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NEW RESULTS ON TYPE 2 SNARKS[1]

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Abstract

Snarks are cyclically 4-edge-connected cubic graphs that admit no proper 3-edge-coloring. A snark is of Type 1 if it has a proper total coloring of its vertices and edges with four colors; it is of Type 2 if any total coloring requires at least five colors. Following an extensive computer search, in 2003, Cavicchioli *et al.* asked whether there exist Type 2 snarks of girth at least 5. This question is still open, however, in 2015, Brinkmann *et al.* described the first known family of Type 2 snarks of girth 4. In this work we provide new families of Type 2 snarks of girth 4, all of which can be constructed by a dot product of two Type 1 snarks. We also show that the previously constructed Type 2 snarks of Brinkmann *et al.* do not have this property.

Keywords: dot product, total coloring, snark.

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1. Introduction

The original motivation for the study of snarks was Tait's theorem [21], which states that the Four-Color Theorem [1, 17] is equivalent to the statement that "every bridgeless cubic graph which is not 3-edge-colorable is non-planar". In 1975, Isaacs [13] showed that it would be enough to prove this last statement for a more restricted class of graphs named snarks by Martin Gardner [11]. However, Isaacs did not give a precise definition of the class and several distinct definitions of snarks were used in subsequent papers. We define a *snark* as a non-3-edge-colorable cubic graph which is cyclically-4-edge connected.

Snarks play an important role in graph theory. Indeed, many conjectures have snarks as minimal possible counterexamples, for instance, the Cycle Double Cover Conjecture [19, 20], Berge-Fulkerson Conjecture [10, 15] and Tutte's 5-Flow Conjecture [22]. We refer to [4] for results about these and other conjectures related to snarks.

In 1971, Rosenfeld [18] proved the validity of the Total Coloring Conjecture [2, 24] for cubic graphs: the total chromatic number of a cubic graph is either 4 (Type 1) or 5 (Type 2). It is natural to ask if the chromatic index and the type of cubic graphs are related and, in particular, to look at the type a snark may have. In 2003, Cavicchioli *et al.* [7] showed by an extensive computer search that all snarks with girth at least 5 and order smaller than 30 are Type 1, and they asked for the smallest order of a Type 2 snark with girth at least 5. Later on, Brinkmann *et al.* [4] have shown that this order should be at least 38. Furthermore, several families of snarks were shown to be Type 1: all members of the infinite families of flower snarks and Goldberg snarks [6] as well as all members of two other infinite families of snarks [8].

In fact, until now no Type 2 cubic graph with girth at least 5 is known, so it is natural to look for Type 2 snarks, withdrawing the girth 5 constraint (notice that our definition of a snark implies girth at least 4). The first family of Type 2 snarks of girth 4 was discovered by Brinkmann *et al.* [5]. The family, which we denote by $\mathcal{S}$, provides such a snark on n vertices for every even integer $n \geq 40$ and contains all currently known cyclically 4-edge-connected Type 2 cubic graphs different from K_4. Furthermore, computer search has shown that the order of a Type 2 snark should be at least 36 [5].

In this work, we present new Type 2 snarks which are obtained from the dot product of two Type 1 snarks. As it will be explained later, the dot product, defined by Isaacs in his seminal paper [13], is a binary operation on snarks which allows us to construct other snarks. Our motivation was based on the two following observations: (i) it is easy to create a Type 2 snark from the dot product of a snark in $\mathcal{S}$ and any other snark; (ii) none of the Type 2 snarks in the family $\mathcal{S}$ defined by Brinkmann *et al.* [5] could be obtained by a dot product of two

Type 1 snarks.

The paper is organized as follows: in Section 2, we introduce the concepts of semi-graphs, bricks, junction, and we present the dot product. In Section 3, we show that snarks in $\mathcal{S}$ cannot be obtained by a dot product of two Type 1 snarks and show that there exist two infinite families of Type 2 snarks that can be obtained by a dot product of two Type 1 snarks.

2. Definitions

A *semi-graph* is a 3-tuple $G = (V(G), E(G), S(G))$ where $V(G)$ is a finite set of vertices of G, $E(G)$ is a set of edges having two distinct endpoints in $V(G)$, and $S(G)$ is a set of *semi-edges* having one endpoint in $V(G)$. When there is no chance of ambiguity, we simply write V, E or S.

