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STRONG AND WEAK PERFECT DIGRAPH THEOREMS FOR PERFECT, α-PERFECT AND STRICTLY PERFECT DIGRAPHS

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Abstract

Perfect digraphs have been introduced in [S.D. Andres and W. Hochstättler, *Perfect digraphs*, J. Graph Theory 79 (2015) 21–29] as those digraphs where, for any induced subdigraph, the dichromatic number and the symmetric clique number are equal. Dually, we introduce a directed version of the clique covering number and define α*-perfect* digraphs as those digraphs where, for any induced subdigraph, the clique covering number and the stability number are equal. It is easy to see that α-perfect digraphs are the complements of perfect digraphs. A digraph is *strictly perfect* if it is perfect and α-perfect. We generalise the Strong Perfect Graph Theorem and Lovász ([*A characterization of perfect graphs*, J. Combin. Theory Ser. B 13 (1972) 95–98]) asymmetric version of the Weak Perfect Graph Theorem to the classes of perfect, α-perfect and strictly perfect digraphs. Furthermore, we characterise strictly perfect digraphs by symmetric chords and non-chords in their directed cycles. As an example for a subclass of strictly perfect digraphs, we show that directed cographs are strictly perfect.

Keywords: perfect digraph, α-perfect digraph, strictly perfect digraph, Strong Perfect Graph Theorem, Weak Perfect Graph Theorem, dichromatic number, perfect graph, directed cograph, filled odd hole, filled odd antihole, acyclic set, clique-acyclic clique.

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1. Introduction

The *dichromatic number* $\chi(D)$ of a digraph D is the minimum size of a partition of the vertex set of D into subsets that induce acyclic digraphs. The study of this generalisation of the chromatic number of a graph was initiated by Neumann-Lara [11]. We consider *perfect digraphs*, which, based on the dichromatic number, were introduced by Andres and Hochstättler [1].

Perfect digraphs are defined as follows. A *symmetric clique* in a digraph D is a subdigraph (V', A') of D such that for any ordered pair $(v, w) \in V' \times V'$ of vertices with $v \neq w$ there is an arc $(v, w) \in A'$. The *symmetric clique number* $\omega(D)$ of D is the number of vertices in a largest symmetric clique of D. It is a trivial lower bound for the dichromatic number. A digraph is *perfect* if, for any induced subdigraph H of D, we have

$$\omega(H) = \chi(H).$$

Here we will define a dual notion, *α-perfect digraphs*, which turn out to be the complements of perfect digraphs. We call a digraph *strictly perfect* if it is perfect and α-perfect.

Figure 1. The perfect digraph $\overline{\vec{C}_4}$ and its complement, the non-perfect but α-perfect digraph $\vec{C}_4$. The numbers indicate a partition into acyclic digraphs (for the digraph $\overline{\vec{C}_4}$) and a covering by clique-acyclic cliques (for the digraph $\vec{C}_4$; for the definition of clique-acyclic cliques see Section 2).

An example of a complementary pair of digraphs is the directed 4-cycle $\vec{C}_4$ and its complement $\overline{\vec{C}_4}$, see Figure 1. It is easy to see that the digraph $\overline{\vec{C}_4}$ is perfect (see the numbering in Figure 1), and the digraph $\vec{C}_4$ is not perfect (since $\chi(\vec{C}_4) = 2 > 1 = \omega(\vec{C}_4)$). We will see that, therefore, $\vec{C}_4$ is α-perfect and $\overline{\vec{C}_4}$ is not α-perfect.

The focus of this paper is on generalising weak and strong Perfect Graph Theorems to perfect, α-perfect and strictly perfect digraphs.

1.1. Motivation

The notions of perfect and α-perfect digraphs are motivated by the distinction between χ-perfect and α-perfect graphs (cf. [8, p. 52]) that was made before Lovász [9] proved the Weak Perfect Graph Theorem. The latter theorem confirmed that these notions coincide for undirected graphs.

Theorem 1 (Weak Perfect Graph Theorem [9]). *A graph is perfect if and only if its complement is perfect.*

The first proof of the Weak Perfect Graph Theorem by Lovász [9] uses sophisticated methods. In order to prove the Weak Perfect Graph Theorem in a second, very elegant way, Lovász [10] proved the following asymmetric form of the Weak Perfect Graph Theorem.

Theorem 2 (Asymmetric Weak Perfect Graph Theorem [10]). *A graph is perfect if and only if, for any induced subgraph H,*

$$\omega(H)\alpha(H) \geq |V(H)|.$$

Here α denotes the stability number and ω the clique number of H. Obviously, Theorem 2 implies Theorem 1.

An explicit characterisation of perfect graphs by forbidden induced subgraphs was proved by Chudnovsky, Robertson, Seymour, and Thomas [6]. Here an *odd hole* is an (induced) cycle of odd length ≥ 5 and an *odd antihole* is an (induced) complement of an odd hole.

Theorem 3 (Strong Perfect Graph Theorem [6]). *A graph is perfect if and only if it contains neither an odd hole nor an odd antihole as an induced subgraph.*

1.2. Key of generalisation and known implications

In order to obtain a generalisation of Theorem 3 to perfect digraphs, Andres and Hochstättler [1] observed that the perfectness of a digraph is related to the perfectness of its symmetric part. The *symmetric part* $S(D)$ of a digraph $D = (V, A)$ is the graph on V, where vw is an edge if and only if $(v, w) \in A$ and $(w, v) \in A$.

Theorem 4 (Symmetrical Reduction Theorem [1]). *A digraph D is perfect if and only if its symmetric part $S(D)$ is perfect and D does not contain any directed cycle with at least 3 vertices as an induced subdigraph.*

The Symmetrical Reduction Theorem will be the key of all our generalisations of Perfect Graph Theorems to digraphs.

