

ENUMERATING THE DIGITALLY CONVEX SETS OF POWERS OF CYCLES AND CARTESIAN PRODUCTS OF PATHS AND COMPLETE GRAPHS

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Abstract

Given a finite set V , a convexity, \mathcal{C} , is a collection of subsets of V that contains both the empty set and the set V and is closed under intersections. The elements of \mathcal{C} are called convex sets. The digital convexity, originally proposed as a tool for processing digital images, is defined as follows: a subset $S \subseteq V(G)$ is digitally convex if, for every $v \in V(G)$, we have $N[v] \subseteq N[S]$ implies $v \in S$. The number of cyclic binary strings with blocks of length at least k is expressed as a linear recurrence relation for $k \geq 2$. A bijection is established between these cyclic binary strings and the digitally convex sets of the $(k-1)^{th}$ power of a cycle. A closed formula for the number of digitally

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convex sets of the Cartesian product of two complete graphs is derived. A bijection is established between the digitally convex sets of the Cartesian product of two paths, $P_n \square P_m$, and certain types of $n \times m$ binary arrays.

Keywords: convexity, enumeration, digital convexity.

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1. INTRODUCTION

Given a finite set V , a collection, \mathcal{C} , of subsets of V is called a *convexity* or *alignment* if it contains \emptyset and V and is closed under intersections. The elements of a convexity \mathcal{C} are called *convex sets* and the ordered pair (V, \mathcal{C}) is an *aligned space*. For any subset $S \subseteq V$, the *convex hull* of S , denoted by $CH_{\mathcal{C}}(S)$, is the smallest convex set that contains S . For any $S \subseteq V$, if $CH_{\mathcal{C}}(S) = S$, then S is a convex set. Van de Vel provides an in-depth study of abstract convex structures in [19].

There are several convexities defined on the vertex set of a graph. The most natural extension of the Euclidean convexity to graphs is defined using an interval notion. For $a, b \in V(G)$, the collection of vertices that are on some a - b geodesic (shortest a - b path) forms the *geodesic interval* between a and b . Then, a set $S \subseteq V(G)$ is *g -convex* if it contains the geodesic interval between every pair of vertices in S . The collection of all g -convex sets in a graph G is a convexity called the *geodesic convexity* of G .

Several other graph convexities defined in terms of different types of intervals between pairs of vertices were studied, for example, in [6, 7, 9]. Interval structures between three or more vertices have led to yet more graph convexities examined, for example, in [4, 5, 12].

In this paper we study the digital convexity of a graph, introduced by Rosenfeld and Pfaltz in [15] as a tool for processing digital images. Rather than using a definition based on an interval structure, the digital convexity is instead defined in terms of neighbourhoods. The *open neighbourhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$ or $N(v)$ when the graph G is obvious, is defined as $N_G(v) = \{x \in V(G) \mid xv \in E(G)\}$. Similarly, the *closed neighbourhood* of v , denoted by $N_G[v]$ or $N[v]$, is defined as $N_G[v] = N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the closed neighbourhood of S , denoted by $N_G[S]$ or $N[S]$, is defined as $N_G[S] = \bigcup_{v \in S} N_G[v]$.

A set $S \subseteq V(G)$ is *digitally convex* if $N_G[v] \subseteq N_G[S]$ implies $v \in S$ for every $v \in V(G)$. For a vertex $v \in V(G)$ and a set $S \subseteq V(G)$, if $N_G[v] - N_G[S - \{v\}] \neq \emptyset$, we say that v has a *private neighbour with respect to S* in G . Thus, S is digitally convex if and only if, for every $v \notin S$, v has a private neighbour with respect to S . Note that private neighbours are not necessarily unique and a vertex v can

be a private neighbour for multiple vertices. For a graph G , the collection of all digitally convex sets in G is the *digital convexity* of G , denoted by $\mathcal{D}(G)$. The number of digitally convex sets in G is denoted by $n_{\mathcal{D}}(G)$. The digital convexity is studied in the context of closure systems in [14]. The relationship between digital convexity and domination in a graph is examined in [3, 13]. In particular, for $v \in V(G)$ and $S \subseteq V(G)$, if $N[v] \subseteq N[S]$, then S is a *local dominating set* for v . Thus, a digitally convex set is a set of vertices containing every vertex for which it is a local dominating set. Figure 1(a) shows a black and white digital image (or grid), with the digital convex hull of the black pixels shown in Figure 1(b). In a black and white digital image, taking the digital convex hull of the set of black pixels is a way of smoothing the digital image. Then, the number of distinct digitally convex sets in a digital image of a given size corresponds to the number of possible distinct smoothed digital images of that size. Generalizing this problem to graphs allows for the study of more abstract structures.

