = G = G(\mathcal{B}(G))$, so we can conclude by Part 4 and Part 6 of Proposition 18 that $\mathcal{B} \not\preccurlyeq \mathcal{B}(G)$ and $\mathcal{B} \not\succcurlyeq \mathcal{B}(G)$. ∎

Now, we are ready to prove Theorem 13 and Theorem 14.

***Proof of Theorem* 13.** Let G be a fixed graph. We show the following chain of implications: $G \in \mathcal{C}^{am}_{bra} \Rightarrow G \in \mathcal{C}^{m}_{bra} \Rightarrow G$ is a distance-hereditary graph $\Rightarrow G \in \mathcal{C}^{am}_{bra}$.

Step 1 ($G \in \mathcal{C}^{am}_{bra} \Rightarrow G \in \mathcal{C}^{m}_{bra}$)**.** This step follows from Observation 10.

Step 2 ($G \in \mathcal{C}^{m}_{bra} \Rightarrow G$ is a distance-hereditary graph)**.** By Proposition 6, it is enough to show that all induced paths in G are geodesics, which follows from Lemma 20.

Step 3 (G is a distance-hereditary graph $\Rightarrow G \in \mathcal{C}^{am}_{bra}$)**.** Let $\mathcal{B}$ be an almost-metric representation of G. In order to prove $\mathcal{B} \preccurlyeq \mathcal{B}(G)$, it suffices to show, by Part 6 of Proposition 18, that every geodesic of $\mathcal{B}$ is a geodesic of $\mathcal{B}(G)$. If P is a geodesic of $\mathcal{B}$, then it is an induced path in $G(\mathcal{B}) = G$. Therefore, P must be a geodesic in G by Proposition 6 and thus P is a geodesic in $\mathcal{B}(G)$ by Part 4 of Proposition 18. ∎

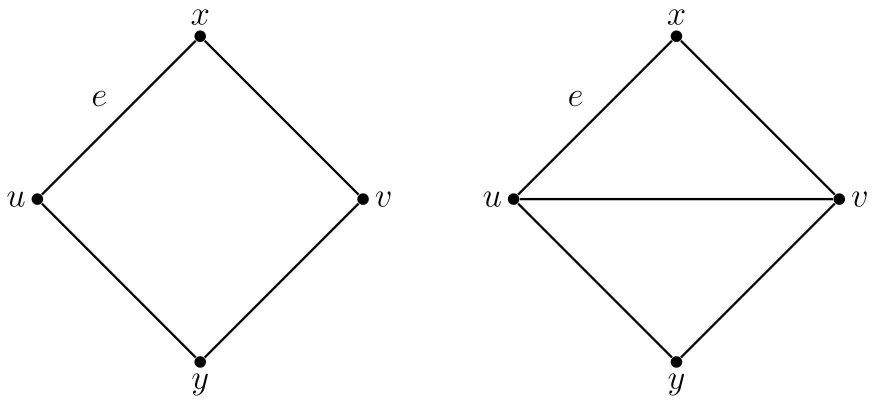
***Proof of Theorem* 14.** Let $G = (V, E)$ be a fixed graph. We show the following chain of implications: $G \in \mathcal{C}^{m}_{ur} \Rightarrow G$ is a block graph $\Rightarrow G \in \mathcal{C}^{am}_{ur} \Rightarrow G \in \mathcal{C}^{am}_{brb} \Rightarrow G \in \mathcal{C}^{m}_{brb} \Rightarrow G \in \mathcal{C}^{m}_{ur}$.

Step 1 ($G \in \mathcal{C}^{m}_{ur} \Rightarrow G$ is a block graph)**.** Assume to the contrary that G is not a block graph. It follows from $G \in \mathcal{C}^{m}_{ur} \subseteq \mathcal{C}^{m}_{bra}$ that G is distance-hereditary by Theorem 13, and we obtain from Proposition 5 and Observation 7 that G contains an induced subgraph H that is either a diamond or a cycle of length 4.

