

ON THE ρ -SUBDIVISION NUMBER OF GRAPHS

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Abstract

For an arbitrary invariant $\rho(G)$ of a graph G the ρ -subdivision number $sd_\rho(G)$ is the minimum number of edges of G whose subdivision results in a graph H with $\rho(H) \neq \rho(G)$. Set $sd_\rho(G) = |E(G)|$ if such an edge set does not exist.

In the first part of this paper we give some general results for the ρ -subdivision number. In the second part we study this parameter for the chromatic number, for the chromatic index, and for the total chromatic number. We show among others that there is a strong relationship to the ρ -edge stability number for these three invariants. In the last part we consider a modification, namely the ρ -multiple subdivision number where we allow multiple subdivisions of the same edge.

Keywords: subdivision number, edge stability number, edge subdivision, chromatic number, chromatic index, total chromatic number.

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1. INTRODUCTION

We consider finite simple graphs $G = (V(G), E(G))$. A (graph) invariant $\rho(G)$ is a function $\rho : \mathcal{I} \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$, where \mathcal{I} is the class of finite simple graphs. An empty graph is a graph with empty edge set.

In this paper we investigate graph invariants which are related to the ρ -edge stability numbers $es_\rho(G)$ (see [6]). For an arbitrary invariant $\rho(G)$ of a graph G

the ρ -edge stability number $es_\rho(G)$ of G is the minimum number of edges of G whose removal results in a graph $H \subseteq G$ with $\rho(H) \neq \rho(G)$ or with $E(H) = \emptyset$ (that is, H is empty and $es_\rho(G) = |E(G)|$).

The operation of removing an edge will be replaced by the operation of subdividing an edge. Subdividing an edge $e = uv$ of a graph G creates a new graph G_e , in which a new vertex w is added and the edge e is removed and replaced by two new edges uw and wv . We write $G_{E'}$ for the graph obtained by subdividing all edges of $E' \subseteq E(G)$. Note that each edge of E' is subdivided exactly once.

Definition. For an arbitrary invariant $\rho(G)$ of a graph G the ρ -subdivision number $sd_\rho(G)$ of G is the minimum number of edges of G to be subdivided such that the resulting graph H fulfills $\rho(H) \neq \rho(G)$. Let $sd_\rho(G) = |E(G)|$ if such an edge set does not exist.

Velammal in 1997 [8] and Arumugam in 2000 (see [4]) studied the domination subdivision number $sd_\gamma(G)$ where $\gamma(G)$ is the domination number of G . This was later the topic of various papers (see e.g. Section III.H in the survey [9]). Subsequently this concept was also investigated for the stability number $\beta(G)$ [4] and other invariants related to the domination number (see e.g. [3]). Moreover, the relationship between edge removing and edge subdivision concerning the domination number was studied, among others, in [3, 4]. For example, $sd_\gamma(K_n) = 1$ and $es_\gamma(K_n) = \lceil n/2 \rceil$ if $n \geq 2$.

Some easy observations follow directly from the definition of the ρ -subdivision number. For example, $sd_\rho(G) = 0$ if and only if G is empty. If G is not empty, then $1 \leq sd_\rho(G) \leq |E(G)|$. If $\rho(G)$ does not change by edge subdivisions (as the number of cycles of a graph G), then $sd_\rho(G) = |E(G)|$.

For some specific invariants $\rho(G)$ it is easy to determine the ρ -subdivision number. In the following we give some examples.

For the order $n(G) = |V(G)|$ and the size $m(G) = |E(G)|$ of a graph G it holds that $sd_n(G) = sd_m(G) = 0$ if G is empty and $sd_n(G) = sd_m(G) = 1$ if G is not empty, since subdividing an edge of a graph increases the order and the size by 1. Therefore, $sd_n(G) = sd_m(G)$ for each graph G .

Subdividing an edge gives a homeomorphic graph, hence several invariants related to drawings of a graph are unaffected by edge subdivisions, for example the crossing number $cr(G)$ or the thickness $\theta(G)$. Therefore, $sd_{cr}(G) = sd_\theta(G) = |E(G)|$.

Subdividing an edge xy does not change the degree of x, y and adds a vertex of degree 2. Therefore, $sd_\delta(G) = 1$ if the minimum degree $\delta(G) \geq 3$, and $sd_\delta(G) = |E(G)|$ otherwise. Moreover, $sd_\Delta(G) = 1$ if the maximum degree $\Delta(G) = 1$, and $sd_\Delta(G) = |E(G)|$ otherwise.

Subdividing an edge does not change the number of cycles but does increase the length of cycles containing the edge. Therefore, for the circumference $c(G)$ (the length of a longest cycle in G or ∞ for acyclic graphs) it holds that $sd_c(G) = 1$ if G has cycles and $sd_c(G) = |E(G)|$ otherwise.

Consider now the girth $g(G)$ of G (the length of a shortest cycle in G or ∞ for acyclic graphs).

Proposition 1. $sd_g(G) = es_g(G)$ for the girth $g(G)$.

Proof. The result for acyclic graphs is obvious since the girth $g(G)$ cannot be changed by edge removals or edge subdivisions.

Let G be a graph with cycles. Let $E' \subseteq E(G)$ with $|E'| = es_g(G)$ and $g(G - E') \neq g(G)$. Every cycle of length $g(G)$ must contain an edge from E' , that is, subdividing these edges increases the girth, hence $sd_g(G) \leq |E'| = es_g(G)$.

Conversely, let $E'' \subseteq E(G)$ with $|E''| = sd_g(G)$ and $g(G_{E''}) \neq g(G)$. Then every cycle of length $g(G)$ must contain an edge from E'' , that is, removing these edges increases the girth, hence $es_g(G) \leq |E''| = sd_g(G)$. ■

In the first part of this paper we investigate the subdivision numbers for arbitrary invariants and we deduce some general results.

After that the χ -subdivision number $sd_\chi(G)$ of G , also called chromatic subdivision number, where $\chi(G)$ is the chromatic number of G , and the χ' -subdivision number $sd_{\chi'}(G)$ of G , also called edge chromatic subdivision number, where $\chi'(G)$ is the chromatic index of G , are considered. It will be shown among others that $sd_\chi(G) = es_\chi(G)$ if $\chi(G) \geq 3$ and $sd_{\chi'}(G) = es_{\chi'}(G)$ if $\chi'(G) = \Delta(G) + 1$, that is, in these cases it does not matter whether edge removals or edge subdivisions will be carried out (as in the example above on the girth). Results on the edge stability numbers $es_\chi(G)$ and $es_{\chi'}(G)$ can be found, e.g., in [5, 6].

In the next part of this paper we investigate the χ'' -subdivision number $sd_{\chi''}(G)$ of G , also called total chromatic subdivision number, where $\chi''(G)$ is the total chromatic number of G . We determine $sd_{\chi''}(G)$ for a large class of type 1 graphs.

In the last part we consider a variation of ρ -subdivision numbers for which multiple subdivisions of a single edge are allowed, that is, each subdivided edge is replaced by a path of length at least 2. It will be shown among others that for the subdivision numbers with respect to the chromatic number $\chi(G)$ and to the chromatic index $\chi'(G)$ it does not matter whether single or multiple edge subdivisions will be carried out while in general the two invariants may differ.

2. GENERAL RESULTS

In this section we consider the ρ -subdivision numbers for invariants $\rho(G)$ that have certain properties.

An invariant $\rho(G)$ is monotone increasing (with respect to subgraphs) if $H \subseteq G$ implies $\rho(H) \leq \rho(G)$, and monotone decreasing (with respect to subgraphs) if $H \subseteq G$ implies $\rho(H) \geq \rho(G)$ (see, e.g., [7] for a study on graph invariants).

