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ON q-CONNECTED CHORDAL GRAPHS WITH MINIMUM NUMBER OF SPANNING TREES

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Abstract

Let k be the largest integer such that $m \geq \frac{(n-k)(n-k-1)}{2} + qk \geq q(n-1)$ for some positive integers n, m, q. Let $S(q, n, m)$ be a set of all q-connected chordal graphs on n vertices and m edges for $\frac{n-k}{2} \geq q \geq 2$. Let $t(G)$ be the number of spanning trees in graph G. We identify $G \in S(q, n, m)$ such that $t(G) < t(H)$ for any H that satisfies $H \in S(q, n, m)$ and $H \ncong G$. In addition, we give a sharp lower bound for the number of spanning trees of graphs in $S(q, n, m)$.

Keywords: chordal graph, spanning tree, enumeration of trees, threshold graph, minimum number of trees, shift transformation.

2020 Mathematics Subject Classification: 05C05, 05C30, 05C17.

1. Introduction

When referring to a graph in this paper, we mean a finite, undirected, simple graph, i.e., a graph without loops or parallel edges. A graph G is *chordal* if every cycle in G that is longer than three has a chord. Chordal graphs have been extensively studied, e.g., [10], and they belong to the family of perfect graphs introduced by Lovász [6]. We say that *vertex v dominates vertex w* if every neighbor of v is also a neighbor of w. Graph $G = (V, E)$ is a *threshold graph* if for any pair of vertices $v, w \in V(G)$ either v dominates w or w dominates v [8]. A graph G is *q-connected* if removing any $q-1$ vertices results in a connected graph, while removing some q vertices results in disconnected graph. Let k be the largest integer such that $|E(G)| \geq \frac{(|V(G)|-k)(|V(G)|-k-1)}{2} + qk \geq q(|V(G)|-1)$. In this paper, we focus on the minimum number of spanning trees $t(G)$ in chordal q-connected graph G for given q such that $\frac{|V(G)|-k}{2} \geq q \geq 2$.

Define the threshold q-connected graph $Q_{n,m,q}$ on n vertices and m edges as follows. $Q_{n,m,q}$ consists of $(n-k)$-clique, joined to $k-1$ vertices of degree q, plus one additional vertex of degree $m - \frac{(n-k)(n-k-1)}{2} - q(k-1) \geq q$, joined to vertices of the clique for some positive integer k. Let x be the largest positive integer such that $m \geq \frac{(n-x)(n-x-1)}{2} + qx \geq q(n-1)$ for some positive integers n, m, q. Define $S(q, n, m)$ to be a set of all q-connected chordal graphs on n vertices and m edges for $\frac{n-x}{2} \geq q \geq 2$. We prove that $G \in S(q, n, m)$ for given n, m, q minimizes the number of spanning trees $t(G)$ if and only if $G \simeq Q_{n,m,q}$. In Figure 1, we illustrate $Q_{n,m,q}$ with vertices $v_2, v_3, \ldots, v_k$ each of degree q, which is less or equal to half the size of $(n-k)$-clique, and with v_1 of degree at least q. Note that if a q-connected simple graph G exists on n vertices and m edges, and x exists that satisfies above inequalities, then $Q_{n,m,q}$ exists for some positive integer $k \leq x$.

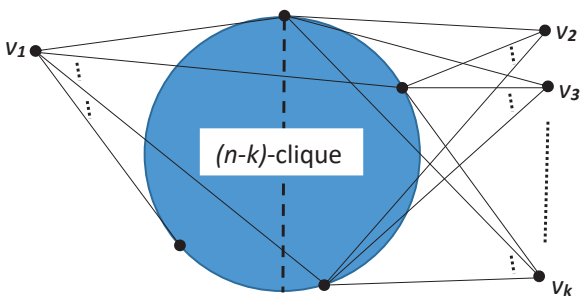

Figure 1. Threshold q-connected graph $Q_{n,m,q}$.

First, Kelmans and Chelnokov [5] proved that $Q_{n,m,q}$ obtained from the complete graph K_n by deleting edges incident to a single vertex minimizes the number of spanning trees over the subset of connected graphs for given n vertices and m edges. Later, Boesch *at al.* [1] conjectured that if $Q_{n,m,q}$ of Kelmans and Chelnokov cannot be obtained, then a special case of $Q_{n,m,q}$, namely $Q_{n,m,1}$, minimizes the number of spanning trees over all connected simple graphs on n vertices and m edges. They pointed out that solving this problem is important in studying the lower bound of network reliability. About 20 years later we proved that conjecture [4]. Even though $Q_{n,m,1}$ is unique among threshold graphs in respect to minimizing the number of spanning trees, it is not unique among connected graphs. Subsequently, we showed in [3] that specific subset of graphs minimizes the number of spanning trees and represents all such graphs. Finally, in [2] it was shown that there is a unique threshold $G \in S(2, n, m)$ such that $t(G) < t(H)$ for any H that satisfies $H \in S(2, n, m)$ and $H \ncong G$. In this paper, we extend this latest result to $G \in S(q, n, m)$, where $\frac{n-k}{2} \geq q \geq 2$.