The complement of a perfect digraph needs not be perfect, as illustrated in Figure 1. Thus, Theorem 1 has no direct analogue in digraphs. However, Andres and Hochstättler [1] remarked that Theorem 1 and Theorem 4 imply the following theorem.

Theorem 5 (Complementary Characterisation of Perfect Digraphs [1]). *A digraph D is perfect if and only if its complement $\overline{D}$ is a clique-acyclic superorientation of a perfect graph.*

A *filled odd hole* is a digraph D such that $S(D)$ is an odd hole (i.e., $S(D)$ is an undirected cycle of odd length ≥ 5). A *filled odd antihole* is a digraph D such that $S(D)$ is an odd antihole. Using the Strong Perfect Graph Theorem [6] and the Symmetrical Reduction Theorem (Theorem 4), perfect digraphs were characterised by forbidden induced subdigraphs in [1] as follows.

Theorem 6 (Strong Perfect Digraph Theorem [1])**.** *A digraph D is perfect if and only if it contains no induced*

- *filled odd holes,*
- *filled odd antiholes,*
- *directed cycles with at least* 3 *vertices.*

We note that also generalisations of the Semi-strong Perfect Graph Theorem (cf. Reed [12]) to digraphs are possible. A specific such generalisation (that also uses the Symmetrical Reduction Theorem in its proof) was given in a recent paper by Andres, Bergold, Hochstättler and Wiehe [2].

The Symmetrical Reduction Theorem is extremely useful: Bang-Jensen, Bellitto, Schweser and Stiebitz [3] used Theorem 4 in order to establish digraph analogs of Hajós and Ore constructions.

1.3. Perfect Digraph Theorems: new results

In this paper, by using the Symmetrical Reduction Theorem we generalise the Weak and the Strong Perfect Graph Theorem in the versions of Theorems 2 and 3 to perfect, α-perfect and strictly perfect digraphs.

In order to generalise Theorem 2 to perfect and α-perfect digraphs, we will introduce parameters $\vec{\alpha}$ and $\vec{\omega}$ that are based on more general concepts of independence and cliques, respectively. The *acyclic independence number* $\vec{\alpha}$ is the maximum size of an acyclic vertex set. The *clique-acyclic clique number* $\vec{\omega}$ is the minimum size of a clique that does not contain complements of proper directed cycles as an induced subdigraph. As generalisations of Theorem 2 we will obtain the following main results.

Theorem 7 (Weak Perfect Digraph Theorem)**.** *For any digraph D the following conditions are equivalent.*

(i) *D is perfect.*

(ii) *For any induced subdigraph H of D, we have*

$$\omega(H)\vec{\alpha}(H) \geq |V(H)|. \tag{1}$$

Theorem 8 (Weak α-perfect Digraph Theorem)**.** *For any digraph D the following conditions are equivalent.*

(i) *D is α-perfect.*

(ii) *For any induced subdigraph H of D, we have*

$$\alpha(H)\vec{\omega}(H) \geq |V(H)|.$$

We will derive the following dual characterisation to Theorem 6.

Theorem 9 (Strong α-perfect Digraph Theorem)**.** *A digraph is α-perfect if and only if it contains no induced*

- *superorientations of odd holes,*
- *superorientations of odd antiholes,*
- *complements of directed cycles with at least* 3 *vertices.*

Theorems 6 and 9 generalise Theorem 3.

Moreover, we generalise Theorems 2 and 3 to strictly perfect digraphs (Corollaries 23 and 24, respectively).

1.4. Remark

Although the results of this paper seem to suggest that there is no big difference between the two dual classes of perfect digraphs and α-perfect digraphs, we would like to mention that some essential differences are known. For instance, by Corollary 18 the α-perfect digraphs are the clique-acyclic superorientations of perfect graphs. A famous result of Boros and Gurvich [4] says that clique-acyclic superorientations of perfect graphs always have a kernel. Thus the kernel existence problem on α-perfect digraphs is trivial (cf. [1]). On the other hand, Andres and Hochstättler [1] proved that the kernel existence problem on perfect digraphs is NP-complete. Therefore all three classes of digraphs studied in this paper, perfect digraphs, α-perfect digraphs and strictly perfect digraphs might be of independent interest for further studies.

1.5. Structure of the paper

The structure of the remaining sections is as follows. In Section 2 we introduce the colouring and covering parameters for digraphs and observe some simple relations. Digraph analogues of the Weak and the Strong Perfect Graph Theorem are proved in Section 3 and Section 4, respectively. Properties of strictly perfect digraphs and some open questions are given in Sections 5 and 6.

2. Notation and Basic Observations

All digraphs we consider are finite and without loops, i.e., a *digraph* D is a pair (V, A) with a finite *vertex* set V and a set A of *arcs* satisfying

$$A \subseteq V_0^2 := (V \times V) \setminus \{(v, v) \mid v \in V\}.$$

The *complement* of D, denoted by $\overline{D}$, is the digraph $(V, \overline{A})$ on the same vertex set V with arc set $\overline{A} := V_0^2 \setminus A$. For a digraph D, by $V(D)$ we denote its vertex set and by $G(D)$ we denote its underlying graph, which is a simple undirected graph that has an edge vw if and only if (v, w) or (w, v) (or both) is an arc of D. A *superorientation* of a graph G is a digraph D with $G = G(D)$.

A *single arc* in a digraph $D = (V, A)$ is an arc $(v, w) \in A$ for which $(w, v) \notin A$. An *orientation* of graph G is a superorientation of G in which every arc is a single arc. The *proper symmetric part* $S'(D)$ of D is the digraph on the vertex set V that can be obtained from D by deleting all single arcs. The digraph D is *symmetric* if $D = S'(D)$. The *symmetric part* $S(D)$ of D is defined as

$$S(D) := G(S'(D)),$$

which is the underlying undirected graph of the proper symmetric part. Undirected graphs are identified with *symmetric digraphs.* In this way, edges are identified with pairs of reversely oriented arcs, see Figure 2. In the following, we will often switch between the graphical and digraphical perspective of graphs, respectively, symmetric digraphs, without further mentioning.