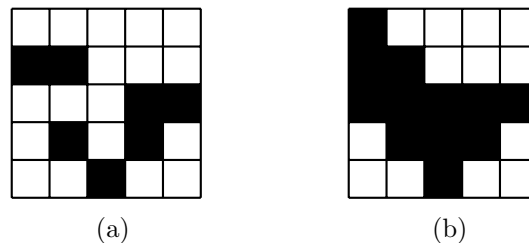


Figure 1. A black and white digital image in (a) and its corresponding digital convex hull in (b).

The problem of determining the number of convex sets of a graph has been studied for several of the above graph convexities. In the case of the geodesic convexity, it has been shown that the number of g -convex sets of a tree is equal to the number of its subtrees, a problem which is explored in [17, 18, 20]. Brown and Oellermann [2] determined that the problem of enumerating the g -convex sets of a cograph can be performed in linear time but, for an arbitrary graph, the problem is $\#P$ -complete. Graphs with a minimal number of g -convex sets or a minimal number of m -convex sets were examined in [1]. An algorithm for generating the digitally convex sets of a tree, as well as sharp upper and lower bounds on the number of digitally convex sets of a tree and a closed formula for the number of digitally convex sets of a path are given in [10].

This paper focuses on the enumeration of digitally convex sets in powers of cycles and in Cartesian products of two complete graphs and of two paths. In Section 2, we derive a recurrence relation for cyclic binary strings, all of whose blocks have length at least $k \geq 2$. We then establish a bijection between the digitally convex sets of the k^{th} power of a cycle and the cyclic binary strings

whose blocks all have length at least $k + 1$. In Section 3, we develop a closed formula for the number of digitally convex sets of the Cartesian product of two complete graphs, $K_n \square K_m$. It is shown that there is a bijection between the number of digitally convex sets of the Cartesian product of two paths, $P_n \square P_m$, and certain types of $n \times m$ arrays. In the special case where $m = 2$, the number of digitally convex sets in the Cartesian product $P_n \square P_2$ can be expressed as a linear recurrence relation of order three.

2. DIGITAL CONVEXITY IN POWERS OF CYCLES

In this section, we establish a recurrence relation for the number of cyclic binary strings with blocks of length at least $k \geq 2$ and we show that the number of digitally convex sets in the $(k-1)^{th}$ power of a cycle satisfies this same recurrence. Note that a cyclic binary string of length n is defined to be a sequence of n 0's and 1's, with the first and last bits in the sequence considered to be adjacent. Throughout this section, for a cycle C_n , we denote the vertices by v_1, v_2, \dots, v_n with $v_i v_{i+1} \in E(C_n)$ for $i = 1, 2, \dots, n-1$ and $v_1 v_n \in E(C_n)$. Recall that for a positive integer d , the d^{th} power of a graph G is the graph $G^d = (V, E')$, such that $uv \in E'$ if and only if u and v are distance at most d apart in G . For $3 \leq n \leq 2k+1$, the graph C_n^k is a complete graph.

For $k \geq 2$, we let $\mathcal{B}_{k,n}$ be the set of cyclic binary strings of length n in which each block (maximal run of 0's or 1's) has length at least k if $n \geq k$, or length exactly n if $n < k$. Let $a_k(n) = |\mathcal{B}_{k,n}|$. Now, we establish a recurrence relation satisfied by $a_k(n)$. Munarini and Salvi use the Schützenberger symbolic method to show this relation for $k = 2$ in [11]. We use this same method to generalize the result of Munarini and Salvi to any $k \geq 2$.