2. Our Approach and Preliminary Results

Our approach in determining that $Q(n, m, q)$ minimizes the number of spanning trees is based on two phases. In phase 1, we will use graph transformations to identify our threshold graph, which will be done in next Section 3. In phase 2, we will identify specific threshold graph, our $Q(n, m, q)$, based on direct comparison of the related functions, which will be done in the last Section 4.

Let $N(v)$ denote the vertices that are neighbors to vertex v. The graph *shift(G,v,w)* is obtained from G by, for all $x \in N(v)\backslash(N(w) \cup \{w\})$, deleting vx and adding wx. It is known that if *shift(G,v,w)* $= G$ for all v, w, then G is a threshold graph [1, 8]. It was also shown in [4, 8, 9] that every connected graph G can be transformed into a threshold graph H using a series of *shift(G,v,w)* transformations. Consequently we have the following.

Theorem 1 [4]**.** *For any connected graph G, there is a series of shift transformations that produces a threshold graph H, with the same numbers of vertices and edges, such that* $t(H) \leq t(G)$.

For 2-connected graphs, such a series of shift transformations that produces a 2-connected threshold graph does not always exists [2]. For chordal q-connected graph G, however, we show in the next section that such a series of shift transformations does exist, and it transforms G to q-connected chordal graph H.

Let d_i be a degree of vertex v_i. Let $H = H(n; d_1, d_2, \ldots, d_k)$ denote a threshold graph consisting of $(n-k)$-clique, with vertices v_{k+1}, $v_{k+2}, \ldots$, v_n, and an independent set on the remaining k vertices, the i'th one of which is joined to $v_{k+1}, v_{k+2}, \ldots, v_{k+d_i}$.

The following result for $H = H(n; d_1, d_2, \ldots, d_k)$ is known.

Theorem 2 [4]**.** *Suppose* $H = H(n; d_1, d_2, \ldots, d_k)$ *is a connected graph, with* $d_1 \geq d_2 \cdots \geq d_k$. *Set* $d_0 = n - k$ *and* $d_{k+1} = 1$. *Then*

$$t(H) = (n-k)^{-2} \prod_{i=0}^{k} \left(d_i (n-k+i)^{d_i - d_{i+1}} \right). \tag{1}$$

Since Karush-Kuhn-Tucker (KKT) conditions [7] cannot be established, in Section 4 we will do a direct comparison of the functions by evaluating the continuous functions corresponding to (1) based on the following result.

Lemma 3 [2]**.** *Let b, c, k, q, be given positive integers with* $b \geq 3$ *and* $kb - k \geq c > k$. *Let* $x_0 = b$, $x_{k+1} = q$, *and let* $f(x_1, x_2, \ldots, x_k) = \prod_{i=0}^{k} \left(x_i (b+i)^{x_i - x_{i+1}} \right)$. *The minimum of f over the region*

$$P := \left\{ x \in \mathbb{R}^k : \sum_{i=1}^{k} x_i = c, b \geq x_1 \geq x_2 \cdots \geq x_k \geq q \right\}$$

occurs at some point $(x_1, x_2, \ldots, x_k)$ *that satifies at most two of the following inequalities strictly*

$$b \geq x_1 \geq x_2 \cdots \geq x_k \geq q.$$

In summary, our approach in identifying chordal graph $G \in S(q, n, m)$ that minimizes $t(G)$ relies on the following refinments.

(1) $G \to H(n; d_1, d_2, \ldots, d_k)$,

(2) $H(n; d_1, d_2, \ldots, d_k) \to H(n; n-k, n-k, \ldots, n-k, d_r, d_r, \ldots, d_r, q, q, \ldots, q)$,

(3) $H(n; n-k, n-k, \ldots, n-k, d_r, d_r, \ldots, d_r, q, q, \ldots, q) \to H(n; n-k, n-k, \ldots, n-k, d_r, q, q, \ldots, q)$,

(4) $H(n; n-k, n-k, \ldots, n-k, d_r, q, q, \ldots, q) \to H(n; d_1, q, q, \ldots, q)$,

which will be done in the next two sections.

3. Generic Threshold Graph with Fewest Spanning Trees

In this section, we use notation $shift(G, v, w) = shift1(G, v, w)$ if v, w are adjacent, and $shift(G, v, w) = shift2(G, v, w)$ if v, w are of distance 2 from each other.

The following three theorems for 2-connected chordal graphs pertaining to $shift1(G, v, w)$ and $shift2(G, v, w)$ were proved in [2].

Theorem 4 [2]**.** *Let $G(V, E)$ be a chordal 2-connected graph. Let $v, w \in V(G)$ be two adjacent vertices and $shift1(G, v, w) = H \neq G$. Then H is also chordal q-connected graph and $t(G) > t(H)$ for $q \geq 2$.*

Theorem 5 [2]**.** *Let $G(V, E)$ be a chordal 2-connected graph. Let $D(G)$ be the set of vertices in G of degree $|V(G)| - 1$ and $G - D(G)$ is a disconnected graph. Let $v, w \in V(G)$ be two vertices, where $d_G(v, w) = 2$ and $shift2(G, v, w) = H \neq G$. Then for $q \geq 2$ H is also chordal q-connected simple graph and $t(G) > t(H)$.*

Theorem 6 [2]**.** *For any chordal 2-connected G that is not a threshold graph, there is a series of shift transformations, consisting of shift1 and shift2, that produces 2-connected threshold graph H, with the same numbers of vertices and edges, such that $t(H) < t(G)$.*

We can extend these three theorems from 2-connected to q-connected chordal graphs as follows.