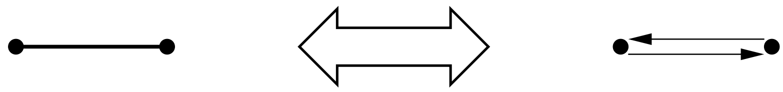


Figure 2. Identifying an edge with the arcs of a digon.

Let $n \in \mathbb{N}$. For $n \geq 1$, by $\vec{P}_n$ we denote the directed path and, for $n \geq 2$, by $\vec{C}_n$ we denote the directed cycle on n vertices. A directed cycle on n vertices is a *proper* directed cycle if $n \geq 3$, and a *digon* if $n = 2$. Recall that a digraph is *acyclic* if it does not contain a directed cycle as an (induced) subdigraph. Indeed, it is enough to forbid induced directed cycles since every non-induced directed cycle has a chord which either is part of a digon (in which case the digon is an induced directed cycle) or is a single arc (in which case a smaller proper directed cycle is formed by the chord arc and one of the half-cycles of the original directed cycle).

A *filled odd hole* is a digraph D such that $S(D)$ is an odd hole (i.e., $S(D)$ is an undirected cycle of odd length ≥ 5). A *filled odd antihole* is a digraph D such that $S(D)$ is an odd antihole. See Figure 3.

Crucial parameters will be based on different kinds of cliques in digraphs and their complements. For us, a *clique* in a digraph D is a set of pairwise adjacent vertices, i.e., any two vertices are either connected by a single arc or induce a digon. We will also call any subdigraph induced by a clique in D a *clique.* Obviously, the complements of cliques are the orientations of graphs.

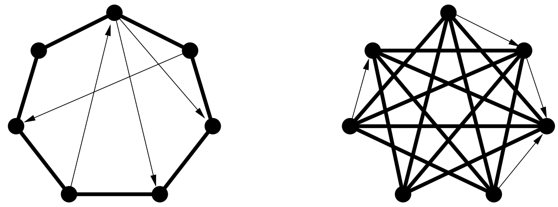


Figure 3. A filled odd hole (left) and a filled odd antihole (right).

A digraph is *clique-acyclic* if it does not contain the complement of a proper directed cycle as an induced subdigraph. Equivalent characterisations of clique-acyclic digraphs are "the subdigraph induced by any clique has no proper directed cycles", or "every clique has a vertex of in-degree $n-1$, where n is the number of vertices of the clique".

A *clique-acyclic clique* is a clique that induces a clique-acyclic subdigraph of D. Dually, a vertex set that induces an acyclic subdigraph of D is called *acyclic set*.

Observation 10. *A digraph is acyclic if and only if its complement is a clique-acyclic clique.*

Proof. Let D be a digraph. The digraph D is acyclic if and only if it contains neither induced proper directed cycles nor digons. D contains no induced proper directed cycles if and only if $\overline{D}$ is clique-acyclic. D contains no digon if and only if $\overline{D}$ is a clique. ■

Symmetric cliques are those clique-acyclic cliques where any pair of vertices induces a digon. Obviously, symmetric cliques of a digraph D induce *stable sets*, i.e., vertex sets of digraphs with an empty arc set, in the complement $\overline{D}$.

The crucial colouring parameter for digraphs, the *dichromatic number*, was introduced by Neumann-Lara [11]. It is based on the concept of *acyclic colourings*, i.e., vertex colourings where each colour class is an acyclic set. We remark that such a colouring needs not be proper. Dually to acyclic colourings, a *clique covering by clique-acyclic cliques* is a (not necessarily proper) vertex colouring in which all colour classes are clique-acyclic cliques.

The numbering of the vertices of $\vec{C}_4$ in Figure 1 indicates a clique covering with clique-acyclic cliques.

For the notions χ and ω related to perfect digraphs and acyclicity, we have the dual notions of k and α related to α-perfect digraphs and clique-acyclicity. We define the following parameters for a digraph D.

$\vec{\omega}(D)$, the *clique-acyclic clique number*, is the cardinality of a largest clique-acyclic clique in D;

$\vec{\alpha}(D)$, the *acyclic independence number*, is the cardinality of a largest acyclic set in D;

$\omega(D)$, the *symmetric clique number*, is the cardinality of a largest symmetric clique in D;

$\alpha(D)$, the *stability number*, is the cardinality of a largest stable set in D;

$\chi(D)$, the *dichromatic number*, is the minimum number of colours needed for an acyclic colouring of D (cf. [11]); and

$k(D)$, the *clique covering number*, is the minimum number of monochromatic clique-acyclic cliques needed to cover all vertices of D in a clique covering by clique-acyclic cliques.

Note that, by definition, $\omega(D) = \omega(S(D))$ and $\alpha(D) = \alpha(G(D))$.

In the case of symmetric digraphs (regarded as undirected graphs), the parameters $\vec{\omega}$ and ω specialise to the usual clique number, $\vec{\alpha}$ and α specialise to the usual stability number, χ corresponds to the usual chromatic number and k corresponds to the usual clique covering number of undirected graphs. Thus the parameters defined above are appropriate to generalise colourings and coverings from graphs to digraphs.

From the definitions, it is clear that

Observation 11. *For any digraph D,*

(i) $\vec{\alpha}(D) = \vec{\omega}(\overline{D})$,

(ii) $\alpha(D) = \omega(\overline{D})$,

(iii) $k(D) = \chi(\overline{D})$.

Furthermore, we observe

Observation 12. *For any digraph D,*

(i) $\omega(D) \leq \chi(D)$,

(ii) $\alpha(D) \leq k(D)$.

Proof. (i) Every vertex in a symmetric clique must be coloured in a different colour, since every digon is a directed cycle. Thus $\chi(D) \geq \omega(D)$.