Let x, y, u and v denote the vertices of H such that $xy \notin E$, and let $e = xu$, which is an edge of H irrespective of whether H is a diamond or a 4-cycle (see Figure 3). Further, let us define a weighted graph $W = (V, E, \omega)$ where $\omega = \mathbf{1}_E + \frac{1}{2}\mathbf{1}_{\{e\}}$, and let $\mathcal{B} = \mathcal{B}(W)$ and $Z = W[\{x, y, u, v\}]$ be an induced subgraph of W.

We show that $\mathcal{B}$ is a representation of G but $\mathcal{B} \neq \mathcal{B}(G)$, in contradiction with $G \in \mathcal{C}^{m}_{ur}$. Observe that W is tight because every path of length at least 2 in W is of weight at least 2, which is greater than any of the edge weights ($\frac{3}{2}$ is an upper bound for ω). Thus, $G(\mathcal{B}) = G(W) = G$ by Proposition 16. Also notice that $(x\ u\ y)_H$ is true but $(x\ u\ y)_Z$ is false, moreover, H is an isometric subgraph of G and Z is an isometric weighted subgraph of W because all distances in H and Z

are less than 3. This implies that $(x\ u\ y)_G$ holds but $(x\ u\ y)_W$ does not, hence, $\mathcal{B} \neq \mathcal{B}(G)$.



(a) $H \simeq C_4$ (b) $H \simeq D$

Figure 3. Graph H in Step 1 of the proof of Theorem 14.

Step 2 (G is a block graph $\Rightarrow G \in \mathcal{C}_{ur}^{am}$)**.** Let G be a block graph and $\mathcal{B}$ be an almost-metric representation of G. By Part 4 and Part 6 of Proposition 18, it suffices to show that G and $\mathcal{B}$ have the same geodesics.

Let x and y be any two vertices of G, and let P and P' be x-y geodesics in G and $\mathcal{B}$, respectively. Note that such a P' exists by Part 3 of Proposition 18. Since both P and P' are induced x-y paths in G, Observation 4 yields $P = P'$, which completes the proof of Step 2.

Step 3 ($G \in \mathcal{C}_{ur}^{am} \Rightarrow G \in \mathcal{C}_{brb}^{am}$)**.** This step follows from Observation 10.

Step 4 ($G \in \mathcal{C}_{brb}^{am} \Rightarrow G \in \mathcal{C}_{brb}^{m}$)**.** Again, this step follows from Observation 10.

Step 5 ($G \in \mathcal{C}_{brb}^{m} \Rightarrow G \in \mathcal{C}_{ur}^{m}$)**.** If $G \in \mathcal{C}_{brb}^{m}$, then because of Lemma 20, every induced path in G must be a geodesic, hence, G is distance-hereditary by Proposition 6. Finally, we obtain from Theorem 13 that $G \in \mathcal{C}_{bra}^{m}$, therefore, $G \in \mathcal{C}_{ur}^{m}$ by Observation 10. ■

5. Conclusion

Motivated by our observations on finite metric spaces, we have defined the class of uniquely representable graphs and the class of graphs that bound their representations from below and from above. These graph-classes have been characterized in Theorem 13 and Theorem 14 in both the metric and the almost-metric sense. In particular, we have shown that the uniquely representable graphs are exactly the block graphs and pointed out that this result generalizes an interesting observation of Dress about trees.

Lastly, we would like to mention two open problems on graph representations that can be potential subjects of future research. By definition, uniquely representable graphs have the minimum number of representations. One may also be interested in the other extreme.

Problem 21. What is the maximum number of metric/almost-metric representations of a graph on n vertices and which graphs realize that maximum?

We conjecture the following.