Lemma 7. *Let $G(V, E)$ be a chordal q-connected graph for $q \geq 2$. Let $v, w \in V(G)$ be two adjacent vertices and $shift1(G, v, w) = H \neq G$. If $shift1(G, v, w) = H \neq G$, then H is also chordal q-connected graph and $t(G) > t(H)$.*

Proof. There must be at least $q-1$ paths of length 2 between any two adjacent vertices v, w in G. Otherwise, there would exist an induced cycle C containing some edge e_i and vertices v, w of length at least 4 — a contradiction. Hence, $shift1(G, v, w) = H \neq G$ produces q-connected graph H. Furthermore, by Theorem 4 such H is also chordal and $t(G) > t(H)$ if $shift1(G, v, w) = H \neq G$ is satisfied. ■

Lemma 8. *Let $G(V, E)$ be a chordal q-connected graph for $q \geq 2$. Let $D(G)$ be the set of vertices in G of degree $|V(G)|-1$ and $G-D(G)$ is a disconnected graph. Let $v, w \in V(G)$ be two vertices, where $d_G(v, w) = 2$ and $shift2(G, v, w) = H \neq G$. Then H is also chordal q-connected simple graph and $t(G) > t(H)$.*

Proof. Let $d_G(v, w) = 2$ and $shift2(G, v, w) = H \neq G$. If removing $D(G)$ from G results in disconnected graph G', then vertices v, w must belong to some connected component W of G'. Then $shift2(W, v, w) = W' \neq W$ is satisfied and W' is connected. This implies that H is q-connected. In addition, by Theorem 5 H is chordal and $t(G) > t(H)$. ■

Based on Lemmas 7–8 we can now state the following.

Theorem 9. *For any chordal q-connected G that is not a threshold graph, there is a series of shift transformations, consisting of shift1 and shift2, that produces q-connected threshold graph H for $q \geq 2$, with the same numbers of vertices and edges, such that $t(H) < t(G)$.*

Proof. Follows directly based on Lemmas 7–8 and Theorem 6. ■

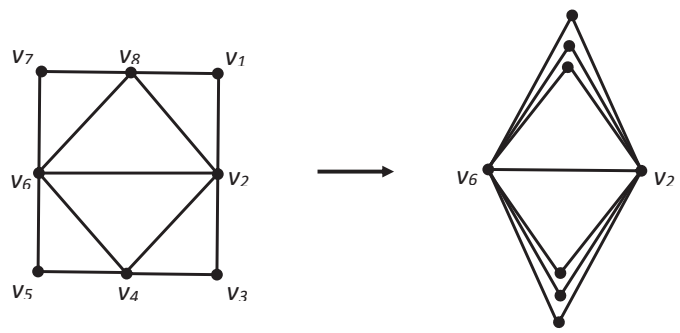

Figure 2. Example of transforming 2-connected chordal graph into $Q_{8,13,2}$.

Consequently, Theorem 9 proves that any q-connected chordal graph G with $q \geq 2$ minimizes the number of spanning trees if and only if G is isomorphic to some q-connected threshold graph. Figure 2 illustrates transformation of 2-connected chordal graph on $n = 8$ vertices and $m = 13$ edges displayed on the

left-hand side into chordal 2-connected graph $Q_{8,13,2}$ displayed on the right-hand side by executing two *shift*1 transformations from vertices v_8, v_4 to vertex v_6, followed by two *shift*1 transformations from vertices v_8, v_4 to vertex v_2.

4. Specific Threshold Graph with Fewest Spanning Trees

Before presenting the main theorem of this section we first prove the following two lemmas based on a direct comparison of the functions, since KKT conditions [7] cannot be applied to (1). In the next lemma we will compare two feasible functions over the feasible region where they are distinct and might be minimum based on Lemma 3. This means that if $c = \sum_{i=1}^{k} d_i$ and $b = n - k$ in a threshold graph $H(n; d_1, d_2, \ldots, d_k)$, then the relation $bk - k \geq c \geq qk + k$ must be satisfied in order for these functions to be distinct.