(ii) follows from (i) by applying Observation 11(ii) and (iii). ■

In [1] perfectness of digraphs was introduced in terms of the dichromatic number. A digraph is *perfect* if, for any induced subdigraph H,

$$\omega(H) = \chi(H).$$

Dually, we define a digraph to be *α-perfect* if, for any induced subdigraph H,

$$\alpha(H) = k(H).$$

Observation 13. *A digraph is perfect* (*respectively, α-perfect*) *if and only if each of its components is perfect* (*respectively, α-perfect*).

Observation 14. *A digraph is α-perfect if and only if its complement is perfect.*

Proof. This follows from the definitions and Observation 11(ii) and (iii). ■

This observation motivates the following definition. A digraph is *strictly perfect* if it is perfect and α-perfect.

Example 15. Some simple examples of strictly perfect digraphs are

(i) all digraphs on at most 3 vertices except for the directed 3-cycle $\vec{C}_3$;

(ii) transitive tournaments;

(iii) the N-digraph and its complement $\overline{N}$ (see Figure 4); or, more generally,

(iv) superorientations of a path and their complements;

(v) orientations of cycles of even length that are not directed proper cycles, and their complements; and

(vi) perfect (undirected) graphs.

Figure 4. The N-digraph and its complement $\overline{N}$.

Proof. (i), (ii) and (iii) are obvious.

To prove (iv) and (v), let D be a superorientation of the path P_n on n vertices with $n \in \mathbb{N} \setminus \{0\}$ or an acyclic orientation of an n-cycle with (even) $n \in 2\mathbb{N} \setminus \{0\}$. By (ii) we may assume without loss of generality $n \geq 2$. The components of any proper subdigraph of D are superorientations of a path. Thus, in order to prove that D is strictly perfect, by Observation 13 it is sufficient to note that

$$\omega(D) = \left\{ \begin{array}{ll} 1 & \text{(in case } D \text{ contains no digon)} \\ 2 & \text{(in case } D \text{ contains a digon)} \end{array} \right\} = \chi(D)$$

and

$$\alpha(D) = \left\lceil \frac{n}{2} \right\rceil = k(D).$$

By Observation 14, strict perfectness follows also for the complement of D.

(vi) Perfect graphs are strictly perfect by the Weak Perfect Graph Theorem (Theorem 1). ■

By our identification of symmetric digraphs and undirected graphs, for any digraph D we have

$$\overline{S(D)} = G(\overline{D}). \tag{2}$$

Equation (2) implies the following lemma.

Lemma 16. (i) *The complements of filled odd holes are the superorientations of odd antiholes.*
(ii) *The complements of filled odd antiholes are the superorientations of odd holes.*

Example 17. Neither proper directed cycles nor their complements are strictly perfect.

(i) Proper directed cycles of even length are α-perfect, but not perfect, and their complements are perfect, but not α-perfect.

(ii) Directed cycles of odd length and their complements are neither perfect nor α-perfect.

(iii) Superorientations of cycles of odd length ≥ 5 are not α-perfect, their complements are not perfect.

(iv) Superorientations of odd antiholes are not α-perfect, their complements are not perfect.

(v) Filled odd holes and filled odd antiholes and their complements, respectively, are not strictly perfect.

Proof. (i) For $n \geq 2$, we have

$$\omega(\vec{C}_{2n}) = 1 < 2 = \chi(\vec{C}_{2n}), \tag{3}$$

thus $\vec{C}_{2n}$ is not perfect, whereas $\overline{\vec{C}_{2n}}$ is perfect, since

$$\omega(\overline{\vec{C}_{2n}}) = n = \chi(\overline{\vec{C}_{2n}})$$

and any induced proper subdigraph of $\overline{\vec{C}_{2n}}$ is the complement of an orientation of a forest of paths, which is α-perfect by Example 15 (iv) and Observation 13.

(ii) By the same argument as in (3), for any $n \geq 1$, the directed cycle $\vec{C}_{2n+1}$ is not perfect. It is not α-perfect, since

$$\alpha(\vec{C}_{2n+1}) = n < n+1 = k(\vec{C}_{2n+1}). \tag{4}$$

(iii) As (4) still holds when the directed cycle $\vec{C}_{2n+1}$ is replaced by any superorientation D of C_{2n+1}, no such D is α-perfect.

(iv) Let $n \geq 0$ and D be a superorientation of an odd antihole on $2n+5$ vertices. Then, by Lemma 16, its complement $\overline{D}$ is a filled odd hole. Thus

$$\alpha(D) = \omega(\overline{D}) = 2 < 3 = \chi(C_{2n+5}) \leq \chi(\overline{D}) = k(D).$$

Thus $\overline{D}$ is not perfect and D is not α-perfect.

The other assertions of (i), (ii), (iii) and (iv) follow by Observation 14. Moreover, by Lemma 16, (iii) and (iv) imply (v). ∎

3. Proof of Weak Perfect Digraph Theorems

As we have seen in Example 17(i), the first form of the Weak Perfect Graph Theorem (Theorem 1) does not generalise directly to digraphs (since the complement of a perfect digraph is not necessarily a perfect digraph), but it generalises in the form of Theorem 5.

Using Observation 14, Theorem 5 can be reformulated in the following way.

Corollary 18. *A digraph is α-perfect if and only if it is a clique-acyclic superorientation of a perfect graph.*

In his second proof of the Weak Perfect Graph Theorem, Lovász [10] reformulated the Weak Perfect Graph Theorem in the asymmetric form of Theorem 2: a graph is perfect if and only if, for any induced subgraph H,

$$\omega(H)\omega(\overline{H}) \geq |V(H)|. \tag{5}$$

The latter condition (5) is obviously equivalent to

$$\omega(H)\alpha(H) \geq |V(H)|. \tag{6}$$

In the following, by weakening the condition (6), we will deduce a set of two generalisations of Theorem 2 to the case of digraphs from Theorem 4 given in Theorems 7 and 8.