An edge having endpoints v and w will be denoted by vw, and a semi-edge with endpoint v will be denoted by $v\cdot$. When vertex v is an endpoint of $e \in E \cup S$ we will say that v and e are *incident*. Two elements of $E \cup S$ incident with the same vertex, respectively two vertices incident with the same edge, will be called *adjacent*.

A *graph* $G = (V, E)$ is a semi-graph with an empty set of semi-edges. Given a semi-graph $G = (V, E, S)$, the *underlying graph of* G is the graph (V, E). All previous semi-graph definitions are also valid for graphs, independently of the existence of semi-edges.

Let $G = (V, E, S)$ be a semi-graph. The *degree* $d(v)$ of a vertex v of G is the number of elements of $E \cup S$ that are incident with v. We say that G is *d-regular* if the degree of each vertex is equal to d. In this paper, we are mainly interested in 3-regular graphs and semi-graphs, also called respectively *cubic graphs* and *cubic semi-graphs*. Given a graph G of maximum degree 3, the semi-graph obtained from G by adding $(3 - d(v))$ semi-edges with endpoint v, for each vertex v of G, is called the *cubic semi-graph generated by* G and is denoted by s-G.

For $k \in \mathbb{N}$, a *k-vertex-coloring* of G is a map $C^V\colon V \to \{1, 2, \ldots, k\}$, such that $C^V(x) \neq C^V(y)$ whenever x and y are two adjacent vertices.

Similarly, a *k-edge-coloring* of G is a map $C\colon E \cup S \to \{1, 2, \ldots, k\}$, such that $C(e) \neq C(f)$ whenever e and f are adjacent elements of $E \cup S$. The *chromatic index* of G, denoted by $\chi'(G)$, is the least k for which G has a k-edge-coloring. By Vizing's theorem [23], we have that $\chi'(G)$ is equal to $\Delta(G)$ or to $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the vertices of G. If $\chi'(G) = \Delta(G)$, then G is said to be *Class* 1, otherwise G is said to be *Class* 2.

A *k-total-coloring* of G is a map $C^T\colon V \cup E \cup S \to \{1, 2, \ldots, k\}$, such that:

- $C^T|_V$ is a vertex-coloring,
- $C^T|_{E \cup S}$ is an edge-coloring,

- $C^T(e) \neq C^T(v)$ whenever $e \in E \cup S$, $v \in V$ and e is incident with v.

The *total chromatic number* of G, denoted by $\chi_T(G)$, is the least k for which G has a k-total-coloring. Clearly $\chi_T(G) \geq \Delta(G) + 1$. The Total Coloring Conjecture [2, 24] claims that $\chi_T(G) \leq \Delta + 2$.

Proposed in 1965, the Total Coloring Conjecture has been proved only for specific classes of graphs, for instance cubic graphs [18]. If $\chi_T(G) = \Delta(G) + 1$ (respectively, $\chi_T(G) = \Delta(G) + 2$), then G is said to be Type 1 (respectively, Type 2). In particular, for cubic graphs, Type 1 (respectively, Type 2) means total chromatic number equals 4 (respectively, 5).

Notice that a 4-total-coloring of a cubic graph is equivalent to a proper 4-edge coloring such that for each edge e the four edges adjacent to e are colored by all four colors. Such a coloring is called strong in [5] (notice that this notion of strong coloring is different from the standard one). We will use this property to display a 4-total-coloring in some figures.

Let $G = (V, E)$ be a graph. Given a proper subset A of V, we denote by $\omega_G(A)$ the set of edges of G with one endpoint in A and the other endpoint in $V \setminus A$. A subset F of edges of G is an *edge cutset* if there exists a proper subset A of V such that $F = \omega_G(A)$ and we will then say that F *is induced by* A. If each of $G[A]$ and $G[V \setminus A]$ (the subgraphs of G induced by A and $V \setminus A$) has at least one cycle then $\omega_G(A)$ is said to be a *c-cutset*. A graph G is said to be *cyclically k-edge-connected* if it has no c-cutset of cardinality smaller than k. Notice that a cubic graph is cyclically 4-edge-connected if and only if each of its edge cutsets of cardinality smaller than 4 is an edge cutset of cardinality 3 induced by a single vertex.

A *brick* is a cubic semi-graph $B = (V, E, S)$ with exactly 4 pairwise non-adjacent semi-edges and whose underlying graph (V, E) is a subgraph of some cyclically 4-edge-connected cubic graph. The smallest brick, called *s-square*, has a chordless cycle on four vertices as its underlying graph [5].