Lemma 10. *Let b, c, k, q be given positive integers with $k, q \geq 2$, $b/2 \geq q$, and $kb - k \geq c = \sum_{i=1}^{k} x_i \geq kq + k$. Let $x_0 = b$, $x_{k+1} = 1$, and let $f(x_1, x_2, \ldots, x_k) = \prod_{i=0}^{k} (x_i(b+i)^{x_i - x_{i+1}})$. Let $f_1 = f(x_1, x_2, \ldots, x_k)$ if $x_1 = x_2 = \cdots = x_k$, and let $f_2 = f(x_1, x_2, \ldots, x_k)$ if $x_1 = x_2 = \cdots = x_{r-1} = b > x_r \geq q$ and $x_{r+1} = x_{r+2} = \cdots = x_k = q$, for $r \geq 1$. Then, $f_1 > f_2$.*

Proof. Let $g_1(b, c, k), g_2(b, c, k, q)$ be two functions corresponding to f_1, f_2, respectively, as follows

$$g_1(b, c, k) = b^{b - \frac{c}{k} + 1} \left(\frac{c}{k}\right)^k (b + k)^{\frac{c}{k} - 1} \tag{2}$$

and

$$g_2(b, c, k, q) = b^r \big(c + q - (b - q)(r - 1) - qk\big) q^{k-r} \times$$

$$(b + r - 1)^{b - (c + q - (b-q)(r-1) - qk)} (b + r)^{c - (b-q)(r-1) - qk} (b + k)^{q-1}, \tag{3}$$

where $r = \left\lfloor \frac{c - qk}{b - q} \right\rfloor + 1$. Let us assume $b, c, k, q, r \in R$. The proof follows by direct comparison of $g_1(b, c, k)$ with $g_2(b, c, k, q)$. First, we compare them for the largest allowed c, and then we compare them with the smallest allowed c.

Claim 1. *If $c = kb - k$, then $g_1(b, c, k) > g_2(b, c, k, q)$.*

Proof. This means that we compare the following functions

$$g_1(b, k) = b^2 (b - 1)^k (b + k)^{b-2}$$

and

$$g_2(b, k, q) = b^r((b-q)(k-r) + b - k) q^{k-r} \left(\frac{b + r}{b + r - 1}\right)^{(b-q)(k-r)-k} (b+r)^{b-q} (b+k)^{q-1},$$

where $r = \frac{k(b-q-1)}{b-q} + 1$. So, we compare $g_1(b,k)$ with the following

$$g_2(b,k,q) = b^{\frac{k(b-q-1)}{b-q}+1} q^{\frac{k}{b-q}} \left(\frac{(b-q)(b+k)-k}{b-q} \right)^{b-q} (b+k)^{q-1}.$$

Define $g_3(b,k,q) = \ln(g_1(b,k,q)/g_2(b,k,q))$. Then,

$$\begin{aligned} g_3(b,k,q) &= 2\ln b + k\ln(b-1) + (b-q-1)\ln(b+k) - \frac{k(b-q-1)+(b-q)}{b-q}\ln b \\ &\quad - \frac{k}{b-q}\ln(q) - (b-q)(\ln((b-q)(b+k)-k) - \ln(b-q)). \end{aligned}$$

We evaluate $g_3(b,k,q)$ as follows. First we verify that for $b = 2q = 4$ and $k = 2$, $g3(b,k,q) \approx 0.08 > 0$. Then,

$$\begin{aligned} &\frac{\partial g_3(b,k,q)}{\partial k} \\ &= \left[\ln(b-1) - \frac{b-q-1}{b-q}\ln(b) - \frac{1}{b-q}\ln(q)\right] + \left[\frac{b-q-1}{b+k} - \frac{(b-q)(b-q-1)}{(b-q)(b+k)-k}\right] \\ &\geq \left[\ln(b-1) - \frac{b-\frac{b}{2}-1}{b-\frac{b}{2}}\ln(b) - \frac{1}{b-\frac{b}{2}}\ln\left(\frac{b}{2}\right)\right] + \left[\frac{b-2-1}{b+k} - \frac{(b-2)(b-2-1)}{(b-2)(b+k)-k}\right] \\ &= \ln(b-1) + \frac{2}{b}\ln(2) - \ln(b) + \frac{b-3}{b+k} - \frac{(b-2)(b-3)}{(b-2)(b+k)-k} = g_4(b,k), \end{aligned}$$

because based on standard evaluation $\frac{\partial}{\partial q}\left(\ln(b-1) - \frac{b-q-1}{b-q}\ln(b) - \frac{1}{b-q}\ln(q)\right) < 0$ and $\frac{\partial}{\partial q}\left(\frac{b-q-1}{b+k} - \frac{(b-q)(b-q-1)}{(b-q)(b+k)-k}\right) > 0$. Furthermore, after lengthy but straithforward evaluation, $\frac{\partial g_4(b,k)}{\partial k} = 0$ for $k = b\sqrt{\frac{b-2}{b-3}}$ that results in minimum of $g_4(b,k)$. Based on standard evaluation,

$$g_4(b) = \ln(b-1) + \frac{2}{b}\ln(2) - \ln(b) + \frac{b-3}{b\left(1+\sqrt{\frac{b-2}{b-3}}\right)} - \frac{(b-2)(b-3)}{(b-2)\left(b\left(1+\sqrt{\frac{b-2}{b-3}}\right)\right) - b\sqrt{\frac{b-2}{b-3}}}$$

is positive for $b = 4$ (i.e., $g_4(4) = 0.016$), $\frac{dg_4(b)}{db}$ asymptotically converges to 0 as b approaches infinity, and $\frac{dg_4(b)}{db} < 0$. This implies that $\frac{\partial g_3(b,k,q)}{\partial k} \geq 0$. Consequently, we may assume $k = 2$ in $g_1(b,k), g_2(b,k,q)$ and focus on comparing the following two functions

$$g_1(b) = b^2(b-1)^2(b+2)^{b-2}$$

and

$$g_2(b,q) = b^{\frac{2(b-q-1)}{b-q}+1} q^{\frac{2}{b-q}} \left(\frac{(b-q)(b+2)-2}{b-q} \right)^{b-q} (b+2)^{q-1}.$$