Note that Theorem 2 does not generalise directly to digraphs. Consider the transitive tournament $\vec{T}_n$ on n vertices. Its complement is isomorphic to $\vec{T}_n$. Since $S(\vec{T}_n) = \overline{K_n}$, the tournament $\vec{T}_n$ is perfect by Theorem 4. But for $n \geq 2$ we have

$$\omega(\vec{T}_n)\omega\left(\overline{\vec{T}_n}\right) = \omega(\vec{T}_n)\alpha(\vec{T}_n) = 1 \ngeq n.$$

However, it is possible to prove the modification of Theorem 2, stated in Theorem 7, which is equivalent to Theorem 2 in the case of undirected graphs. For that we use the notion of the *acyclic independence number* $\vec{\alpha}(D)$ of a digraph D: recall that it is the maximum number of vertices of an induced acyclic subdigraph of D. We start with two observations.

Observation 19. *For any digraph H,* $\alpha(H) \leq \vec{\alpha}(H) \leq \alpha(S(H))$.

Observation 20. *For any digraph $H = (V, A)$, we have*

$$\chi(H)\vec{\alpha}(H) \geq |V|.$$

Proof. Let $c : V \longrightarrow \{1, 2, \ldots, \chi(H)\}$ be an acyclic colouring of H with $\chi(H)$ colours. Then

$$|V| = \sum_{i=1}^{\chi(H)} |c^{-1}(i)| \leq \sum_{i=1}^{\chi(H)} \vec{\alpha}(H) = \chi(H)\vec{\alpha}(H).$$

■

***Proof of Theorem* 7 *using Theorem* 2 *and Theorem* 4.** Let D be a perfect digraph and $H = (V, A)$ be an induced subdigraph of D. By the perfectness of D and by Observation 20, we have

$$\omega(H)\vec{\alpha}(H) = \chi(H)\vec{\alpha}(H) \geq |V|,$$

thus condition (1) in (ii) holds, which proves the implication (i)$\Longrightarrow$(ii).

For the (nontrivial) implication (ii)$\Longrightarrow$(i), let D be a non-perfect digraph. By Theorem 4, either the digraph D contains an induced subdigraph H that is a directed cycle of length at least 3 or the symmetric part $S(D)$ of D is an imperfect graph.

In the former case, if H is a directed cycle $\vec{C}_n$ for some integer $n \geq 3$, then $\omega(H) = 1$ and $\vec{\alpha}(H) = n - 1$. Thus

$$\omega(H)\vec{\alpha}(H) = n - 1 < n = |V(H)|.$$

In the latter case, by Theorem 2 the symmetric part $S(D)$ has an induced subgraph $H' = (V', E')$ with the property

$$\omega(H')\alpha(H') < |V'|. \tag{7}$$

Let in this case $H = D[V']$ be the subdigraph of D induced by the vertex set V' of H'. In particular, we have $S(H) = H'$. Thus, trivially, we have

$$\omega(H) = \omega(S(H)) = \omega(H'), \tag{8}$$

and by the second inequality in Observation 19 we have

$$\vec{\alpha}(H) \leq \alpha(S(H)) = \alpha(H'). \tag{9}$$

Therefore,

$$\omega(H)\vec{\alpha}(H) \overset{(8),(9)}{\leq} \omega(H')\alpha(H') \overset{(7)}{<} |V'| = |V(H)|.$$

In both cases, (1) is violated, which proves the implication (ii)$\Longrightarrow$(i). ■

In line with the above notion of $\vec{\alpha}$ corresponding to acyclicity we have defined a dual notion $\vec{\omega}$ corresponding to clique-acyclicity: recall that the *clique-acyclic clique number* $\vec{\omega}(D)$ of a digraph D is the maximum number of vertices of an induced subdigraph of D that is a clique-acyclic superorientation of a clique in $G(D)$.

Dually to Observations 19 and 20 and Theorem 7, we have the following.

Observation 21. *For any digraph H, $\omega(H) \leq \vec{\omega}(H) \leq \omega(G(H))$.*

Observation 22. *For any digraph H, we have*

$$k(H)\vec{\omega}(H) \geq |V(H)|.$$

***Proof of Theorem* 8.** This follows easily from Theorem 7 by applying Observation 11 (respectively, Observation 14).

D is α-perfect.

$\overset{\text{Obs 14}}{\Longleftrightarrow}$ $\overline{D}$ is perfect.

$\overset{\text{Thm 7}}{\Longleftrightarrow}$ For any induced subdigraph H of $\overline{D}$

$$\omega(H)\vec{\alpha}(H) \geq |V(H)|.$$

$\overset{\text{Obs 11}}{\Longleftrightarrow}$ For any induced subdigraph H of D

$$\alpha(H)\vec{\omega}(H) \geq |V(H)|.$$

■

Combining Theorem 7 and Theorem 8 we obtain the following corollary.

Corollary 23 (Weak Strictly Perfect Digraph Theorem)**.** *For any digraph D, the following conditions are equivalent.*

(i) *D is strictly perfect.*

(ii) *For any induced subdigraph $H = (V, A)$ of D, we have*

$$\omega(H)\vec{\alpha}(H) \geq |V| \tag{10}$$

and

$$\alpha(H)\vec{\omega}(H) \geq |V|. \tag{11}$$

4. Proof of Strong Perfect Digraph Theorems

Theorem 6 was already proved by Andres and Hochstättler [1]. Here we prove its dual, Theorem 9.

In order to be able to apply Theorem 6 to prove Theorem 9, we need the obvious Lemma 16 as a key.

***Proof of Theorem* 9.** For any digraph D, we have the following equivalences.

D is α-perfect

$\overset{\text{Obs 14}}{\Longleftrightarrow}$ $\overline{D}$ is perfect

$\overset{\text{Thm 6}}{\Longleftrightarrow}$ $\overline{D}$ contains no filled odd hole, filled odd antihole, or proper directed cycle as an induced subdigraph.