We call $\rho(G)$ non-increasing (with respect to edge subdivisions) if $\rho(G_{E'}) \leq \rho(G)$ for every $E' \subseteq E(G)$, and non-decreasing (with respect to edge subdivisions) if $\rho(G_{E'}) \geq \rho(G)$ for every $E' \subseteq E(G)$.

$\rho(G)$ is called ss-monotone (for subgraph and subdivisions) if it is monotone increasing with respect to subgraphs and non-increasing with respect to edge subdivisions, or monotone decreasing with respect to subgraphs and non-decreasing with respect to edge subdivisions, respectively. Note that the class of ss-monotone invariants is not empty; for example, constant invariants or the number of cycles in a graph are ss-monotone.

If H_1 and H_2 are disjoint graphs, then an invariant is called additive if $\rho(H_1 \cup H_2) = \rho(H_1) + \rho(H_2)$ and maxing if $\rho(H_1 \cup H_2) = \max\{\rho(H_1), \rho(H_2)\}$.

Some easy observations follow directly from the definitions.

Proposition 2. *If $\rho(G_{E'}) \neq \rho(G)$, then $sd_\rho(G) \leq |E'|$.*

Proposition 3. *If $\rho(G)$ is ss-monotone or does not change by edge subdivisions, then $es_\rho(G) \leq sd_\rho(G)$.*

Proof. If $\rho(G)$ does not change by edge subdivisions, then $sd_\rho(G) = |E(G)|$ and $es_\rho(G) \leq |E(G)| = sd_\rho(G)$ follows.

On the other hand, we may assume that there is an edge set $E' \subseteq E(G)$ with $|E'| = sd_\rho(G)$ and $\rho(G_{E'}) \neq \rho(G)$.

If $\rho(G)$ is monotone increasing with respect to subgraphs and non-increasing with respect to edge subdivisions, then $G - E' \subseteq G_{E'}$ implies $\rho(G - E') \leq \rho(G_{E'}) < \rho(G)$. If $\rho(G)$ is monotone decreasing with respect to subgraphs and non-decreasing with respect to edge subdivisions, then $\rho(G - E') \geq \rho(G_{E'}) > \rho(G)$. In both cases it follows that $es_\rho(G) \leq |E'| = sd_\rho(G)$. ■

The following two theorems on disjoint unions of graphs can be obtained by transferring results on the edge stability number (see [6]).

Theorem 4. *Let $\rho(G)$ be additive, $G = H_1 \cup \dots \cup H_k$ a graph whose subgraphs H_1, \dots, H_k and the integer $s \geq 0$ are defined in such a way that $\rho(H_i)$ can be changed by edge subdivisions if and only if $1 \leq i \leq s$. Then $sd_\rho(G) = |E(G)|$ if $s = 0$ and $sd_\rho(G) = \min\{sd_\rho(H_i) : 1 \leq i \leq s\}$ if $s \neq 0$.*

Proof. If $s = 0$, then $\rho(H_i)$ cannot be changed by edge subdivisions for every subgraph H_i , which implies by the additivity that also $\rho(G) = \rho(H_1) + \dots + \rho(H_k)$ cannot be changed by edge subdivisions, and $sd_\rho(G) = |E(G)|$ follows.

If $s \neq 0$, then let H_j be a subgraph with $sd_\rho(H_j) = \min\{sd_\rho(H_i) : 1 \leq i \leq s\}$ and $E' \subseteq E(H_j)$ be a set of edges with $|E'| = sd_\rho(H_j)$ and $\rho((H_j)_{E'}) \neq \rho(H_j)$. By

the additivity, $\rho(G_{E'}) = \rho(H_1) + \dots + \rho(H_{j-1}) + \rho((H_j)_{E'}) + \rho(H_{j+1}) + \dots + \rho(H_k) \neq \rho(H_1) + \dots + \rho(H_{j-1}) + \rho(H_j) + \rho(H_{j+1}) + \dots + \rho(H_k) = \rho(G)$, which implies $sd_\rho(G) \leq |E'| = sd_\rho(H_j)$.

Let $E'' \subseteq E(G)$ be a set of edges with $|E''| < sd_\rho(H_j)$. By the minimality of $sd_\rho(H_j)$, $\rho((H_i)_{E'' \cap E(H_i)}) = \rho(H_i)$ for $i = 1, \dots, k$, which implies $\rho(G_{E''}) = \rho(G)$ since $\rho(G)$ is additive. Therefore, $sd_\rho(G) = sd_\rho(H_j)$. ■

For maxing invariants $\rho(G)$ we need an additional property.

Theorem 5. *Let $\rho(G)$ be maxing and non-increasing with respect to edge subdivisions, let $G = H_1 \cup \dots \cup H_k$ be a graph whose subgraphs H_1, \dots, H_k and the integer $s \geq 1$ are defined such that $\rho(H_i) = \rho(G)$ if and only if $1 \leq i \leq s$. Then $sd_\rho(G) = |E(G)|$ if there is a subgraph H_j , $1 \leq j \leq s$, such that $\rho(H_j)$ cannot be changed by edge subdivisions, and $sd_\rho(G) = \sum_{i=1}^s sd_\rho(H_i)$ otherwise.*

Proof. If there is a subgraph H_j , $1 \leq j \leq s$, such that $\rho(H_j)$ cannot be changed by edge subdivisions, then $\rho(G) = \rho(H_j) = \rho(G_{E'})$ for every $E' \subseteq E(G)$, since the invariant is maxing and non-increasing with respect to edge subdivisions. Therefore, $sd_\rho(G) = |E(G)|$.

Otherwise, let $E' = E'_1 \cup \dots \cup E'_s$ with $E'_i \subseteq E(H_i)$, $|E'_i| = sd_\rho(H_i)$, and $\rho((H_i)_{E'_i}) \neq \rho(H_i)$ for $i = 1, \dots, s$. Since the invariant is maxing, $\rho(G_{E'}) = \max\{\rho((H_i)_{E'_i}) : 1 \leq i \leq s\} \cup \{\rho(H_i) : s+1 \leq i \leq k\} \neq \rho(G)$ which implies $sd_\rho(G) \leq |E'| = \sum_{i=1}^s sd_\rho(H_i)$.

If $G_{E''}$ is considered where $E'' \subseteq E(G)$ with $|E''| < |E'|$, then there is a subgraph H_j , $1 \leq j \leq s$, in which less than $sd_\rho(H_j)$ edges are subdivided, which implies $\rho((H_j)_{E'' \cap E(H_j)}) = \rho(H_j)$ and thus, since the invariant is maxing and non-increasing, $\rho(G_{E''}) = \rho(H_j) = \rho(G)$. Therefore, $sd_\rho(G) = |E'| = \sum_{i=1}^s sd_\rho(H_i)$. ■

Theorems 4 and 5 imply that the ρ -subdivision number $sd_\rho(G)$ can be computed by the ρ -subdivision numbers of the components of G if the invariant is additive or if it is maxing and non-increasing with respect to edge subdivisions. Therefore, it is sufficient to consider connected graphs G in these cases.

A lower bound for $es_\rho(G)$ given in [6] can be transferred as follows.