Define $g_3(b,q) = \ln(g_1(b)/g_2(b,q))$. We obtain,

$$\begin{aligned} g_3(b,q) &= (b-q-1)\ln(b+2) - \frac{b-q-2}{b-q}\ln b + 2\ln(b-1) \\ &\quad -\frac{2}{b-q}\ln(q) - (b-q)(\ln((b-q)(b+2)-2) - \ln(b-q)). \end{aligned}$$

Then,

$$\begin{aligned} \frac{\partial g_3(b,q)}{\partial q} &= \left[\frac{2}{(b-q)^2}\ln\left(\frac{b}{q}\right)\right] + \left[\ln\left(1 - \frac{2}{(b-q)(b+2)}\right)\right] \\ &\quad + \frac{(b-q)(b+2)}{(b-q)(b+2)-2} - \frac{2}{q(b-q)} - 1 \\ &< \left[\frac{2}{(b-q)^2}\ln\left(\frac{b}{q}\right)\right] + \left[-\frac{2}{b-q}\left(\frac{1}{b+2} + \frac{1}{(b+2)^2(b-q)}\right)\right] \\ &\quad + \frac{(b-q)(b+2)}{(b-q)(b+2)-2} - \frac{2}{q(b-q)} - 1 \\ &= \left[\frac{2}{(b-q)^2}\ln\left(\frac{b}{q}\right)\right] + \left[\frac{(2b^2q+8bq+12q+8)-(2b^3+8b^2+4b)}{q(b-q)(b+2)((b-q)(b+2)-2)}\right] \\ &< \left[\frac{2}{(b-q)^2}\ln\left(\frac{b}{q}\right)\right] + \left[\frac{(2b^2q+8bq+12q+8)-(2b^3+8b^2+4b)}{q(b-q)^2(b+2)^2}\right] \\ &= \frac{2}{(b-q)^2}\left[\ln\left(\frac{b}{q}\right) - \frac{(b^3+4b^2+2b)-(b^2q+4bq+6q+4)}{q(b+2)^2}\right] \\ &= \frac{2}{(b-q)^2}\left[\ln\left(\frac{b}{q}\right) - \left(\frac{b}{q} - 1 - \frac{2(b+q+2)}{q(b+2)^2}\right)\right] = \frac{2}{(b-q)^2}g_4(b,q) < 0, \end{aligned}$$

for $q \le b/2$, since $g_4(b,q) < 0$ for $b = 2q = 4$ and $\partial g_4(b,q)/\partial b = 1/b - 1/q - 2(b+2)^{-2}/q - 4(b+2)^{-3} < 0$. Hence, we may assume $q = b/2$ and compare functions $g_1(b), g_2(b) = g_2(b, b/2)$. Let $g_3(b) = \ln(g_1(b)/g_2(b))$. Then,

$$\begin{aligned} g_3(b) &= 2\ln(b-1) + \frac{b-2}{2}\ln(b+2) - \frac{b-4}{b}\ln b - \frac{4}{b}\ln\left(\frac{b}{2}\right) \\ &\quad - \frac{b}{2}\left(\ln\left(\frac{b(b+2)}{2} - 2\right) - \ln\left(\frac{b}{2}\right)\right) \\ &= 2\ln(b-1) + \left(\frac{b}{2} - 1\right)\ln(b+2) - \frac{b}{2}\ln\left(b^2+2b-4\right) \\ &\quad + \frac{4}{b}\ln(2) + \left(\frac{b}{2} - 1\right)\ln(b). \end{aligned}$$

For $b = 4$ we verify that $g_3(b) \approx 0.077 > 0$. We also verify that

$$\begin{aligned}\tfrac{dg_3(b)}{db} &= \frac{1}{2}\ln(b+2) + \frac{1}{2}\ln(b) - \frac{1}{2}\ln(b^2+2b-4) - \frac{4}{b^2}\ln(2)\\ &+ \frac{2}{b-1} + \frac{(b-2)(b+1)}{b(b+2)} - \frac{b(b+1)}{b^2+2b-4},\end{aligned}$$

and for $b < 4.1456$ we have $dg_3(b)/db > 0$, while for $b > 4.1456$ we have $dg_3(b)/db < 0$ (with local maximum for $g_3(b)$ at $b \approx 4.1456$) and that $\lim_{b\to\infty} dg_3(b)/db = \lim_{b\to\infty} g_3(b) = 0$. Consequently, $g_1(b,c,k) > g_2(b,c,k,q)$ for $c = bk - k$. □