$\overset{\text{Lem 16}}{\Longleftrightarrow}$ D contains no superorientation of an odd antihole, superorientations of an odd hole, or complement of a proper directed cycle as an induced subdigraph. ■

As an immediate consequence of Theorem 6 and its dual, Theorem 9, we obtain the following.

Corollary 24 (Strong Strictly Perfect Digraph Theorem)**.** *A digraph is strictly perfect if and only if it contains no induced*

- *filled odd holes,*
- *filled odd antiholes,*
- *superorientations of odd holes,*
- *superorientations of odd antiholes,*
- *directed cycles with at least* 3 *vertices,*
- *complements of directed cycles with at least* 3 *vertices.*

5. Characterisation of Strictly Perfect Digraphs by Semi-Filled Directed Cycles

In the following two sections, we turn our attention to the concept of strictly perfect digraphs.

Firstly, in this section we would like to mention some analogues of the Symmetrical Reduction Theorem (Theorem 4) for α-perfect and strictly perfect digraphs. Though trivial consequences of Theorem 4, the latter one implies a characterisation of strictly perfect digraphs by a concept that we call "semi-filled directed cycles". This characterisation might be of independent interest for further studies on strictly perfect digraphs.

In line with the symmetric part of a digraph, we define the *co-symmetric part* co$S(D)$ of a digraph D as the graph $\overline{G(D)}$ having an edge (digon) for every pair of non-adjacent vertices of D, or equivalently, as $S(\overline{D})$ (see equation (2)).

Definition and Remark 25. co$S(D) := \overline{G(D)} = S(\overline{D})$.

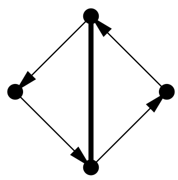
The *oriented* (or *asymmetric*) *part* $O(D)$ of a digraph $D = (V, A)$ is the digraph $(V, \vec{A})$, where $\vec{A}$ is the set of all single arcs of D.

Theorem 26 (Symmetrical Reduction Theorem for α-perfect digraphs)**.** *A digraph D is α-perfect if and only if its co-symmetric part co$S(D)$ is a perfect graph and D contains no induced complement of a proper directed cycle.*

Proof. By Observation 14, the statement is equivalent to saying that "$\overline{D}$ is perfect if and only if co$S(D)$ is a perfect graph and $\overline{D}$ contains no induced proper directed cycle". This is true by the Symmetrical Reduction Theorem (Theorem 4) for perfect digraphs applied to $\overline{D}$, since co$S(D) = S(\overline{D})$. ∎

Theorem 27 (Symmetrical Reduction Theorem for strictly perfect digraphs)**.** *A digraph D is strictly perfect if and only if its symmetric part $S(D)$ and its co-symmetric part co$S(D)$ are perfect graphs and D contains no induced proper directed cycle or complement of a proper directed cycle.*

Proof. This is an immediate consequence from the combination of Theorem 4 and Theorem 26. ∎



$\vec{D}_0$

Figure 5. The semi-diamond $\vec{D}_0$.

A *semi-filled directed cycle* is a digraph on n vertices for some $n \geq 3$ such that $O(D)$ is the directed n-cycle $\vec{C}_n$ and D is neither $\vec{C}_n$ nor (isomorphic to) its complement $\overline{\vec{C}_n}$. Obviously, there does not exist a semi-filled directed cycle for $n = 3$ and a unique one, which we call *semi-diamond*, for $n = 4$, see Figure 5. The 6 non-isomorphic semi-filled directed cycles on 5 vertices are depicted in Figure 6.

By the definition of semi-filled directed cycles, the statement of the following main result of this section is a corollary of Theorem 27.

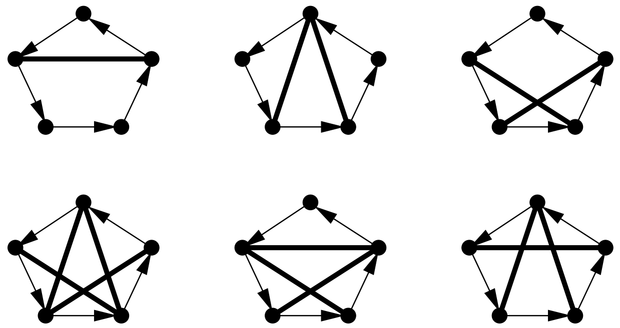


Figure 6. The semi-filled cycles on 5 vertices.

Theorem 28 (Semi-filled directed cycle characterisation)**.** *A digraph D is strictly perfect if and only if its symmetric part $S(D)$ and its co-symmetric part $coS(D)$ are perfect graphs and the vertex set of every* (*not necessarily induced*) *minimal proper directed cycle in $O(D)$ induces a semi-filled directed cycle in D.*

We remark that, in the formulation of Theorem 28, we cannot waive the precondition of minimality of the proper directed cycles. For example, the strictly perfect digraph from Figure 7 has a directed 5-cycle which does not induce a semi-filled directed cycle, but the directed 4-cycle, which is a minimal proper directed cycle here, induces a semi-filled directed cycle (the semi-diamond).

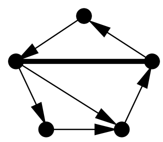


Figure 7. A strictly perfect digraph that is not a semi-filled directed cycle.

6. Cuts in Strictly Perfect Digraphs

In this section we prove that directed cographs and Cartesian products of directed 3-cycles are strictly perfect. We further discuss some open problems towards characterising strictly perfect digraphs.