Theorem 6. *Let $\rho(G)$ be ss-monotone and let G be a nonempty graph with $\rho(G) = k$. If G contains s nonempty subgraphs G_1, \dots, G_s with $\rho(G_1) = \dots = \rho(G_s) = k$ such that $a \geq 0$ is the number of edges that occur in at least two of these subgraphs and $q \geq 1$ is the maximum number of these subgraphs with a common edge, then both $sd_\rho(G) \geq \frac{1}{q} \sum_{i=1}^s sd_\rho(G_i) \geq s/q$ and $sd_\rho(G) \geq \sum_{i=1}^s sd_\rho(G_i) - a(q-1)$ hold.*

Proof. Let $\rho(G)$ be monotone increasing with respect to subgraphs and non-increasing with respect to edge subdivisions.

Let E' be a set of edges of G with $|E'| = sd_\rho(G)$ such that $\rho(G_{E'}) < k$ or $E' = E(G)$. If $\rho(G_{E'}) < k$, then the set E' must contain at least $sd_\rho(G_i)$ edges of each graph G_i , $1 \leq i \leq s$, since otherwise $k > \rho(G_{E'}) \geq \rho((G_j)_{E' \cap E(G_j)}) = k$ for some j , $1 \leq j \leq s$, a contradiction. If $E' = E(G)$, then all edges of G_i are in E' for $1 \leq i \leq s$.

Therefore, $b = \sum_{i=1}^s |E' \cap E(G_i)| \geq \sum_{i=1}^s sd_\rho(G_i) \geq s$.

On the other hand, at most $\bar{a} = \min\{a, |E'|\}$ edges of E' are counted at most q times in b , every other edge of E' is counted at most once, so $b \leq \bar{a} \cdot q + (|E'| - \bar{a}) \cdot 1 = |E'| + \bar{a}(q - 1)$.

Since $\bar{a} \leq |E'|$, $b \leq q|E'|$ and therefore $sd_\rho(G) = |E'| \geq b/q \geq \frac{1}{q} \sum_{i=1}^s sd_\rho(G_i) \geq s/q$. On the other hand, $\bar{a} \leq a$ implies $sd_\rho(G) = |E'| \geq b - a(q - 1) \geq \sum_{i=1}^s sd_\rho(G_i) - a(q - 1)$.

The proof for a monotone decreasing with respect to subgraphs and non-decreasing with respect to edge subdivisions invariant $\rho(G)$ runs analogously. ■

Corollary 7. *Let $\rho(G)$ be ss-monotone and let G be a nonempty graph with $\rho(G) = k$. If G contains s nonempty subgraphs G_1, \dots, G_s with $\rho(G_1) = \dots = \rho(G_s) = k$ and pairwise disjoint edge sets, then $sd_\rho(G) \geq \sum_{i=1}^s sd_\rho(G_i) \geq s$.*

Proof. Each edge of G is contained in at most $q = 1$ of the given subgraphs since they are pairwise edge disjoint. The result follows from Theorem 6. ■

Corollary 8. *Let $\rho(G)$ be ss-monotone. If $H \subseteq G$ and $\rho(H) = \rho(G)$, then $sd_\rho(H) \leq sd_\rho(G)$.*

Proof. If H is empty, then $sd_\rho(H) = 0 \leq sd_\rho(G)$; otherwise Corollary 7 with $s = 1$ implies the result. ■

Note that in general $sd_\rho(G)$ must not be monotone increasing with respect to subgraphs even if $\rho(G)$ has this property.

3. CHROMATIC SUBDIVISION NUMBER

In this section we consider the chromatic subdivision number $sd_\chi(G)$ where $\chi(G)$ is the chromatic number of G , that is, the minimum number of colors in a proper vertex coloring of G . This problem was communicated by Arumugam [1].

Theorem 9. *If $\chi(G) \geq 3$ or if G is acyclic, then $sd_\chi(G) = es_\chi(G)$. If $\chi(G) = 2$ and G has cycles, then $sd_\chi(G) = 1$ but $es_\chi(G) = |E(G)|$.*

Proof. If $\chi(G) \leq 2$, then $es_\chi(G) = |E(G)|$ by definition. Note that after removing all edges of a non-empty graph the chromatic number drops from 2 to 1. On the other hand, if G is acyclic, then edge subdivisions do not change the

chromatic number, so $sd_\chi(G) = es_\chi(G) = |E(G)|$. If $\chi(G) = 2$ and G has cycles, then subdividing an edge from a necessarily even cycle creates an odd cycle with chromatic number 3, so $sd_\chi(G) = 1$.

In the following, let G be a graph with $\chi(G) \geq 3$. Let $E' \subseteq E(G)$ with $|E'| = es_\chi(G)$ and $\chi(G - E') = \chi(G) - 1$ and consider the graph G' obtained from G by subdividing all edges of E' .

Since $G - E'$ is an induced subgraph of G' , a proper vertex coloring of $G - E'$ with $\chi(G) - 1$ colors gives a partial vertex coloring of G' where only the subdivision vertices are uncolored. Note that each edge e of E' must connect vertices of the same color, otherwise $E' \setminus \{e\}$ would be an edge set with $\chi(G - E' \setminus \{e\}) = \chi(G) - 1$ which contradicts the minimality of $es_\chi(G)$. We complete the vertex coloring of G' by coloring the subdivision vertices in an arbitrary order. Each subdivision vertex is adjacent to two vertices of G of the same color, hence there is an unused color among the $\chi(G) - 1 \geq 2$ available colors. It follows that $\chi(G') = \chi(G) - 1$. By the minimality of $sd_\chi(G)$, $|E'| = es_\chi(G) \geq sd_\chi(G)$.

Consider now $E'' \subseteq E(G)$ with $|E''| = sd_\chi(G)$ such that subdividing all edges of E'' gives a graph G'' with $\chi(G'') = \chi(G) - 1$. Then $G - E'' \subseteq G''$ which implies that $\chi(G - E'') \leq \chi(G'') = \chi(G) - 1$. By the minimality of $es_\chi(G)$, $|E''| = sd_\chi(G) \geq es_\chi(G)$. ■

The last inequality $es_\chi(G) \leq sd_\chi(G)$ also follows from Proposition 3 since $\chi(G)$ is ss-monotone if $\chi(G) \geq 3$.

4. EDGE CHROMATIC SUBDIVISION NUMBER

In this section we consider the χ' -subdivision number $sd_{\chi'}(G)$ with respect to the chromatic index $\chi'(G)$ of G which is the minimum number of colors in a proper edge coloring of G . By Vizing's Theorem (see [2], p. 251), the chromatic index can only attain one of two values, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. Graphs with $\chi'(G) = \Delta(G)$ are called class 1 graphs and graphs with $\chi'(G) = \Delta(G) + 1$ are called class 2 graphs.

First note that the maximum degree $\Delta(G)$ does not change by edge subdivisions except if $\Delta(G) = 1$. In this case, the chromatic index increases from 1 to 2. Therefore, $sd_{\chi'}(G) = 1$ if $\Delta(G) = 1$ and, by the definition, $sd_{\chi'}(G) = 0$ if $\Delta(G) = 0$, that is, if G is empty.

Assume in the following that $\Delta(G) \geq 2$. Since $\Delta(G_{E'}) = \Delta(G)$ for any set of edges E' of G , we have by Vizing's Theorem

$$\Delta(G) \leq \chi'(G), \chi'(G_{E'}) \leq \Delta(G) + 1,$$

which implies that $\chi'(G_{E'}) \neq \chi'(G)$ if and only if $G_{E'}$ and G have different classes (in this case the difference between the chromatic indices is always 1).

We give some examples.

For cycles C_n it holds that $\chi'(C_n) = \Delta(C_n) = 2$ if n is even and $\chi'(C_n) = \Delta(C_n) + 1 = 3$ if n is odd, that is, C_n is in class 1 if n is even and in class 2 if n is odd. Since subdividing an edge of a cycle increases its length by 1, it follows that $sd_{\chi'}(C_n) = 1$.