Claim 2. *If $c = qk + k$, then $g_1(b,c,k) > g_2(b,c,k,q)$.*

Proof. We now compare $g_1(b,c,k)$ with $g_2(b,c,k,q)$ for least allowed c for given b, k, q. This means that we compare the following functions

$$g_1(b,k,q) = b^{b-q}(q+1)^k(b+k)^q$$

and

$$g_2(b,k,q) = b^r(q+k-(b-q)(r-1))q^{k-r}\left(\frac{b+r}{b+r-1}\right)^{k-(b-q)(r-1)}(b+r-1)^{b-q}(b+k)^{q-1},$$

where $r = \frac{k}{b-q} + 1$. So, we compare $g_1(b,k,q)$ with the following

$$g_2(b,k,q) = b^{\frac{k}{b-q}+1}q^{\frac{k(b-q-1)}{b-q}}\left(b + \frac{k}{b-q}\right)^{b-q}(b+k)^{q-1}.$$

Define $g_3(b,k,q) = \ln(g_1(b,k,q)/g_2(b,k,q))$. Then, after straightforward evaluation we obtain

$$\begin{aligned}g_3(b,k,q) &= \left(b - q - \frac{k}{b-q} - 1\right)\ln b - \frac{k(b-q-1)}{b-q}\ln(q) + k\ln(q+1)\\ &- (b-q)\ln\left(b + \frac{k}{b-q}\right) + \ln(b+k).\end{aligned}$$

Then,

$$\frac{\partial g_3(b,k,q)}{\partial k} = \left[\ln(q+1) - \frac{1}{b-q}\ln(b) - \frac{b-q-1}{b-q}\ln(q)\right] + \left[\frac{1}{b+k} - \frac{1}{b+\frac{k}{b-q}}\right].$$

After further evaluation, we obtain $\partial^2 g_3(b,k,q)/\partial k^2 = 1/((b-2)(b+k)/(b-2))^2 - 1/(b+k)^2 = 0$ for $k = b(b-q)(\sqrt{b-q}-1)/(b-q-\sqrt{b-q})$ that results

in minimum of $\partial g_3(b,k,2)/\partial k$, and $\partial^2 g_3(b,k,q)/\partial q^2\big|_{k=\frac{b(b-q)(\sqrt{b-q}-1)}{b-q-\sqrt{b-q}}} < 0$. Hence, after substitutions $k = b(b-q)(\sqrt{b-q}-1)/(b-q-\sqrt{b-q})$ and $q = b/2$ we obtain

$$g_4(b) = \ln(b+2) - \ln(b) - \frac{2}{b}\ln(2) + \frac{b-2\sqrt{\frac{b}{2}}}{b(b-2)\sqrt{\frac{b}{2}}} - \frac{b-2\sqrt{\frac{b}{2}}}{b(b-2)},$$

which satisfies $dg_4(b)/db < 0$, $\lim_{b\to\infty} g_4(b) = 0$, and $g_4(4) \approx 0.016$. So, we conclude that $\partial g_3(b,k,q)/\partial k > 0$. Consequently, we may assume $k = 2$ in $g_3(b,k,q)$ and evaluate the following

$$\begin{aligned} g_3(b,q) &= (b-q-\frac{2}{b-q}-1)\ln(b) - \frac{2(b-q-1)}{b-q}\ln(q) + 2\ln(q+1) \\ &\quad - (b-q)\ln\left(b+\frac{2}{b-q}\right) + \ln(b+2). \end{aligned}$$

Then, after standard evaluation,

$$\begin{aligned} \frac{\partial g_3(b,q)}{\partial q} &= -\frac{2}{(b-q)^2}(\ln(b)-\ln(q)) + \ln\left(b+\frac{2}{b-q}\right) \\ &\quad - \ln(b) - \frac{2}{b(b-q)+2} + \frac{2}{q(b-q)} + \frac{2}{q+1} - \frac{2}{q} \\ &= \left[-\frac{2}{(b-q)^2}\ln\left(\frac{b}{q}\right)\right] + \left[\ln\left(1+\frac{2}{b(b-q)}\right)\right] \\ &\quad - 2\frac{(b-2q-1)(b(b-q)+2)+q(q+1)(b-q)}{q(q+1)(b(b-q)+2)(b-q)} \\ &< \left[-\frac{2}{(b-q)^2}\left(\frac{b-q}{b}\right)\right] + \left[\frac{2}{b(b-q)}\right] \\ &\quad - 2\frac{(b-2q-1)(b(b-q)+2)+q(q+1)(b-q)}{q(q+1)(b(b-q)+2)(b-q)} \\ &= -2\frac{(b-2q-1)(b(b-q)+2)+q(q+1)(b-q)}{q(q+1)(b(b-q)+2)(b-q)} < 0, \end{aligned}$$

for $q \le b/2$. Hence, we may assume $q = b/2$ in $g_3(b,q)$ and evaluate the following

$$\begin{aligned} g_3(b) &= \left(\frac{b}{2} - \frac{2}{\frac{b}{2}} - 1\right)\ln(b) - \frac{b-2}{\frac{b}{2}}\ln\left(\frac{b}{2}\right) + 2\ln\left(\frac{b}{2}+1\right) \\ &\quad - \frac{b}{2}\ln\left(b+\frac{2}{\frac{b}{2}}\right) + \ln(b+2) \end{aligned}$$

$$= \left(\frac{b}{2} - \frac{4}{b} - 1\right)\ln(b) - \frac{2(b-2)}{b}\ln\left(\frac{b}{2}\right) + 2\ln\left(\frac{b+2}{2}\right)$$
$$- \frac{b}{2}\ln\left(b + \frac{4}{b}\right) + \ln(b+2)$$
$$= \frac{b-6}{2}\ln(b) - \frac{4}{b}\ln(2) + 3\ln(b+2) - \frac{b}{2}\ln\left(b + \frac{4}{b}\right).$$