Let $D = (V, A)$ be a digraph and $V = V_1 \cup V_2$, $V_1 \cap V_2 \neq \emptyset$, $V_1 \neq \emptyset$, $V_2 \neq \emptyset$. The *cut* with *sides* V_1 and V_2 is the arc set

$$\{(v, w) \in A \mid (v \in V_1 \wedge w \in V_2) \vee (v \in V_2 \wedge w \in V_1)\}.$$

A cut C is *empty* if $C = \emptyset$, respectively, *forward directed complete* if $C = V_1 \times V_2$, respectively, *symmetric complete* if $C = (V_1 \times V_2) \cup (V_2 \times V_1)$. More generally, a cut C is *forward directed* if $C \subseteq V_1 \times V_2$ and *forward mixed semi-complete* if

$$V_1 \times V_2 \subseteq C \subseteq (V_1 \times V_2) \cup (V_2 \times V_1).$$

By changing the roles of V_1 and V_2 we might define analog types of cuts, which are *backward* instead of forward.

Thus directed cuts and mixed semi-complete cuts are complementary notions. In particular, empty cuts and symmetric complete cuts are complementary notions, and the notion of directed complete cuts is self-complementary (only switching between forward and backward).

Theorem 29. *Let $D = (V, A)$ be a digraph and C be a cut that is either empty, symmetric complete or forward directed complete and let D_1 and D_2 be the subdigraphs induced by the two sides of the cut. Then the following conditions are equivalent.*

(i) *D is strictly perfect.*

(ii) *Both D_1 and D_2 are strictly perfect.*

Proof. The assertion (i)$\Longrightarrow$(ii) is trivial by definition of perfect and α-perfect, since D_1 and D_2 are induced subdigraphs of D.

To prove (ii)$\Longrightarrow$(i), assume that D is a counterexample to the theorem with a minimal number of vertices and $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ are strictly perfect and H is a subdigraph of D. We have to prove that H is strictly perfect. This is obvious if H is a subdigraph of either D_1 or D_2. Otherwise, there is a cut C' that is either empty, symmetric complete or forward directed complete and partitions H into H_1 (a nonempty subdigraph of D_1) and H_2 (a nonempty subdigraph of D_2). Since H_1 and H_2 are strictly perfect, by the minimality of D we may assume that $H = D$ and $H_1 = D_1$ and $H_2 = D_2$ and need only to show that $\omega(D) = \chi(D)$ and $\alpha(D) = k(D)$.

We consider three cases.

Case 1. C is an empty cut. In this case, Observation 13 implies that, since D_1 and D_2 are strictly perfect, the digraph D is strictly perfect.

Case 2. C is a symmetric complete cut. Since $\overline{D_1}$ and $\overline{D_2}$ are strictly perfect and the cut $\overline{C}$ with sides V_1 and V_2 in $\overline{D}$ is an empty cut, the assertions follow easily from Case 1 (applied to $\overline{D}$) by Observation 11:

$$\omega(D) = \alpha(\overline{D}) = k(\overline{D}) = \chi(D),$$
$$\alpha(D) = \omega(\overline{D}) = \chi(\overline{D}) = k(D).$$

Case 3. C is a forward directed complete cut. Since D_1 and D_2 are perfect, for any $i \in \{1, 2\}$, there are acyclic colourings f_i of D_i with colours $\{1, \ldots, \omega(D_i)\}$.

Then the function f, defined by $f(v) = f_i(v)$ if $v \in V_i$ (for some $i \in \{1, 2\}$), is an acyclic colouring with

$$\omega(D) = \max\{\omega(D_1), \omega(D_2)\}$$

colours, i.e., (using Observation 12 (i)) we have $\omega(D) = \chi(D)$.

Since $\overline{D_1}$ and $\overline{D_2}$ are perfect and the cut $\overline{C}$ with sides V_1 and V_2 in $\overline{D}$ is a backward directed complete cut, we conclude in a similar way that $\omega(\overline{D}) = \chi(\overline{D})$.

By Observation 11, this implies

$$\alpha(D) = \omega(\overline{D}) = \chi(\overline{D}) = k(D).$$

Thus in any case, we have a contradiction to our assumption that D is a counterexample to the theorem. ■

One might conjecture that Theorem 29 generalises to forward directed cuts and forward mixed semi-complete cuts. However, this is not true in general, as the the example in Figure 8 shows: note that the digraph D_+ in the figure consists of two strictly perfect digraphs joined by a directed mixed semi-complete cut. But D is not perfect, since $S(D)$ is isomorphic to the non-perfect graph C_5. Therefore, $\overline{D_+}$ is not α-perfect. Note that $\overline{D_+}$ consists of two strictly perfect digraphs joined by a directed cut.

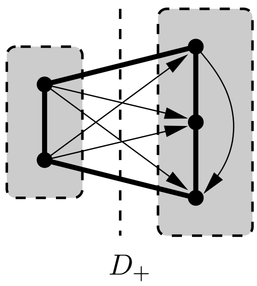


Figure 8. An imperfect digraph built from two strictly perfect digraphs joined by a directed mixed semicomplete cut.

A *directed cograph* is a digraph that can be constructed recursively from K_1 (the graph with one isolated vertex) by taking two directed cographs D_1 and D_2 and joining them by either an empty cut, a symmetric complete cut, or a forward directed complete cut. These kinds of joins are denoted by $D_1 \cup D_2$ (the *union*), $D_1 \vee D_2$ (the *complete join*), and $D_1 \vec{\nabla} D_2$ (the *forward directed complete join*), respectively.

Theorem 29 immediately implies the following.

Corollary 30. *Directed cographs are strictly perfect.*

The explicit notion of directed cographs has been introduced by Crespelle and Paul [7]. This concept and its undirected analogue, *cographs*, were also the initial motivation in a recent paper [2], where Reed's Semi-strong Perfect Graph Theorem [12] is generalised in a certain way to perfect digraphs.

We remark that there are many strictly perfect digraphs that are no directed cographs. The smallest examples are the directed path $\vec{P}_3$ on three vertices and its complement. Other examples are $\vec{P}_3^-$ and $\vec{P}_3^+$, see Figure 9. Among the examples with four vertices are the N and its complement $\overline{N}$ (see Figure 4).

Figure 9. Two self-complementary strictly perfect digraphs that are no directed cographs.