Conjecture 22. *The number of metric representations of a graph of order n is maximized by the balanced complete bipartite graph of order n.*

Note that the balanced complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ on n vertices have at least $2^{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil - n + 1}$ metric representations. Namely, pick one vertex from both classes of $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, and set the weight of the edges adjacent to these vertices to 1. For all other edges, choose a weight arbitrarily from the set $\{1, 2\}$. It can be easily seen that different weighted graphs obtained in this way induce different metric representations of $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

We have weakened the property of unique representability by only requiring $\mathcal{B}(G)$ to be the smallest/largest element in the poset of representations of G, where elements are ordered according to the relation $\preccurlyeq$. We can further weaken this condition by saying that $\mathcal{B}(G)$ is a minimal/maximal element in the poset of representations.

Problem 23. Characterize graphs G for which $\mathcal{B}(G)$ is a minimal/maximal element in the poset of its metric/almost-metric representations.

Acknowledgment

We are grateful to Pierre Aboulker for sharing his thoughts on the results presented here, as well as on other, related results. We thank the anonymous reviewers whose suggestions helped improve this manuscript.

This work was supported by the National Research, Development and Innovation Office — NKFIH, No. 108947.

References

[1] P. Aboulker, X. Chen, G. Huzhang, R. Kapadia and C. Supko, *Lines, betweenness and metric spaces*, Discrete Comput. Geom. **56** (2016) 427–448. https://doi.org/10.1007/s00454-016-9806-2

[2] P. Aboulker and R. Kapadia, *The Chen-Chvátal conjecture for metric spaces induced by distance-hereditary graphs*, European J. Combin. **43** (2015) 1–7. https://doi.org/10.1016/j.ejc.2014.06.009

[3] P. Aboulker, M. Matamala, P. Rochet and J. Zamora, *A new class of graphs that satisfies the Chen-Chvátal conjecture*, J. Graph Theory **87** (2018) 77–88. https://doi.org/10.1002/jgt.22142

[4] K. Balakrishnan, M. Changat, A.K. Lakshmikuttyamma, J. Mathew, H.M. Mulder, P.G. Narasimha-Shenoi and N. Narayanan, *Axiomatic characterization of the interval function of a block graph*, Discrete Math. **338** (2015) 885–894. https://doi.org/10.1016/j.disc.2015.01.004

[5] H.-J. Bandelt and A.W.M. Dress, *A canonical decomposition theory for metrics on a finite set*, Adv. Math. **92** (1992) 47–105. https://doi.org/10.1016/0001-8708(92)90061-O

[6] H.-J. Bandelt and H.M. Mulder, *Distance-hereditary graphs*, J. Combin. Theory Ser. B **41** (1986) 182–208. https://doi.org/10.1016/0095-8956(86)90043-2

[7] L. Beaudou, A. Bondy, X. Chen, E. Chiniforooshan, M. Chudnovsky, V. Chvátal, N. Fraiman and Y. Zwols, *A de Bruijn-Erdős theorem for chordal graphs*, Electron. J. Combin. **22** (2015) #P1.70. https://doi.org/10.37236/3527

[8] L. Beaudou, G. Kahn and M. Rosenfeld, *Bisplit graphs satisfy the Chen-Chvátal conjecture*, Discrete Math. Theor. Comput. Sci. **21** (2019) #5.

[9] B. Brešar, M. Changat, J. Mathews, I. Peterin, P.G. Narasimha-Shenoi and A.T. Horvat, *Steiner intervals, geodesic intervals, and betweenness*, Discrete Math. **309** (2009) 6114–6125. https://doi.org/10.1016/j.disc.2009.05.022

[10] P. Buneman, *A note on the metric properties of trees*, J. Combin. Theory Ser. B **17** (1974) 48–50. https://doi.org/10.1016/0095-8956(74)90047-1

[11] L. Burigana, *Tree representation of qualitive proximities*, Math. Social Sci. **185** (2009) 5–36. https://doi.org/10.4000/msh.11000