Then,

$$\frac{dg_3(b)}{db} = \frac{4}{b^2+4} + \frac{4\ln(2)}{b^2} + \frac{3}{b+2} - \frac{1}{2}\ln\left(1 + \frac{4}{b^2}\right) - \frac{3}{b}.$$

Based on straithforward evaluation, $dg_3(b)/db > 0$ for $b < 5.763$, $dg_3(b)/db < 0$ for $b > 5.763$, and maximum for $g_3(b)$ is obtained at $b = 5.763$. Furthermore, $\lim_{b\to\infty} g_3(b) = 0$, and $g_3(4) \approx 0.077$. Hence, we conclude that $g_1(b,c,k) > g_2(b,c,k,q)$ for $c = qk + k$. □

Based on (2) and (3), define $g_3(b,c,k,q) = \ln(g_1(b,c,k)/g_2(b,c,k,q))$. From Claims 1–2, $g_3(b,c,k,q) > 0$ for $c = bk - k$ and for $c = qk + k$. By examining $\partial g_3(b,c,k,q)/\partial c = 0$ over extended region for c, $bk \geq c \geq qk$, we conclude that there must be at most two extreme points between $c = qk$ and $c = bk$. For given b, k, q we have $g_1(b,qk,k) = g_2(b,qk,k,q) = b^{b-q+1}q^k(b+k)^{q-1}$ and $g_1(b,bk,k) = g_2(b,bk,k,q) = b^{k+1}(b+k)^{b-1}$. This means that $g_3(b,c,k,q) = 0$ for $c = qk$ and for $c = bk$. So, there must be exactly one extreme point and it must be maximum for $kb - k \geq c \geq kq + k$. This proves that $g_1(b,c,k) > g_2(b,c,k,q)$ for $kb - k \geq c \geq kq + k$. ■

We note here that if $q > c/2 = (n-k)/2$, then $f_1 < f_2$ is possible for f_1, f_2 from Lemma 10. The smallest such example is for $|V(G)| = 13$ by comparing $G' = H(13; 5,5,5,5,5,5,5)$ corresponding to f_1 with $G'' = H(13; 6,6,6,5,4,4,4)$ corresponding to f_2. In this case, $k = 7, n-k = 6, q = 4$, and the corresponding number of spanning trees are $t(G') = 2231328125 < 2277849600 = t(G'')$. In spite of this, we conjecture that $t(Q_{n,m,q}) < t(G)$ for every G such that $G \in S(q,n,m), G \ncong Q_{n,m,q}$, and $n - k > q \geq 1$. In the rest of this paper we prove this conjecture for $(n-k)/2 \geq q \geq 2$.

Based on Lemmas 3 and 10 we obtain the following result.

Lemma 11. *Let b, c, k, be given positive integers with $b \geq 2q \geq 4$, $kb - k \geq c \geq qk + k$. Let $x_0 = b$, $x_{k+1} = q$, and $g(x_1, x_2, \ldots, x_k) = \prod_{i=0}^{k}(x_i(b+i)^{x_i - x_{i+1}})$. The minimum of g over the region*

$$P := \left\{x \in \mathbb{N}^k : \sum_{i=1}^{k} x_i = c, b \geq x_1 \geq x_2 \cdots \geq x_k \geq q\right\}$$

occurs at some point $(x_1, x_2, \ldots, x_k)$ *if and only if* $x_1 = x_2 = \cdots = x_{r-1} = b$, $x_r > q$ *and* $x_{r+1} = x_{r+2} = \cdots = x_k = q$, *for some* $r \geq 1$.

Proof. Suppose a minimum of g occurs at some point $(x_1, x_2, \ldots, x_k)$ where $b > x_i \geq x_{i+1} > q$ is satisfied. Let t be the least index for which $x_t < b$, and let r be the largest index for which $x_r > q$. Then we have the following two cases to consider.

Case 1. $x_t = x_r$ is satisfied. In this case $b > x_t = x_{t+1} = \cdots = x_r > q$. Consider corresponding function $f_1(x_t, x_{t+1}, \ldots, x_r)$ from Lemma 10, where $b = x_{t-1}, q = x_{r+1}$, $x \in \mathbb{R}^{r-t+1}$. Since $b/2 \geq q$, by Lemma 10, $f_1 > f_2$ — a contradiction.