The most important basic families of strictly perfect digraphs that are no directed cographs are directed paths $\vec{P}_n$ with $n \geq 3$ vertices and symmetric paths P_n with $n \geq 4$ vertices. The latter characterise cographs (in the undirected case): it is well-known that cographs are exactly the P_4-free graphs.

Crespelle and Paul [7] characterised directed cographs as those digraphs that contain no induced

$$\vec{P}_3, \overline{\vec{P}_3}, N, \overline{N}, \vec{P}_3^-, \vec{P}_3^+, P_4, \vec{C}_3.$$

As remarked above, the first seven of them are strictly perfect, whereas the latter is not.

Extending the constructive approach. In view of constructively characterising strictly perfect digraphs, one might define the *basic set* $\mathcal{B}$ as the inclusion-minimial set of isomorphism classes of strictly perfect digraphs, such that every strictly perfect digraph can be constructed recursively from digraphs from this set by applying unions, complete joins and forward directed joins.

By the above, we have

$$\mathcal{B} \supseteq \{K_1, \vec{P}_3, \overline{\vec{P}_3}, N, \overline{N}, \vec{P}_3^-, \vec{P}_3^+, P_4\}. \tag{12}$$

Problem 31. Determine $\mathcal{B}$.

Problem 32. Is $\mathcal{B}$ an infinite set?

Proposition 33. *For any $n \in \mathbb{N}$, the Cartesian product*

$$\vec{P}_3^n := \underbrace{\vec{P}_3 \square \cdots \square \vec{P}_3}_{n \text{ times}}$$

is strictly perfect.

Proof. The digraph $\vec{P}_3^n$ is perfect, since it is acyclic. Since it is an orientation of a bipartite graph, the symmetric part of its complement is the complement of a bipartite graph, which is a perfect graph. Note further that $\vec{P}_3^n$ does not contain complements of proper directed cycles. Thus, by Theorem 4, the complement $\overline{\vec{P}_3^n}$ is a perfect digraph. Therefore, $\vec{P}_3^n$ is strictly perfect for any $n \in \mathbb{N}$. ■

The example of the series $(\vec{P}_3^n)_{n\in\mathbb{N}}$ given in Proposition 33 seems to suggest that $\mathcal{B}$ is an infinite set. In particular, a construction of $\vec{P}_3^n$ from $\vec{P}_3^0, \ldots, \vec{P}_3^{n-1}$ seems not to be possible. However, there might be some finite number of other members of $\mathcal{B}$ from which such a construction of $\vec{P}_3^n$ might be possible. The author did not yet find a formal proof of an affirmative answer to Problem 32.

We conjecture a generalisation of Theorem 29.

Conjecture 34. *Let $m \in \mathbb{N}$ and $D = (V, A)$ be a digraph such that D or its complement $\overline{D}$ is composed by digraphs $D_1, \ldots, D_{m+1}$ arranged in a line. Let C_i $(i = 1, \ldots, m)$ be the cut that separates D_i and D_{i+1} and let furthermore C_i be empty, symmetric complete or forward (or backward) directed complete. Then the following are equivalent.*

(i) *D is strictly perfect.*

(ii) *For every $i \in \{1, \ldots, m + 1\}$, D_i is strictly perfect.*

This conjecture is motivated by Example 15(iv): superorientations of paths, which are composed of isolated vertices separated by symmetric complete, forward or backward directed complete cuts on a line, are strictly perfect.

If Conjecture 34 is true, then the seven obstructions to directed cographs occurring in (12) in the basic set $\mathcal{B}$ can be constructed. However, still other strictly perfect digraphs cannot be constructed, namely such ones that are based on a tree that is not a path. An example of such a complementary pair is given in Figure 10.

Figure 10. Strictly perfect digraphs that cannot be constructed by means of Conjecture 34.

Refined constructive methods. Even if one might prove a generalisation of Conjecture 34 based on an arrangement of the D_i on a forest instead of a path (line), there are still strictly perfect digraphs that cannot be constructed by disjoint unions, complete joins and forward directed complete joins based on a tree structure. One example is the *semi-diamond*, which is depicted in Figure 5.

One reason might be that unions, complete joins and forward directed joins are not sufficient to easily characterise strictly perfect digraphs constructively. In a recent paper, Bang-Jensen, Bellitto, Schweser, and Stiebitz [3] considered some other types of joins, so-called directed Hajós joins, directed Ore joins and bidirected Ore joins, in order to characterise m-dichromatic critical digraphs. Their proof strongly relies on the characterisation of perfect digraphs given in Theorem 4. Therefore, using such kind of joins and inventing some dual join concepts for α-perfect digraphs might enable us to get a finer constructive method for strictly perfect.

However, since the concepts of joins considered by Bang-Jensen *et al.* [3] involve glueing vertices together and replacements of arcs, it is not clear whether these operations can be backtracked efficiently. We wonder whether any constructive method for strictly perfect digraphs might be impossible by reasons of complexity of the recognition of strictly perfect digraphs.

Complexity. Using the Strong Perfect Graph Theorem by Chudnovsky, Robertson, Seymour, and Thomas [6], the characterisation of Berge graphs by Chudnovsky, Cornuéjols, Liu, Seymour, and Vušković [5] implies that the recognition problem for perfect graphs is in polynomial time.

By a reduction from 3SAT Andres and Hochstättler [1] showed that the recognition problem for perfect digraphs (and thus also for α-perfect digraphs) is coNP-complete. However, the reduction seems not to be transferable in an obvious way to strictly perfect digraphs. The following questions are open.

Problem 35. Determine the complexity of recognising strictly perfect digraphs.

Problem 36. Determine the complexity of recognising strictly perfect digraphs with no induced $\vec{D}_0$.

Problem 37. Determine the complexity of recognising strictly perfect digraphs with no induced $\vec{D}_0$, F_3, $\overline{F_3}$.

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