Given two disjoint bricks $B = (V, E, S)$ and $B' = (V', E', S')$, any graph $G = (V \cup V', E \cup E' \cup E'')$ with E'' being a set of 4 disjoint edges xy with $x\cdot \in S$ and $y\cdot \in S'$, is called a *junction of B and B'*. In [5] it is shown that a cubic semi-graph B with exactly four pairwise non-adjacent semi-edges is a brick if and only if any junction of B and an s-square is cyclically 4-edge-connected.

Let G be a cyclically 4-edge-connected cubic graph. By removing two non-adjacent edges of G, one obtains a cubic semi-graph B generated by G, with exactly four pairwise non-adjacent semi-edges and which is, by definition, a brick. Such brick is called a *direct-brick of G*.

Two non-adjacent vertices of a direct-brick B of G that are adjacent in G form a *pair of B* (with respect to G). Two semi-edges incident with vertices of the same pair of B are also called a *pair*. By definition, a direct-brick of G contains two pairs of semi-edges. Similarly, removing two adjacent vertices of G

and all their incident edges, one obtains a cubic semi-graph B generated by G, with exactly four pairwise non-adjacent semi-edges and which is, by definition, a brick. Such brick is called an *edge-brick of* G. Two vertices of B that are adjacent in G to the same vertex not in B form a *pair of* B (with respect to G). Two semi-edges incident with vertices of the same pair of B are also called a *pair*. By definition, an edge-brick of G contains two pairs of semi-edges.

In [13], Isaacs defined a *dot product* $G \cdot H$ of snarks G and H as a cubic graph obtained by a pair-to-pair junction of a direct-brick of G and an edge-brick of H (see Figure 1). In the same paper, Isaacs proved that a dot product of two snarks is a snark, using the following well known and useful lemma.

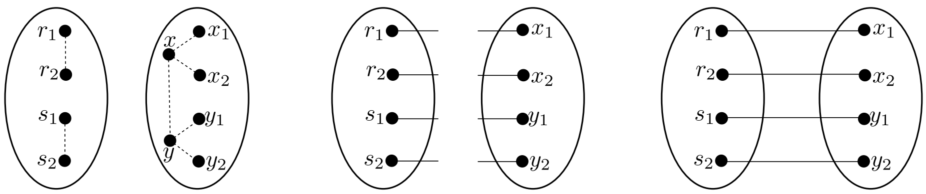

Figure 1. A representation of a dot product of snarks G and H.

Lemma 1 (Parity Lemma, Blanuša, 1946 [3], Descartes, 1948 [9])**.** *Let G be a cubic semi-graph containing exactly k semi-edges, C be a 3-edge-coloring of G, and k_1, k_2, k_3 be the numbers of semi-edges of G colored respectively* 1, 2, 3 *by C. Then*

$$k_1 \equiv k_2 \equiv k_3 \equiv k \bmod (2).$$

Implicitly, Isaacs also used the following lemma.

Lemma 2 (Brinkmann *et al.*, 2015 [5])**.** *Any junction of two bricks is a cyclically* 4*-edge-connected graph.*

3. Construction of Type 2 Snarks

The principle of the construction of Type 2 snarks in [5] is based on the following easy remark.

Observation 3.1. *Let G be a graph and let H be a subgraph of G such that $\Delta(G) = \Delta(H)$. If H is Class* 2 (*respectively, Type* 2) *then G is Class* 2 (*respectively, Type* 2)*.*

Indeed, in order to obtain a Type 2 snark, it is then "enough" to make a junction of a Class 2 brick and a Type 2 brick. Class 2 bricks are easily obtained by what is called in [5] a *semi-dot product* of two snarks, which is very similar to the dot product, except that only one pair-junction is made. The smallest such bricks have 18 vertices and there are five of them [5], they are obtained by a semi-dot product of two copies of the Petersen graph, one is displayed in Figure 2. We denote by $\mathcal{P}^*$ the set of these five Class 2 bricks.

On the other hand, Type 2 bricks are not so easy to obtain. The smallest Type 2 brick B^* was found by a computer search, and it has 22 vertices (see Figure 2).