Case 2. $x_t > x_r$ is satisfied. Consider corresponding function $f(x_t, x_r)$ from Lemma 3, where $b = x_{t-1}, q = x_{r+1}$, $x \in \mathbb{R}^{r-t+1}$. Clearly, there must be index i, $t \geq i > r$, such that $x_i > x_{i+1}$. Consequently, $b > x_t$, $x_i > x_{i+1}$, and $x_r > q$ are satisfied. So, by Lemma 3 f is not minimum — a contradiction.

Since both cases are not feasible, then by Lemma 3 the minimum of g must occur at some point $(x_1, x_2, \ldots, x_k)$, where $x_1 = x_2 = \cdots = x_{r-1} = b$, $x_r > q$ and $x_{r+1} = x_{r+2} = \cdots = x_k = q$, for some $r \geq 1$. ■

According to the introduced notations for our specific threshold graph, $Q_{n,m,q} = H(n; d_1, d_2, \ldots, d_k) = H(n; d_1, q, \ldots, q)$, we can now present the main result of this paper.

Theorem 12. *Let* $G \in S(q, n, m)$. *If* $G \ncong Q_{n,m,q}$, *then* $t(G) > t(Q_{n,m,q})$.

Proof. Suppose $G \ncong Q_{n,m,q}$, $G \in S(q, n, m)$, and $t(G) \leq t(Q_{n,m,q})$. By Theorem 9 G must be a q-connected threshold graph. Furthermore, by Lemma 11, G must be of form $H\big(n; d_1, d_2, \ldots, d_i, \ldots, d_k\big) = H\big(n; n-k, \ldots, n-k, d_i, q, \ldots, q\big)$ for some $i, k \geq 2$ and $n - k \geq d_i > q$. If $i = 2$, then $G = H\big(n; n-k, d_2, q, \ldots, q\big) \simeq H\big(n; d'_1, q, \ldots, q\big) = H\big(n; d'_1, d'_2, \ldots, d'_{k-1}\big) \simeq Q_{n,m,q}$, where $d'_j = d_{j+1}$ for $j \leq k-1$ — a contradiction. So, $i > 2$ must be satisfied. In this case $G = H(n; n-k, \ldots, n-k, d_i, q, \ldots, q) \simeq H\big(n; n-k, \ldots, n-k, d'_{i-1}, q, \ldots, q\big) = H\big(n; d'_1, d'_2, \ldots, d'_{k-1}\big)$, where $d'_j = d_{j+1}$ for $j \leq k-1$. Furthermore, $H(n; n-k, \ldots, n-k, d'_{i-1}, q, \ldots, q)$ can be transformed to $H'\big(n; n-k+1, \ldots, n-k+1, d''_j, q, \ldots, q\big)$ for some $j \leq i$, which is not isomorphic to $H\big(n; n-k, \ldots, n-k, d'_{i-1}, q, \ldots, q\big)$. Then by Lemma 11

$$t\big(H'\big(n; n-k+1, \ldots, n-k+1, d''_j, q, \ldots, q\big)\big) < t\big(H\big(n; n-k, \ldots, n-k, d'_{i-1}, q, \ldots, q\big)\big)$$

— a contradiction. This contradiction proves Theorem 12. ■

If $m \geq \frac{(n-1)(n-2)}{2} + q$, then $Q_{n,m,q}$ represents a threshold graph obtained from K_n by removing k edges adjacent to a single vertex, where $n - q - 1 \geq k \geq 1$.

In this case $S(q-1,n,m)=\emptyset$, which implies that $Q_{n,m,q-1}$ does not exist. Conversely, if $m<\frac{(n-1)(n-2)}{2}+q$, then $Q_{n,m,q-1}$ does exist. So, we can state the following corollary to Theorem 12.

Corollary 13. *Let p be the largest integer such that $m \geq \frac{(n-p)(n-p-1)}{2}+qp$, $\frac{n-p}{2} \geq q \geq 2$, and let $m<\frac{(n-1)(n-2)}{2}+q$ for given positive integers n,m,q. If $G \in S(q,n,m)$, then $t(G)>t(Q_{n,m,q-1})$.*

Proof. If $G \in S(q,n,m)$ then by Theorem 12 $t(G) \geq t(Q_{n,m,q})$. Furthermore, by Lemma 11 $t(Q_{n,m,q})>t(Q_{n,m,q-1})$. ■

In closing we note that by setting $r=1$ and dividing formula (3) by b^2 we obtain,

$$(c+q-qp)q^{p-1}b^{b-c-q+qp-1}(b+1)^{c-qp}(b+p)^{q-1},$$

which represents a sharp lower bound for the number of spanning trees of q-connected graphs $H \in S(q,n,m)$ from Theorem 12, where $c=m-(n-p)(n-p-1)/2$, $b=n-p$, and $p=k$ represents the number of vertices not included in clique of $Q_{n,m,q}$ (i.e., vertices $v_1,v_2,\ldots,v_k$ in Figure 1 illustrating $Q_{n,m,q}$). Note also that this sharp lower bound on the number of spanning trees pertains to $(n-p)/2 \geq q \geq 1$ since for $q=1$ it is implied from the result in [4].

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