Let G be an acyclic graph with $\Delta(G) \geq 2$. By the Theorem of König (see [2], p. 257), $\chi'(G) = \Delta(G)$ since G is bipartite. Subdividing edges of G does not create cycles, so for any set of edges E' it holds that $\chi'(G_{E'}) = \Delta(G_{E'}) = \Delta(G)$. Therefore, $sd_{\chi'}(G) = |E(G)|$.

Now we consider class 1 graphs G and we ask when it is possible to increase the chromatic index of G by edge subdivisions.

Proposition 10. *Let G be a class 1 graph and $e = uv$ an edge with $d(v) < \Delta(G)$. Then $\chi'(G_e) = \Delta(G_e) = \Delta(G)$.*

Proof. The condition $d(v) < \Delta(G)$ implies $\Delta(G) \geq 2$, thus $\Delta(G_e) = \Delta(G)$.

Let c be a $\Delta(G)$ -edge coloring of G . Consider G_e with subdivision vertex w and color all edges of G_e except uw, vw with the color used in c , color uw with the color $c(e)$, and vw with one color not used to color any edge incident with v in G (which exists since $d(v) < \Delta(G)$). This gives a $\Delta(G)$ -edge coloring of G_e , thus $\chi'(G_e) = \Delta(G) = \Delta(G_e)$. ■

Therefore, subdivision edges in a minimal set of edges must always connect vertices of maximum degree. If there are no adjacent vertices of maximum degree, then $G_{E'}$ is always in class 1 for any set of edges $E' \subseteq E(G)$, thus the following holds.

Corollary 11. *If G is a class 1 graph with no adjacent vertices of maximum degree, then $sd_{\chi'}(G) = |E(G)|$.*

Example 12. If $G \cong K_{a,b}$ is a complete bipartite graph with $a < b$, then G is in class 1 by the Theorem of König. Since adjacent vertices have different degrees a and b , there are no adjacent vertices of maximum degree, thus $sd_{\chi'}(G) = |E(G)|$.

The following proposition shows that the above result may also hold if a class 1 graph contains two adjacent vertices of maximum degree.

A generalized θ -graph θ_{l_1, \dots, l_m} , $l_1 \leq \dots \leq l_m$, is a graph with two vertices connected by m internally disjoint paths of length l_1, \dots, l_m . If $m = 1$, then θ_{l_1} is a path of length l_1 and if $m = 2$, then θ_{l_1, l_2} is a cycle of length $l_1 + l_2$ which have been discussed in the above examples, so in the following we may assume $m \geq 3$. Note that the two vertices of maximum degree m are adjacent if and only if $l_1 = 1$. Moreover, since only simple graphs are considered, at most one path has length 1 which implies $l_2 \geq 2$.

Proposition 13. *If G is a generalized θ -graph θ_{l_1, \dots, l_m} with $m \geq 3$, then G is of class 1 and $sd_{\chi'}(G) = |E(G)| = l_1 + \dots + l_m$.*

Proof. At first we show that every generalized θ -graph $\theta_{l_1, l_2, \dots, l_m}$ with $m \geq 3$ is in class 1.

If $l_1 \geq 2$, then this follows by consecutively applying Proposition 10 on the complete bipartite graph $K_{2, m}$ which is in class 1 by the Theorem of König. If $l_1 = 1$ and $m \geq 4$ or if $l_1 = 1$, $m = 3$, and $l_2 + l_3$ even, then the subgraph θ_{l_2, \dots, l_m} is a class 1 graph (by the above fact or since the subgraph is an even cycle, respectively) whose edges can be properly colored by colors $1, \dots, m-1$; in these cases color the additional edge of the path of length $l_1 = 1$ by m . If $l_1 = 1$, $m = 3$, and $l_2 + l_3$ odd, then $l_3 \geq 3$ since $l_2 \geq 2$; color one edge not incident to a vertex of maximum degree and the edge of the path of length $l_1 = 1$ by $m = 3$ and the remaining edges of the graph (which form a path) alternately by 1 and by 2.

It follows that $\chi'(G) = \Delta(G) = m$. Since subdividing edges of G always gives generalized θ -graphs with longer paths and same maximum degree which are also in class 1, $sd_{\chi'}(G) = |E(G)| = l_1 + \dots + l_m$ by definition. ■

The next result shows that the edge chromatic subdivision number may also be small for class 1 graphs.

Proposition 14. *If G is a regular class 1 graph with $\Delta(G) \geq 2$, then G_e is in class 2 for any edge e of G , and $sd_{\chi'}(G) = 1$ follows.*

Proof. The edge set of G is partitioned into $\Delta(G)$ perfect matchings, thus the order n of G must be even and the size is $m = n\Delta(G)/2$.

Subdividing an arbitrary edge e of G gives a graph G_e of odd order $n+1$, size $m+1$, and maximum degree $\Delta(G_e) = \Delta(G)$. Since $m+1 > \Delta(G_e) \lfloor (n+1)/2 \rfloor = m$, the graph G_e is overfull and thus in class 2. This implies $sd_{\chi'}(G) = 1$. ■

In the following we apply this result on some graph classes.

Even cycles are 2-regular class 1 graphs, thus $sd_{\chi'}(C_n) = 1$ if n is even.

If K_n is a complete graph of even order $n \geq 4$, then K_n is a regular class 1 graph and $sd_{\chi'}(K_n) = 1$ follows.

If $K_{a,a}$ is a complete bipartite graph with $a \geq 2$, then $K_{a,a}$ is regular and in class 1 by the Theorem of König. Therefore, $sd_{\chi'}(K_{a,a}) = 1$.

We now consider class 2 graphs. The next result implies that it is always possible to lower the chromatic index by edge subdivisions, just as by edge deletions. Indeed, the two invariants $sd_{\chi'}(G)$ and $es_{\chi'}(G)$ are equal for class 2 graphs.

Theorem 15. *If G is a class 2 graph, then $sd_{\chi'}(G) = es_{\chi'}(G)$.*

Proof. Let $E' \subseteq E(G)$ such that $|E'| = es_{\chi'}(G)$ and $\chi'(G - E') = \Delta(G)$. Let c be a proper $\Delta(G)$ -edge coloring of $G - E'$.

If a vertex v of $G - E'$ is incident to x edges of E' , then $d_{G-E'}(v) \leq \Delta(G) - x$, that is, vertex v has at least x missing colors.

Consider an arbitrary edge $e = uv \in E'$. If the same color a is missing at both end vertices u, v , then the edge e could be colored by a , contradicting the minimality of E' with $|E'| = es_{\chi'}(G)$. Therefore, the set of missing colors at u and at v are disjoint for each edge $e = uv \in E'$.

We extend c to a $\Delta(G)$ -edge coloring \bar{c} of $G_{E'}$ as follows. First, $\bar{c}(e) = c(e)$ for each edge $e \in E(G) \setminus E'$. If x edges of E' are incident with a vertex v of G , then v has at least x missing colors which can be used to color the x subdivided edges incident to v . Note that the colors of the two edges incident with a subdivision vertex are different as mentioned above. Therefore, \bar{c} is a proper edge coloring of $G_{E'}$, and $sd_{\chi'}(G) \leq |E'| = es_{\chi'}(G)$ follows. This implies that it is possible to decrease the chromatic index of a class 2 graph by edge subdivisions.