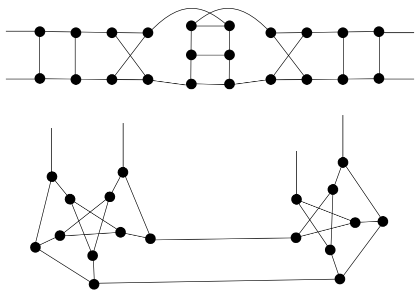

Figure 2. Type 2 brick $B^\star$ (top) and one Class 2 brick $P^\star$ (bottom).

Thus the smallest known Type 2 snarks have 40 vertices. A computer study showed that there are exactly 11 Type 2 snarks of order 40 that can be obtained in this way, one of them is displayed in Figure 3. Furthermore, as noticed in [5], from any Type 2 or Class 2 brick it is easy to obtain a new one with two more vertices, say v and w: delete two semi-edges $x\cdot$, $y\cdot$ and add edges xv, vw, wy and semi-edges $v\cdot$, $w\cdot$. Any brick obtained from a brick B by repeating this procedure is called an *augmentation* of B. The authors of [5] considered the class $\mathcal{S}$ of graphs that are obtained by a junction of a brick in $\mathcal{P}^*$ and an augmentation of the brick B^*. From the previously mentioned results, $\mathcal{S}$ contains Type 2 snarks of any even order at least 40. It is not known whether there exists a Type 2 snark of order 36 or 38 [5].

It is easy to show that one can obtain a snark of Type 2 from a dot product of a Type 2 snark $S \in \mathcal{S}$ and any snark; for example: by definition S contains a copy of B^*, let B be a direct-brick obtained from S by removing two non-adjacent edges that are not in B^*, the junction of B with any edge-brick issued from any snark will contain B^* and hence, be a Type 2 snark. Notice that in case we remove edges in B^*, we cannot derive such a conclusion on the type of the result of the dot product.

But is it possible to obtain a Type 2 snark by a dot product of two Type 1 snarks? In the sequel we show that no snark in $\mathcal{S}$ can be obtained in this way, but that nevertheless, the question has a positive answer. Before showing these results, we need some more definitions and notation.

It is sometimes useful to be able to distinguish the semi-edges of a brick; this

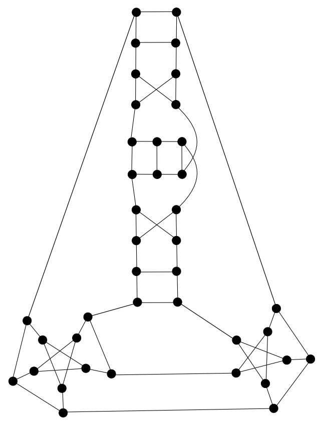

Figure 3. A Type 2 snark of girth 4 obtained by a junction of bricks $B^\star$ and $P^\star$.

can be done by a numbering of the semi-edges. In that case, we say that the brick is *numbered*. Let B be a brick with four numbered semi-edges s_1, s_2, s_3, s_4. By the Parity Lemma, every 3-edge-coloring C of B is such that either all semi-edges of B get the same color, or one color is used for two of the semi-edges and one other color for the two other semi-edges. We use a vector $C(B)$ to characterize the different possible cases of the coloring of the semi-edges of B: $C(B) = (a, a, a, a)$ if $C(s_1) = C(s_2) = C(s_3) = C(s_4)$, and $C(B) = (a, a, b, b)$ (respectively, (a, b, a, b), (a, b, b, a)) if $C(s_1) = C(s_2) \neq C(s_3) = C(s_4)$ (respectively, $C(s_1) = C(s_3) \neq C(s_2) = C(s_4)$, $C(s_1) = C(s_4) \neq C(s_2) = C(s_3)$).

We say that the vector (x, y, z, t) with components in $\{a, b\}$ is a *coloring of the semi-edges of* B if there exists a 3-edge-coloring C of B such that $C(B) = (x, y, z, t)$. As we have seen, there exist at most four possible colorings of the semi-edges of a numbered brick. It is important to remark that if a brick is 3-edge-colorable then its semi-edges have at least two colorings.

Indeed, it is known that from any 3-edge-coloring of a brick, by exchanging two colors in one Kempe bicolored chain connecting two semi-edges we always get at least one other coloring of the semi-edges [13, 16].