[12] M. Changat, S. Klavžar and H.M. Mulder, *The all-paths transit function of a graph*, Czechoslovak Math. J. **51** (2001) 439–448. https://doi.org/10.1023/A:1013715518448

[13] M. Changat, A.K. Lakshmikuttyamma, J. Mathews, I. Peterin, P.G. Narasimha-Shenoi, G. Seethakuttyamma and S. Špacapan, *A forbidden subgraph characterization of some graph classes using betweenness axioms*, Discrete Math. **313** (2013) 951–958. https://doi.org/10.1016/j.disc.2013.01.013

[14] M. Changat and J. Mathew, *Induced path transit function, monotone and Peano axioms*, Discrete Math. **286** (2004) 185–194.
https://doi.org/10.1016/j.disc.2004.02.017

[15] M. Changat, J. Mathew and H.M. Mulder, *Induced path transit function, betweenness and monotonicity*, Electron. Notes Discrete Math. **15** (2003) 60–63.
https://doi.org/10.1016/S1571-0653(04)00531-1

[16] M. Changat, J. Mathew and H.M. Mulder, *The induced path function, monotonicity and betweenness*, Discrete Appl. Math. **158** (2010) 426–433.
https://doi.org/10.1016/j.dam.2009.10.004

[17] M. Changat, P.G. Narasimha-Shenoi and G. Seethakuttyamma, *Betweenness in graphs: A short survey on shortest and induced path betweenness*, AKCE Int. J. Graphs Comb. **16** (2019) 96–109.
https://doi.org/10.1016/j.akcej.2018.06.007

[18] M. Changat, F.H. Nezhad and N. Narayanan, *Axiomatic characterization of the interval function of a bipartite graph*, in: Algorithms Discrete Appl. Math., D. Gaur and N.S. Narayareswany (Ed(s)) (2018) 96–106.
https://doi.org/10.1007/978-3-319-53007-9_9

[19] M. Changat, I. Peterin, A. Ramachandran and A. Tepeh, *The induced path transit function and the Pasch axiom*, Bull. Malays. Math. Sci. Soc. (2) **39** (2016) 123–134.
https://doi.org/10.1007/s40840-015-0285-z

[20] X. Chen, *The Sylvester-Chvátal theorem*, Discrete Comput. Geom. **35** (2006) 193–199.
https://doi.org/10.1007/s00454-005-1216-9

[21] X. Chen and V. Chvátal, *Problems related to a de Bruijn-Erdős theorem*, Discrete Appl. Math. **156** (2008) 2101–2108.
https://doi.org/10.1016/j.dam.2007.05.036

[22] V. Chvátal, *A de Bruijn-Erdős theorem for* 1–2 *metric spaces*, Czechoslovak Math. J. **64** (2014) 45–51.
https://doi.org/10.1007/s10587-014-0081-1

[23] V. Chvátal, *Sylvester-Gallai theorem and metric betweenness*, Discrete Comput. Geom. **31** (2004) 175–195.
https://doi.org/10.1007/s00454-003-0795-6

[24] A. Dress, *The Category of X-Nets*, in: Networks: From Biology to Theory, J. Feng, J. Jost, M. Qian (Ed(s)), (Springer London, 2007) 3–22.
https://doi.org/10.1007/978-1-84628-780-0_1

[25] A. Dress, K.T. Huber, J. Koolen, V. Moulton and A. Spillner, *Characterizing block graphs in terms of their vertex-induced partitions*, Australas. J. Combin. **66** (2016) 1–9.