Consider now $E'' \subseteq E(G)$ such that $|E''| = sd_{\chi'}(G)$ and $\chi'(G_{E''}) = \Delta(G)$. Then $G - E'' \subseteq G_{E''}$ which implies $\chi'(G - E'') \leq \chi'(G_{E''}) = \Delta(G)$. Therefore, $es_{\chi'}(G) \leq |E''| = sd_{\chi'}(G)$. ■

For example, the complete graphs K_n of odd order $n \geq 3$ and the Petersen graph P are class 2 graphs. Therefore, $sd_{\chi'}(K_n) = es_{\chi'}(K_n) = (n-1)/2$ if $n \geq 3$ odd and $sd_{\chi'}(P) = es_{\chi'}(P) = 2$ follow by Theorem 15 and by [6] for the latter equations.

Let $t'(G)$ be the minimum number of edges in a color class of the graph G where the minimum is taken over all edge colorings of G with $\chi'(G)$ colors.

Proposition 16. *If G is a class 2 graph, then $sd_{\chi'}(G) \leq t'(G)$.*

Proof. Let c be a $(\Delta(G) + 1)$ -edge coloring of G with a color class C of minimal cardinality $|C| = t'(G)$. Without loss of generality, let $C = c^{-1}(\Delta(G) + 1)$.

Consider an edge $e = uv$ of C . Since c uses $\Delta(G) + 1$ colors, at least one color is missing at each vertex. If the same color, say a , is missing at both u and v , then the edge e could be recolored by a , which contradicts the minimality of C . Therefore, the sets of missing colors at u, v are disjoint.

We construct from c a $\Delta(G)$ -edge coloring c_C of G_C in the following way: $c_C(e) = c(e)$ for each edge $e \notin C$. For an edge $e = uv \in C$, say with subdivision vertex w , we color uw with a missing color at u and wv with a missing color at v (according to the edge coloring c). These two new colors are different and distinct from $\Delta(G) + 1$ which was used at the edge e .

Note that the edges of C are independent and therefore the coloring of the subdivided edges can be done independently of each other. The obtained edge coloring c_C does not use color $\Delta(G) + 1$, that is, $\chi'(G_C) = \Delta(G) = \Delta(G_C)$ which implies that $sd_{\chi'}(G) \leq |C| = t'(G)$. ■

This result also directly follows from Theorem 15 and the fact that removing all edges of a color class reduces the chromatic index.

5. TOTAL CHROMATIC SUBDIVISION NUMBER

In this section we consider the χ'' -subdivision number $sd_{\chi''}(G)$ with respect to the total chromatic number $\chi''(G)$ of G . A proper total coloring of G is an assignment of colors to the vertices and edges of G (together called the elements of G) such that neighbored elements—two adjacent vertices or two adjacent edges or a vertex and an incident edge—are colored differently. A k -total coloring is a proper total coloring with k colors. The total chromatic number $\chi''(G)$ of G is defined as the minimum k in a k -total coloring of G . Obviously, $\chi''(G) \geq \Delta(G) + 1$ by definition, and the Total Coloring Conjecture states that $\chi''(G) \leq \Delta(G) + 2$ for every graph G (see [2], p. 282). Therefore, the truth of this conjecture would imply that $\chi''(G)$ attains one of two values for every graph G . Graphs G are called type 1 graphs if $\chi''(G) = \Delta(G) + 1$ and type 2 graphs if $\chi''(G) = \Delta(G) + 2$, respectively.

By the definition, $sd_{\chi''}(G) = 0$ if $\Delta(G) = 0$. If $\Delta(G) = 1$, then subdividing an edge does not change the total chromatic number of the graph (which is 3), thus $sd_{\chi''}(G) = |E(G)|$ if $\Delta(G) = 1$. Assume in the following that $\Delta(G) \geq 2$, which implies $\Delta(G_{E'}) = \Delta(G)$ for any set E' of edges of G . If the Total Coloring Conjecture is true, then

$$\Delta(G) + 1 \leq \chi''(G), \chi''(G_{E'}) \leq \Delta(G) + 2,$$

which implies that $\chi''(G_{E'}) \neq \chi''(G)$ if and only if $G_{E'}$ and G have different types (in this case the difference between the total chromatic numbers is 1).

We will ask when it is possible to increase the total chromatic number of a type 1 graph by edge subdivisions and when it is possible to decrease the total chromatic number of a type 2 graph by edge subdivisions.

For example, for cycles C_n of order n it holds that $\chi''(C_n) = \Delta(C_n) + 1 = 3$ if n is divisible by 3 and $\chi''(C_n) = \Delta(C_n) + 2 = 4$ otherwise. Since subdividing an edge of a cycle increases its length by 1, it follows that $sd_{\chi''}(C_n) = 1$ if $n \equiv 0 \pmod{3}$ or $n \equiv 2 \pmod{3}$, and $sd_{\chi''}(C_n) = 2$ if $n \equiv 1 \pmod{3}$.

If G is an acyclic graph with $\Delta(G) \geq 2$, then $\chi''(G) = \Delta(G) + 1$ (proof by induction on the order of G). Subdividing edges of G does not create cycles, so for any set of edges E' it holds that $\chi''(G_{E'}) = \Delta(G_{E'}) + 1 = \Delta(G) + 1 = \chi''(G)$. Therefore, $sd_{\chi''}(G) = |E(G)|$.

The χ'' -subdivision number of graphs G with $\Delta(G) \leq 2$ can be determined by the above examples.

Proposition 17. *If G is a graph with $\Delta(G) \leq 2$ and n_i cycles of length congruent to i modulo 3, $i \in \{0, 1, 2\}$, then $sd_{\chi''}(G) = |E(G)|$ if G is acyclic, $sd_{\chi''}(G) = 1$ if $n_1 = n_2 = 0$ and $n_0 \geq 1$, and $sd_{\chi''}(G) = 2n_1 + n_2$ otherwise.*

In the proof of the next proposition we will use the following result on total colorings of paths.

Lemma 18. *Let $P_n = (v_1, \dots, v_n)$ be a path of order $n \geq 3$ with a partial total coloring c of the vertices v_1 and v_n and the edges v_1v_2 and $v_{n-1}v_n$. Then c can be extended to a 4-total coloring of P_n except if $n = 3$ and the four precolored elements use 4 distinct colors, or if $n = 4$, $c(v_1) = c(v_{n-1}v_n)$, and $c(v_n) = c(v_1v_2)$. In these cases c can be extended to a 5-total coloring of P_n .*

Proof. Denote the colors of the precoloring by $c(v_1) = \alpha$, $c(v_1v_2) = \beta$, $c(v_{n-1}v_n) = \gamma$, and $c(v_n) = \delta$ and assume without loss of generality that $\alpha, \beta, \gamma, \delta \in \{1, 2, 3, 4\}$.

If $n = 3$, then all elements of P_n are already properly colored except for the vertex v_2 which is adjacent or incident to all 4 other elements. Therefore, c can be extended to a 4-total coloring of P_n if at most 3 colors were used in the precoloring, whereas a fifth color is needed to color v_2 if the 4 precolored elements use 4 distinct colors.

Let $n = 4$. If $\alpha = \gamma$ and $\beta = \delta$, then the remaining three elements v_2, v_2v_3, v_3 must be colored with three pairwise distinct colors different from α and β , that is, c cannot be extended to a 4-total coloring but to a 5-total coloring of P_4 . If $\alpha \neq \gamma$, then color v_2v_3 by α , then v_3 and last v_2 in a greedy manner. If $\beta \neq \delta$, then color v_2v_3 by δ , then v_2 and last v_3 in a greedy manner.