Let B and B' be two bricks with numbered semi-edges respectively $s_1 = t\cdot$, $s_2 = u\cdot$, $s_3 = v\cdot$, $s_4 = w\cdot$ and $s'_1 = t'\cdot$, $s'_2 = u'\cdot$, $s'_3 = v'\cdot$, $s'_4 = w'\cdot$. The numbered junction of B and B' is the junction of these two bricks having edges tt', uu', vv', ww'. Any junction of B and B' is a numbered junction for some numbering of the semi-edges of B and B'. The numbered junction of two bricks B and B' is non-3-edge-colorable if and only if the numbered B and B' have no common semi-edge colorings. We call *dot-brick* a brick which is a direct-brick or an edge-brick of a snark. From the previous facts, we have the following remark.

Observation 3.2. *A brick B is a dot-brick only if it is not* 3*-edge-colorable or* *it has exactly two semi-edges colorings. Furthermore, any* 3*-edge-coloring of a direct-brick of a snark is such that each of its pairs of semi-edges is colored with two distinct colors, any* 3*-edge-coloring of an edge-brick of a snark is such that each of its pairs of semi-edges is colored with one single color, and no* 3*-edge-colorable dot-brick can be both a direct-brick and an edge-brick of a snark.*

Proposition 3.3. *A brick B' obtained by an augmentation of a brick B is a dot-brick if and only if B is a dot-brick.*

Proof. It is enough to verify the property for B' obtained by deleting two semi-edges $x\cdot$, $y\cdot$ of B, and adding edges xv, vw, wy and semi-edges $v\cdot$, $w\cdot$. From any 3-edge-coloring C of B there is a straightforward way to extend it to a 3-edge-coloring C' of B'. So obviously, B' is 3-edge-colorable if and only if B is 3-edge-colorable. Furthermore, two distinct colorings of the semi-edges of B will produce distinct colorings of the semi-edges of B'. So the numbers of semi-edge colorings of B and B' are equal. By Observation 3.2 this concludes the proof. ■

Theorem 3. *Snarks that belong to $\mathcal{S}$ cannot be obtained by a dot product of two Type* 1 *snarks.*

Proof. Let G be a graph in $\mathcal{S}$ obtained by a junction of $P^* \in \mathcal{P}^*$ and an augmentation B^*_+ of B^*. Before beginning the proof, we notice several facts.

Fact 1. *Each semi-edge $x\cdot$ of B^* corresponds in G to a path from x to a vertex of P^*. Furthermore, the paths P_1, P_2, P_3, P_4 corresponding to the four semi-edges of B^* have the following property: for $i \in \{1, 2, 3, 4\}$, every interior vertex of P_i is connected to an interior vertex of P_j for some $j \neq i$ in $\{1, 2, 3, 4\}$.*

Fact 2. *All* 2*-edge cuts of B^* consist in two horizontal parallel edges of B^* as represented in Figure* 2*.*

Fact 3. *It is easy to verify that all possible bricks whose vertices are included in B^* have three distinct semi-edge colorings.*

Suppose that G is a dot product of two Type 1 snarks S_1 and S_2. Thus, G contains a cutset $\omega_G(A)$ of four pairwise non-adjacent edges joining two dot-bricks B_A and $B_{V\setminus A}$ induced respectively by $G[A]$ and $G[V \setminus A]$. As B^* which is contained in B^*_+ is of Type 2, we get that the vertices of B^* are separated by $\omega_G(A)$ and so, since there is no bridge in B^*, at least two edges of B^* belong to $\omega_G(A)$. Hence, $\omega_G(A)$ contains at most two other edges. We consider now two cases.

Case I. The vertices of P^* are not separated by $\omega_G(A)$. Without loss of generality we can assume that $A \subseteq V(G) \setminus V(P^*)$, so the vertices of A all belong to the augmentation B^*_+ of B^*.

If $A \subseteq V(B^*)$ then, by Fact 3, we get a contradiction to the assumption that B_A is a dot-brick. So, A contains at least one vertex that is not in B^*. Thus, this vertex should be an interior vertex of some path P_i and there should exist an edge xy of P_i, which is in $\omega_G(A)$, and such that x is an interior vertex of P_i. As $\omega_G(A)$ contains only non-adjacent edges, the edge xz, incident with x and not in P_i, is inside B_A. Furthermore, by Fact 1, z should belong to some other path P_j, and this path would also intersect $\omega_G(A)$. Hence, $\omega_G(A)$ consists in one edge of P_i, one edge of P_j, and two edges of a 2-edge cut of B^*, as described in Fact 2. So, B_A is an augmentation of a brick induced by a subgraph of B^* containing two endpoints of semi-edges of B^*. By Observation 3.2, Proposition 3.3, and Fact 3 we get a contradiction to the assumption that B_A is a dot-brick. An illustration of this case is presented in Figure 4.