[26] A. Dress and M. Krüger, *Parsimonious phylogenetic trees in metric spaces and simulated annealing*, Adv. in Appl. Math. **8** (1987) 8–37.
https://doi.org/10.1016/0196-8858(87)90003-0

[27] F. Harary, *A characterization of block-graphs*, Canad. Math. Bull. **6** (1963) 1–6. https://doi.org/10.4153/CMB-1963-001-x

[28] E. Howorka, *On metric properties of certain clique graphs*, J. Combin. Theory Ser. B **27** (1979) 67–74. https://doi.org/10.1016/0095-8956(79)90069-8

[29] E. Howorka, *A characterization of distance-hereditary graphs*, Q. J. Math **28** (1977) 417–420. https://doi.org/10.1093/qmath/28.4.417

[30] J.R. Isbell, *Six theorems about injective metric spaces*, Comment. Math. Helv. **39** (1964) 65–76. https://doi.org/10.1007/BF02566944

[31] I. Kantor and B. Patkós, *Towards a de Bruijn-Erdős theorem in the L_1-metric*, Discrete Comput. Geom. **49** (2013) 659–670. https://doi.org/10.1007/s00454-013-9496-y

[32] D.C. Kay and G. Chartrand, *A characterization of certain ptolemaic graphs*, Canad. J. Math. **17** (1965) 342–346. https://doi.org/10.4153/CJM-1965-034-0

[33] V.B. Le and N.N. Tuy, *The square of a block graph*, Discrete Math. **310** (2010) 734–741. https://doi.org/10.1016/j.disc.2009.09.004

[34] M.A. Morgana and H.M. Mulder, *The induced path convexity, betweenness, and svelte graphs*, Discrete Math. **254** (2002) 349–370. https://doi.org/10.1016/S0012-365X(01)00296-5

[35] H.M. Mulder, The interval function of a graph (Mathematical Centre Tracts 132, Mathematisch Centrum, Amsterdam, 1980). https://doi.org/10.21136/CMJ.1994.128449

[36] H.M. Mulder and L. Nebeský, *Axiomatic characterization of the interval function of a graph*, European J. Combin. **30** (2009) 1172–1185. https://doi.org/10.1016/j.ejc.2008.09.007

[37] H.M. Mulder, *Transit functions on graphs* (*and posets*), in: Convexity in Discrete Structures, M. Changat, S. Klavžar, H.M. Mulder and A. Vijayakumar (Ed(s)), (Ramanujan Math. Soc. Lect. Notes Ser. 5, 2008) 117–130.

[38] L. Nebeský, *A characterization of the interval function of a connected graph*, Czechoslovak Math. J. **44** (1994) 173–178. https://doi.org/10.21136/CMJ.1994.128449

[39] L. Nebeský, *A characterization of the set of all shortest paths in a connected graph*, Math. Bohem. **119** (1994) 15–20. https://doi.org/10.21136/MB.1994.126208

[40] L. Nebeský, *Characterizing the interval function of a connected graph*, Math. Bohem. **123** (1998) 137–144. https://doi.org/10.21136/MB.1998.126307

[41] L. Nebeský, *A characterization of the interval function of a (finite or infinite) connected graph*, Czechoslovak Math. J. **51** (2001) 635–642.
https://doi.org/10.1023/A:1013744324808

[42] L. Nebeský, *The interval function of a connected graph and a characterization of geodetic graphs*, Math. Bohem. **126** (2001) 247–254.
https://doi.org/10.21136/MB.2001.133909

[43] L. Nebeský, *The induced paths in a connected graph and a ternary relation determined by them*, Math. Bohem. **127** (2002) 397–408.
https://doi.org/10.21136/MB.2002.134072

[44] R. Schrader and L. Stenmans, *A de Bruijn-Erdös theorem for* $(q, q-4)$*-graphs*, Discrete Appl. Math. **279** (2019) 198–201.
https://doi.org/10.1016/j.dam.2019.11.008

[45] M. Sholander, *Trees, lattices, order, and betweenness*, Proc. Amer. Math. Soc. **3** (1952) 369–381.
https://doi.org/10.1090/S0002-9939-1952-0048405-5

Received 14 July 2020
Revised 21 May 2021
Accepted 24 May 2021
Available online 19 June 2021