Let $n \geq 5$. Color the elements of the path periodically with three pairwise distinct colors α , β , and ϵ up to v_{n-3} , where $\epsilon \in \{1, 2, 3, 4\} \setminus \{\alpha, \beta\}$, and then $v_{n-3}v_{n-2}$ with a color from $\{1, 2, 3, 4\} \setminus \{c(v_{n-4}v_{n-3}), c(v_{n-3}), \delta\}$, in order to reduce this case to the case $n = 4$. ■

Proposition 19. *Let G be a type 1 graph and $e = uv$ be an edge with $d(v) < \Delta(G)$. Then $\chi''(G_e) = \chi''(G) = \Delta(G) + 1$ with the possible exception that $\Delta(G) = 3$, $d(v) = 2$, and v is adjacent to two vertices of maximum degree.*

Proof. The condition $1 \leq d(v) < \Delta(G)$ implies $\Delta(G) \geq 2$, thus $\Delta(G_e) = \Delta(G)$.

If $\Delta(G) = 2$, then $d(v) = 1$ and $e = uv$ is the first edge of a component P_p , $p \geq 2$, of G . Subdividing e gives a component P_{p+1} of G_e . The assertion follows from the fact that $\chi''(P_n) = 3$ for $n \geq 2$.

If $\Delta(G) = 3$, then we need to consider several cases. If e is an edge of an (attached) path P_p in G (including the case $d(v) = 1$), then we obtain an (attached) path P_{p+1} in G_e , whose elements can be colored in a greedy manner with the available 4 colors. Hence $d(v) = 2$ and e is an edge of a path $P_p =$

$(v_1, \dots, u = v_{i-1}, v = v_i, \dots, v_p)$ that connects two vertices v_1, v_p of maximum degree. According to the assumption let the order p of the path be at least 4. Then a path P_{p+1} of order at least 5 connects in G_e v_1 with v_p . Color all elements of G_e except the subdivision vertex w and its incident edges as in a 4-total coloring of G , color uw as uv , then recolor the elements of the path according to Lemma 18.

Let $\Delta(G) \geq 4$. Let c be a $(\Delta(G) + 1)$ -total coloring of G . Consider G_e with subdivision vertex w and color all elements of G_e except w, uw, vw with the color used in c . Color uw with the color $c(e)$, color vw with a color not used to color v or any edge incident to v in G (which exists since $d(v) < \Delta(G)$), and color w with a color different from the colors of u, v, uw, vw (which exists since $\Delta(G) \geq 4$). This gives a $(\Delta(G) + 1)$ -total coloring of G_e , thus $\chi''(G_e) = \Delta(G) + 1 = \Delta(G_e) + 1$. ■

Under the same assumption as in Proposition 19 subdivision edges in a minimal set of edges must always connect vertices of maximum degree. If there are no adjacent vertices of maximum degree, then $G_{E'}$ is always of type 1 for any set of edges $E' \subseteq E(G)$, thus the following holds.

Corollary 20. *If G is a type 1 graph with no adjacent vertices of maximum degree, then $sd_{\chi''}(G) = |E(G)|$, with the possible exception that $\Delta(G) = 3$ and G has vertices of degree 2 which are adjacent to two vertices of maximum degree.*

The following example shows that the result of Proposition 19 may also hold if $\Delta(G) = 3$, $d(v) = 2$, and v is adjacent to two vertices of maximum degree.

Example 21. The complete bipartite graph $K_{2,3}$ is a type 1 graph, that is, $\chi''(K_{2,3}) = 4$. By coloring the vertices of degree 3 in $K' = (K_{2,3})_e$ with color 4 and the remaining elements with colors 1, 3, 2 and 2, 1, 3 for the paths of length 2 and 3, 2, 4, 3, 1 for the path of length 3 we obtain a 4-total coloring of K' .

This example can be extended to $K_{2,m}$, $m \geq 3$, and even to generalized θ -graphs θ_{l_1, \dots, l_m} , $m \geq 3$. Note that if $m = 1$, then θ_{l_1} is a path of length l_1 and if $m = 2$, then θ_{l_1, l_2} is a cycle of length $l_1 + l_2$ which were discussed in Proposition 17.

Proposition 22. *If G is a generalized θ -graph θ_{l_1, \dots, l_m} with $m \geq 3$, then G is of type 1 and $sd_{\chi''}(G) = |E(G)| = l_1 + \dots + l_m$.*

Proof. At first we show that every generalized θ -graph θ_{l_1, \dots, l_m} with $m \geq 3$ is of type 1.

Without loss of generality, let $l_1 \leq \dots \leq l_m$, let x, y be the vertices of degree m in G and $P_i = (x = v_{i,0}, v_{i,1}, \dots, v_{i,l_i} = y)$ be the vertices of the i th path of length l_i in G ($i = 1, \dots, m$).

If $l_1, \dots, l_m \geq 2$, then color vertices x and y by $m+1$ (which is possible since x and y are not adjacent), color edges $xv_{i,1}$ by i and edges $v_{i,l_i-1}y$ by $(i \bmod m) + 1$, $i \in \{1, \dots, m\}$. Then apply Lemma 18 on each path P_i , $i \in \{1, \dots, m\}$, in order to properly color the remaining elements with $m+1$ colors. Note that the two exceptional cases of Lemma 18 do not occur with the above precolored elements.

If $l_1 = 1$, $l_2 = \dots = l_{a+1} = 2$ where $a \geq 0$ is the number of paths of length 2, and $l_i \geq 3$ for $i = a+2, \dots, m$, then vertices x and y must be colored differently.

Color vertices x by $m+1$ and y by 2, color edges $xv_{i,1}$ by i , $i \in \{1, \dots, m\}$, and edges $v_{i,l_i-1}y$ by $i+1$, $i \in \{2, \dots, m\}$. If $a \geq 1$, then color vertices $v_{i,1}$ by 1 for $i \in \{2, \dots, a+1\}$ which completes the coloring of paths of length at most 2. Then apply Lemma 18 on each path P_i , $i \in \{a+2, \dots, m\}$, in order to properly color the elements of the paths of length at least 3. Note again that the exceptional cases of Lemma 18 do not occur.

It follows in both cases that $\chi''(G) = \Delta(G) + 1 = m+1$. Since subdividing edges of G always gives generalized θ -graphs with longer paths and same maximum degree which are also of type 1, $sd_{\chi''}(G) = |E(G)| = l_1 + \dots + l_m$ by definition. ■

If a graph of type 1 has adjacent vertices of maximum degree, then the determination of its χ'' -subdivision number is in general open. This includes for example the class of regular graphs of type 1. Let us determine $sd_{\chi''}(K_{2k+1})$ for odd complete graphs as an example.

Example 23. Complete graphs of odd order K_{2k+1} , $k \geq 1$, are of type 1: $\chi''(K_{2k+1}) = \Delta(K_{2k+1}) + 1 = 2k+1$.

Subdividing an edge $e = uv$ of K_{2k+1} gives a graph K' with $2k+2$ vertices, $(2k+1)k+1$ edges, and maximum degree $\Delta(K') = \Delta(K_{2k+1}) = 2k$. If we assume that K' is also of type 1, then each of the $2k+1$ color classes may contain no vertex and at most $k+1$ edges, 1 vertex and at most k edges, or 2 vertices and at most k edges. Note that this last case may occur at most twice, namely for the subdivision vertex and a second vertex different from u, v , or for the two vertices u, v (in which case the color class may only contain at most $k-1$ edges).

If no color class has 2 vertices, then at least $2k+2$ color classes are needed in order to color all vertices. If one color class has 2 vertices, then the other $2k$ color classes must contain a vertex each, and these classes contain at most $(2k+1)k < |E(K')|$ edges. If two color classes have 2 vertices, then $2k-2$ classes contain a vertex each and one no vertex, and these classes contain at most $k+k-1 + (2k-2)k + k+1 = (2k+1)k < |E(K')|$ edges.