Case II. The vertices of P^* are separated by $\omega_G(A)$. In this case, $\omega_G(A)$ should consist of two edges belonging to a 2-edge cut of B^* (Fact 3) and the unique 2-edge cut of P^*. Assume, without loss of generality, that A contains the part P_d of P^*, which is a direct-brick of the Petersen graph. So, B_A should be a brick obtained from P_d and a subgraph of B^* by connecting the two endpoints of a pair of P_d and two adjacent vertices that are endpoints of semi-edges of B^*_+ (see Figure 4).

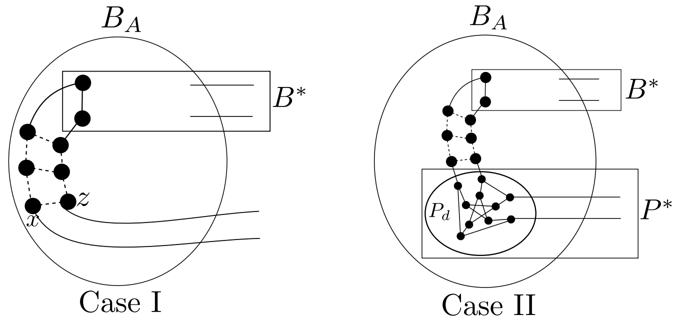

Figure 4. Proof of Theorem 3: Cases I and II.

Observe that P_d is a 3-edge-colorable direct-brick of a snark with one pair corresponding to a pair of semi-edges of B_A. By this observation, Fact 3, Proposition 3.3, we get that B_A is 3-edge-colorable. Furthermore by the Parity Lemma, in every 3-edge-coloring of B_A, each of its pair of semi-edges is bicolored. As we have assumed that B_A is a dot-brick we get, by Observation 3.2, that B_A should be a direct-brick of the non-3-edge-colorable graph G obtained by adding one edge between the endpoints of the semi-edges that are both in B^*, respectively P^*. Thus, G is not a snark as it has at least one 2-edge cut (see Figure 4). This contradicts the assumption that B_A is a dot-brick.

In all cases, we have got a contradiction, therefore the decomposition $S = S_1 \cdot S_2$ does not exist. ■

Theorem 4. *There exist two infinite families of Type* 2 *snarks that can be obtained by a dot product of two Type* 1 *snarks.*

Proof. Let S_1 and S_2 be the graphs depicted in Figure 5. It is easy to verify that both are cyclically 4-edge-connected cubic graphs. The 4-total-colorings indicated in the figure itself, show that they are Type 1 graphs.

Furthermore, both S_1 and S_2 have the underlying graph of $P^\star$ as a subgraph and therefore, by Observation 3.1, they are Class 2. Hence, graphs S_1 and S_2 are Type 1 snarks.

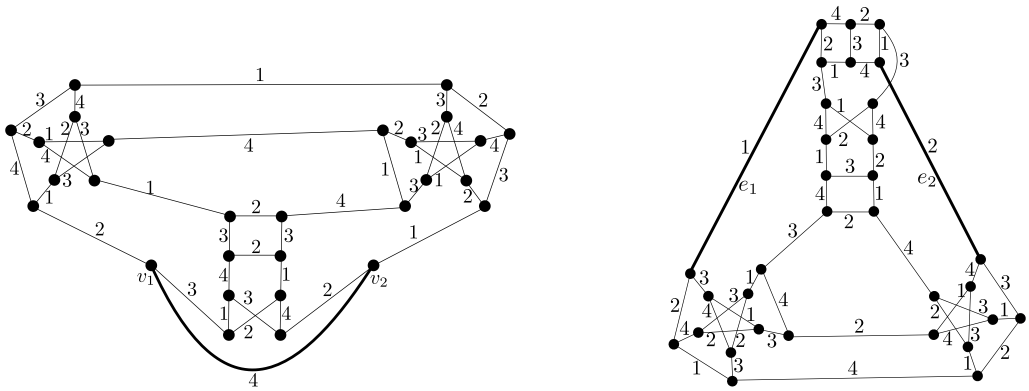

Figure 5. Snarks S_1 (left) and S_2 (right).