Therefore, a contradiction to the assumption follows, and K' is of type 2. This implies $sd_{\chi''}(K_{2k+1}) = 1$ for $k \geq 1$.

The determination of the χ'' -subdivision number of type 2 graphs is still an

open question. Complete graphs of even order are of type 2. We proved as partial result that $sd_{\chi''}(K_4) = 1$, $sd_{\chi''}(K_6) = 2$, and $2 \leq sd_{\chi''}(K_{2k}) \leq k - 1$ for $k \geq 3$.

6. MULTIPLE SUBDIVISION NUMBERS

Instead of subdividing each selected edge exactly once as in the definition of the ρ -subdivision number (which we call in the following single edge subdivisions) one could allow multiple subdivisions of the same edge, that is, each selected edge is replaced by a path of length at least 2 instead of by a path of length exactly 2. Note that when talking about the number of multiple edge subdivisions we mean the number of inserted subdivision vertices instead of the number of edges of the original graph that have been subdivided.

Definition. For an arbitrary invariant $\rho(G)$ of a graph G , the ρ -multiple subdivision number $\overline{sd}_\rho(G)$ of G is the minimum number of subdivision vertices (where multiple subdivisions of the same edge are allowed) such that the resulting graph H fulfills $\rho(H) \neq \rho(G)$. Let $\overline{sd}_\rho(G) = \infty$ if $\rho(G)$ does not change by multiple edge subdivisions or if G is empty.

Note that if G is empty or if $\rho(G)$ does not change by multiple edge subdivisions, then $sd_\rho(G) = |E(G)| < \infty = \overline{sd}_\rho(G)$. If $\rho(G)$ can be changed by single edge subdivisions, then $\overline{sd}_\rho(G) \leq sd_\rho(G)$, since single edge subdivisions are allowed in the determination of $\overline{sd}_\rho(G)$. If $\rho(G)$ does not change by single edge subdivisions but by multiple edge subdivisions, then $sd_\rho(G) = |E(G)|$ and $\overline{sd}_\rho(G)$ may be smaller or larger.

Example 24. 1. Let $\rho(G)$ be the number of components of order at least 4 in $G = aP_4 \cup P_2$. Then $\rho(G) = \rho(G_{E'}) = a$ for every $E' \subseteq E(G)$, hence $sd_\rho(G) = |E(G)| = 3a + 1$. On the other hand, it suffices to subdivide the single edge of the component P_2 twice to obtain an additional component P_4 , hence $\overline{sd}_\rho(G) = 2$ which is larger than $sd_\rho(G)$ for $a = 0$ and smaller for $a \geq 1$.

2. The eccentricity $e(v)$ of a vertex v in a graph G is $e(v) = \max\{d(v, u) : u \in V(G)\}$ if G is connected, where $d(v, u)$ is the distance between the vertices v and u , that is, the length of a shortest v - u path in G , and $e(v) = \infty$ if G is not connected. The radius $\text{rad}(G)$ of G is the minimum eccentricity $e(v)$ over all $v \in V(G)$.

For a path P_n it holds that $\text{rad}(P_n) = \lfloor n/2 \rfloor$, thus $sd_{\text{rad}}(P_n) = \overline{sd}_{\text{rad}}(P_n) = 1$ if $n \geq 3$ odd, $sd_{\text{rad}}(P_n) = \overline{sd}_{\text{rad}}(P_n) = 2$ if $n \geq 4$ even, whereas $sd_{\text{rad}}(P_2) = 1$ and $\overline{sd}_{\text{rad}}(P_2) = 2$ since the single edge of P_2 must be subdivided twice in order to increase the radius.

3. For the domination number of a path P_n it holds that $\gamma(P_n) = \lceil n/3 \rceil$, thus $sd_\gamma(P_n) = \overline{sd}_\gamma(P_n) = 1$ if $n \equiv 0 \pmod{3}$, $sd_\gamma(P_n) = \overline{sd}_\gamma(P_n) = 2$ if $n \equiv 2$

(mod 3) and $n \geq 5$, $sd_\gamma(P_n) = \overline{sd}_\gamma(P_n) = 3$ if $n \equiv 1 \pmod{3}$ and $n \geq 4$, whereas $sd_\gamma(P_2) = |E(P_2)| = 1$ by definition and $\overline{sd}_\gamma(P_2) = 2$ since the single edge of P_2 must be subdivided twice in order to increase the domination number.

Some easy observations follow directly from the definition.

Proposition 25. *If G' is a graph obtained from a graph G by multiple edge subdivisions with k subdivision vertices, then $\overline{sd}_\rho(G) \leq \overline{sd}_\rho(G') + k$. Moreover, if $\rho(G) \neq \rho(G')$, then $\overline{sd}_\rho(G) \leq k$.*

Proof. If $\rho(G) \neq \rho(G')$, then $\overline{sd}_\rho(G) \leq k \leq \overline{sd}_\rho(G') + k$.

Therefore, assume in the following that $\rho(G) = \rho(G')$. If $\rho(G')$ does not change by multiple edge subdivisions, then $\overline{sd}_\rho(G') = \infty$ by definition and $\overline{sd}_\rho(G) \leq \overline{sd}_\rho(G') + k$ follows.

Otherwise, let G'' be a graph obtained from G' by multiple edge subdivisions with $k' = \overline{sd}_\rho(G')$ subdivision vertices such that $\rho(G'') \neq \rho(G') = \rho(G)$. Since $\rho(G)$ can be changed by multiple edge subdivisions with $k' + k$ subdivision vertices, $\overline{sd}_\rho(G) \leq k' + k = \overline{sd}_\rho(G') + k$ follows. ■

We consider in the following the multiple subdivision numbers for the chromatic number, the chromatic index, and the total chromatic number.

Theorem 26. *If G is acyclic, then $\overline{sd}_\chi(G) = \infty$ and $sd_\chi(G) = |E(G)|$. Otherwise, if $\chi(G) = 2$ and G has cycles or if $\chi(G) \geq 3$, then $\overline{sd}_\chi(G) = sd_\chi(G)$.*

Proof. The first assertion follows from the fact that the chromatic number of an acyclic graph does not change by multiple edge subdivisions.

If $\chi(G) = 2$ and G has cycles, then subdividing an edge from a necessarily even cycle creates an odd cycle with chromatic number 3, hence $\overline{sd}_\chi(G) = sd_\chi(G) = 1$.

Assume in the following that $\chi(G) \geq 3$. Since subdividing all edges of G once gives a bipartite graph with $V(G)$ as one partition set and all subdivision vertices as the second partition set, $\chi(G)$ can change by single edge subdivisions, and $\overline{sd}_\chi(G) \leq sd_\chi(G)$ follows. Moreover, the chromatic number cannot increase by edge subdivisions, since each subdivision vertex can be properly colored with the $\chi(G) \geq 3$ available colors.

Let G' be a graph obtained from G by multiple edge subdivisions of the edges E' of G with $\overline{sd}_\chi(G)$ subdivision vertices such that $\chi(G') = \chi(G) - 1$. Consider a proper vertex coloring of G' with $\chi(G) - 1$ colors. Each edge e of E' is incident to vertices of the same color since otherwise the multiple subdivision of e could be removed from the set of subdivisions which contradicts the minimality of $\overline{sd}_\chi(G)$. Moreover, a multiple edge subdivision of $e \in E'$ can be replaced by a single edge subdivision, since each new subdivision vertex is adjacent to two vertices of G of the same color which implies that there is an unused color among the $\chi(G) - 1 \geq 2$

available colors that can be used to color the new subdivision vertex. Therefore we can assume that only single edge subdivisions of E' occur, that is, $G' \cong G_{E'}$, which implies that $\overline{sd}_\chi(G) \geq sd_\chi(G)$ by the minimality of $sd_\chi(G)$. It follows that $\overline{sd}_\chi(G) = sd_\chi(G)$ if $\chi(G) \geq 3$. ■

Theorem 27. *If $\chi'(G)$ does not change by multiple edge subdivisions, then $\overline{sd}_{\chi'}(G) = \infty$ and $sd_{\chi'}(G) = |E(G)|$. Otherwise, $\overline{sd}_{\chi'}(G) = sd_{\chi'}(G)$.*

Proof. The first assertion follows by the definitions. This holds for example for acyclic graphs G with $\Delta(G) \neq 1$.

Assume in the following that $\chi'(G)$ can be changed by multiple edge subdivisions.

If $\Delta(G) = 1$, then subdividing an edge of G once gives an acyclic graph with maximum degree 2, thus $\overline{sd}_{\chi'}(G) = sd_{\chi'}(G) = 1$. If $\Delta(G) = 2$ and G has n_0 even cycles and n_1 odd cycles, then it is easy to see that $\overline{sd}_{\chi'}(G) = sd_{\chi'}(G) = 1$ if $n_1 = 0$, $n_0 > 0$ and $\overline{sd}_{\chi'}(G) = sd_{\chi'}(G) = n_1$ if $n_1 > 0$.

If $\Delta(G) \geq 3$, then we consider two cases.

Case 1. G is in class 1. Let G' be a class 2 graph obtained from G by multiple edge subdivisions with $\overline{sd}_{\chi'}(G)$ subdivision vertices. By the minimality of $\overline{sd}_{\chi'}(G)$, performing multiple edge subdivisions with exactly $\overline{sd}_{\chi'}(G) - 1$ of these vertices gives a class 1 graph, which implies by Proposition 10 that every considered edge subdivision must be a single edge subdivision (since subdivision vertices have a degree of $2 < \Delta(G)$). Therefore, $\overline{sd}_{\chi'}(G) \geq sd_{\chi'}(G)$ by the minimality of $sd_{\chi'}(G)$. On the other hand, $\overline{sd}_{\chi'}(G) \leq sd_{\chi'}(G)$ holds since single edge subdivisions are allowed in the determination of $\overline{sd}_{\chi'}(G)$. It follows that $\overline{sd}_{\chi'}(G) = sd_{\chi'}(G)$ for class 1 graphs.

Case 2. G is in class 2. As in the proof of Theorem 26, subdividing all edges of G once gives a bipartite graph of the same maximum degree which is in class 1. Therefore, $\chi'(G)$ decreases by single edge subdivisions which implies that $\overline{sd}_{\chi'}(G) \leq sd_{\chi'}(G)$.

Let G' be a class 1 graph obtained from G by multiple edge subdivisions of the edges E' of G with $\overline{sd}_{\chi'}(G)$ subdivision vertices. Consider a proper edge coloring of G' with $\Delta(G)$ colors. Each edge e of E' is replaced in G' by a path of length at least 2. If the first and the last edge of the path are colored the same, then the subdivisions of e could be removed and e could be colored by the now unused color, which contradicts the minimality of $\overline{sd}_{\chi'}(G)$. On the other hand, if the first and the last edge of the path are colored differently, then multiple subdivisions of e could be replaced by a single subdivision whose edges could be colored by the two different colors. Consequently, we can assume that only single edge subdivisions of E' occur, that is, $G' \cong G_{E'}$. Therefore, $\overline{sd}_{\chi'}(G) \geq sd_{\chi'}(G)$ by the minimality of $sd_{\chi'}(G)$ which implies that $\overline{sd}_{\chi'}(G) = sd_{\chi'}(G)$ for class 2 graphs. ■

For the total chromatic number we present a partial result.

Theorem 28. *If $\chi''(G)$ does not change by multiple edge subdivisions, then $\overline{sd}_{\chi''}(G) = \infty$ and $sd_{\chi''}(G) = |E(G)|$. Otherwise, if $\Delta(G) = 2$ or if G is of type 1 and $\Delta(G) \geq 4$, then $\overline{sd}_{\chi''}(G) = sd_{\chi''}(G)$.*

Proof. The first assertion follows by the definitions. This holds for example for acyclic graphs. Note that $\chi''(P_2) = \chi''(P_3) = 3$ but the type changes from 2 to 1.

Assume in the following that $\chi''(G)$ can be changed by multiple edge subdivisions of G .

If $\Delta(G) = 2$, then $sd_{\chi''}(G)$ is determined in Proposition 17, and the same result and proof also hold for $\overline{sd}_{\chi''}(G)$, since each cycle of G must be subdivided at most twice in order to change its type, and multiple edge subdivisions can therefore be replaced by single edge subdivisions.

Assume in the following that G is of type 1 and $\Delta(G) \geq 4$. Let G' be a type 2 graph obtained from G by multiple edge subdivisions with $\overline{sd}_{\chi''}(G)$ subdivision vertices. By the minimality of $\overline{sd}_{\chi''}(G)$, performing multiple edge subdivisions with exactly $\overline{sd}_{\chi''}(G) - 1$ of these vertices gives a type 1 graph, which implies by Proposition 19 that every considered edge subdivision must be a single edge subdivision (since subdivision vertices have a degree of $2 < \Delta(G)$ and since we excluded graphs of maximum degree 3).

Therefore, $\overline{sd}_{\chi''}(G) \geq sd_{\chi''}(G)$ by the minimality of $sd_{\chi''}(G)$. On the other hand, $\overline{sd}_{\chi''}(G) \leq sd_{\chi''}(G)$ holds since single edge subdivisions are allowed in the determination of $\overline{sd}_{\chi''}(G)$. It follows that $\overline{sd}_{\chi''}(G) = sd_{\chi''}(G)$ for type 1 graphs with $\Delta(G) \geq 4$. ■

7. CONCLUDING REMARKS

In this paper we investigated the ρ -subdivision numbers $sd_{\rho}(G)$ of graphs G for arbitrary invariants $\rho(G)$ of G . Starting with some general results we proved subsequently that the ρ -subdivision number coincides with the ρ -edge stability number in the case that $\rho(G)$ is the chromatic number $\chi(G)$ and $\chi(G) \geq 3$.

In the next section we could prove the same coincidence in case that $\rho(G)$ is the chromatic index $\chi'(G)$ and G is class 2. For class 1 graphs we found some partial results, but in general this case remains open.

In the case that $\rho(G)$ is the total chromatic number $\chi''(G)$ we determined the ρ -subdivision number for a large class of type 1 graphs. For type 2 graphs the problem of determining $sd_{\chi''}(G)$ remains nearly completely open.

A subdivision of an edge is a replacement of this edge by a path of length 2. If we replace an edge by a path of arbitrary length, then we obtain a so-called multiple subdivision of this edge. If we compare the ρ -subdivision number $sd_{\rho}(G)$ with the analogously defined ρ -multiple subdivision number $\overline{sd}_{\rho}(G)$, then it turns

out that these two parameters coincide if $\rho(G)$ is the chromatic number or the chromatic index, or the total chromatic number if the graph G is of type 1 and $\Delta(G) \geq 4$. We conjecture that this coincidence holds in general for the total chromatic number. On the other hand, $sd_\rho(G)$ and $\overline{sd}_\rho(G)$ may differ for other invariants.

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