Now, let B_1 be the edge-brick associated with the edge v_1v_2 of S_1, and B_2 the direct-brick of S_2 associated with the non-adjacent edges e_1 and e_2 of S_2 (v_1v_2, e_1 and e_2 are indicated on Figure 5). The junction S of B_1 and B_2, displayed on Figure 6, is a dot product of S_1 and S_2, and S contains the underlying graph of $B^\star$ as a subgraph. Since $B^\star$ is Type 2, we get that S is a Type 2 snark.

We notice that "replacing" the two top horizontal edges of S_1, by any brick B', we still get a snark. Indeed, the direct-brick B_t obtained by breaking these two edges is Class 2 since two of its semi-edges are a pair of a direct-brick of the Petersen graph, and the two other semi-edges are a pair of an edge-brick of the Petersen graph. So, by Observation 3.2, in any 3-edge-coloring of B_t one should have one pair of semi-edges colored with two distinct colors and the other pair colored with a single color. This is not compatible with the Parity Lemma. Similarly, the direct-brick B_b obtained by breaking the two bottom horizontal edges of S_2 is Class 2. It remains to show that we can choose bricks that have 4-total-colorings which are compatible with the 4-total-colorings indicated in Figure 5. For instance, for every integer $k \geq 1$ we have the brick $B'^{(k)}$ with $6k$ vertices; for $k = 1$ it is represented in Figure 7. Let S_1^k be the snark obtained by attaching $B'^{(k)}$ to B_t: the dot product of S_1^k and S_2, similar to the one used to obtain S from S_1 and S_2, is a snark of Type 2 obtained by a dot product of two Type 1 snarks, and this is our first family. Similarly, we obtain a snark S_2^k by attaching $B'^{(k)}$ to B_b, and it is easy to verify that the 4-total-coloring of S_2, presented in Figure 5, can be extended to a 4-total-coloring of S_2^k.

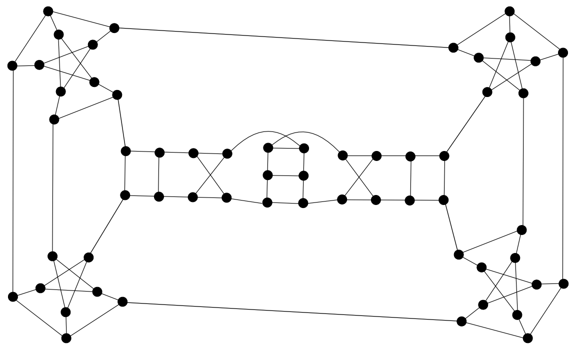

Figure 6. Type 2 snark S on 58 vertices.

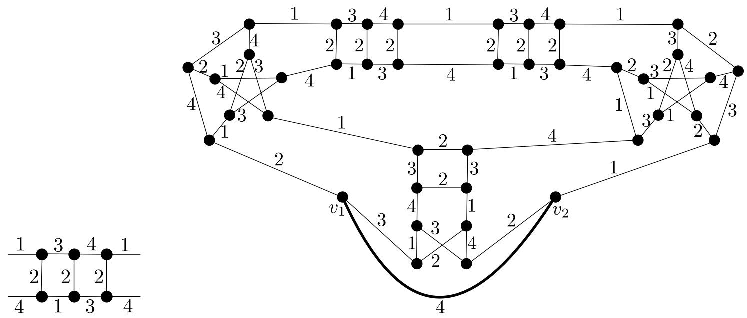

Figure 7. The brick $B'^{(1)}$ with a compatible 4-total-coloring used to construct snarks S_1^k.

We also construct another infinite family of Type 2 snarks obtained from the dot product of two Type 1 snarks. For that purpose we use another kind of bricks, called $L_k^\star$ (see Figure 8) defined for any integer $k \geq 0$ and consisting of $2k+3$ copies of L' the semi-graph with 5 semi-edges obtained from the Petersen graph minus a path on 3 vertices. Loupekine [14] and Goldberg [12] have shown that $L_k^\star$ bricks are Class 2. As a consequence, the graphs T_1 and T_2 displayed in Figure 9, obtained from L_0, are snarks. As shown in the figure, they are also Type 1.

Furthermore, the dot product of T_1 and T_2 based on the edge v_1v_2 of T_1 and the non-adjacent edges e_1 and e_2 of T_2 is a Type 2 snark (see Figure 10). Inserting in T_2 a chain of k copies of the semi-graph of Figure 11, between the left copy of L' and the right two copies of L', we obtain a snark T_2^k containing L_k. Moreover, using the coloring indicated in Figure 11, we obtain a 4-total-coloring

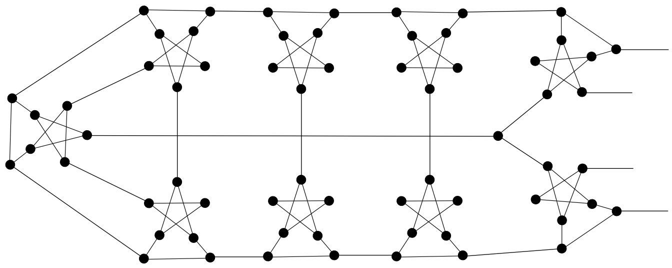

Figure 8. The brick $L_3^\star$.

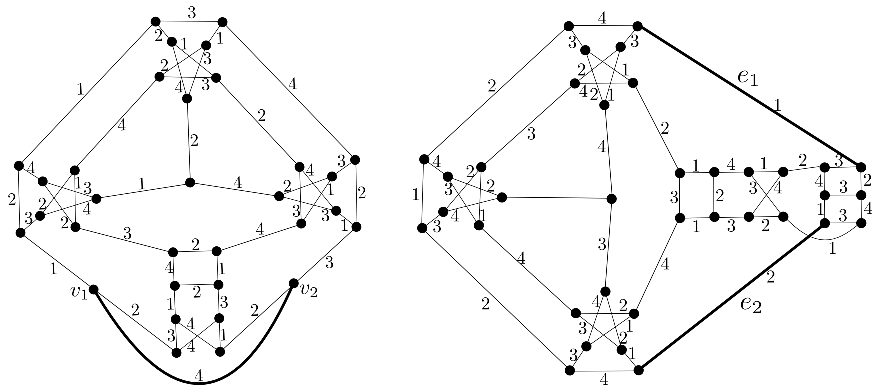

Figure 9. Graphs T_1 (left) and T_2 (right).

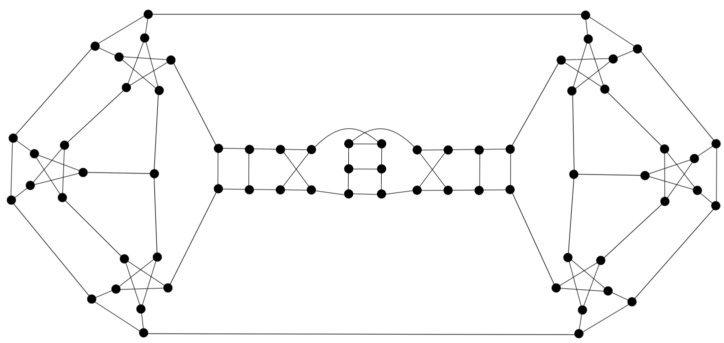

Figure 10. Type 2 snark T on 66 vertices.

of T_2^k. As before, the dot product of T_2^k and T_1 based on the edge v_1v_2 of T_1 and the non-adjacent edges e_1 and e_2 of T_2^k is a Type 2 snark. ■

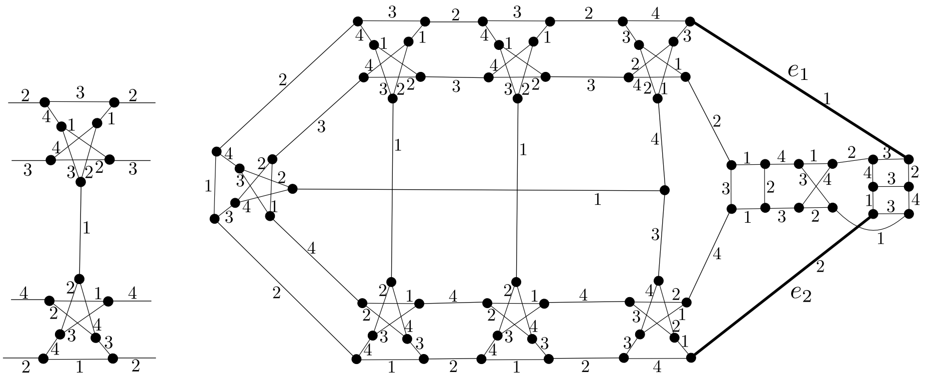

Figure 11. The semi-graph with a 4-total-coloring used to construct snark T_2^k and the snark T_2^3.

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