Lemma 1. *Let $k \geq 2$. Then $a_k(i) = 2$ for $3 \leq i \leq 2k-1$, $a_k(j) = 2 + j(j-2k+1)$ for $2k \leq j \leq 2k+2$ and, for $n \geq 2k+3$,*

$$a_k(n) = 2a_k(n-1) - a_k(n-2) + a_k(n-2k).$$

Proof. First, we establish the initial conditions. If $3 \leq n < k$, then the only cyclic binary strings of length n with each block of length exactly n are clearly $(00 \dots 0)$ and $(11 \dots 1)$. Thus, $a_k(n) = 2$ for $3 \leq n < k$. Now suppose $k \leq n \leq 2k-1$. Any cyclic binary string with at least two blocks must have one block of length $\ell \leq n/2 \leq (2k-1)/2 < k$. So the only strings in $\mathcal{B}_{k,n}$ are $(00 \dots 0)$ and $(11 \dots 1)$ when $k \leq n \leq 2k-1$.

If $2k \leq n \leq 2k+2$, then clearly both $(00 \dots 0)$ and $(11 \dots 1)$ are cyclic binary strings in $\mathcal{B}_{k,n}$. The remaining strings in $\mathcal{B}_{k,n}$ are those with two blocks, one of length ℓ , with $k \leq \ell \leq n-k$, and the other of length $n-\ell$. Without loss of generality, let the block of length ℓ be ℓ consecutive 1's. There are n

distinct cyclic shifts of these two blocks, giving n distinct cyclic binary strings with ℓ consecutive 1's. There are $n - 2k + 1$ possible values of ℓ , so there are $n(n - 2k + 1)$ cyclic binary strings in $\mathcal{B}_{k,n}$ with exactly two blocks. Overall, we have $a_k(n) = 2 + n(n - 2k + 1)$ for $2k \leq n \leq 2k + 2$. Therefore, the initial conditions hold.

Now, we find the generating function to show the desired recurrence. A cyclic binary string in $\mathcal{B}_{k,n}$ contains either exactly one block of length at least k or at least two blocks. In the former case, the generating function for these cyclic binary strings is $\frac{2x}{1-x}$.

In the latter case, we count half of the desired cyclic binary strings by assuming that the first element of the string is 0. Now, these strings can be decomposed as follows:

1. A nonempty string s_0 of 0's.
2. A (possibly empty) sequence of pairs of binary strings, with the first element of the pair a string of at least k 1's and the second element a string of at least k 0's.
3. A string of at least k 1's.
4. A string s_f of 0's such that the sum of the lengths of s_0 and s_f is at least k .

Now, we obtain the generating function for each component of the cyclic binary string, as described above.

- 1, 4. The pair (s_0, s_f) is a string of at least k 0's, with one element marked as the first element in the string. Thus, the generating function is

$$\sum_{j=k}^{\infty} jx^j = x \left(\frac{x^k}{1-x} \right)' = \frac{(1-k)x^{k+1} + kx^k}{(1-x)^2}.$$

2. Using the symbolic method, the generating function for a sequence of pairs of binary strings of this type is

$$\sum_{\ell=0}^{\infty} \left(\frac{x^{2k}}{(1-x)^2} \right)^{\ell} = \frac{1}{1 - \frac{x^{2k}}{(1-x)^2}} = \frac{(1-x)^2}{1 - 2x + x^2 - x^{2k}}.$$

3. The generating function for a string of at least k 1's is $\frac{x^k}{1-x}$.

Now, using the symbolic method, we multiply these three functions to get the generating function for all cyclic binary strings with at least two blocks of length at least k and first element 0. Multiplying by two to account for the cyclic binary strings with first element 1 and adding the above generating function for the cyclic binary strings containing exactly one block, we get the full generating

function for $\mathcal{B}_{k,n}$.

$$\begin{aligned} \sum_{n=0}^{\infty} a_k(n)x^n &= \frac{2x}{1-x} + 2 \frac{(1-k)x^{k+1} + kx^k}{(1-x)^2} \cdot \frac{(1-x)^2}{1-2x+x^2-x^{2k}} \cdot \frac{x^k}{1-x} \\ &= \frac{2x}{1-x} + \frac{2x^{2k+1} - 2kx^{2k+1} + 2kx^{2k}}{(1-x)(1-2x+x^2-x^{2k})} \\ &= \frac{2x - 4x^2 + 2x^3 - 2kx^{2k+1} + 2kx^{2k}}{(1-x)(1-2x+x^2-x^{2k})} = \frac{2x - 2x^2 + 2kx^{2k}}{1-2x+x^2-x^{2k}}. \end{aligned}$$

From the form of the generating function, we know that $a_k(n) - 2a_k(n-1) + a_k(n-2) - a_k(n-2k) = 0$. Rearranging this, we get the desired recurrence. ■

By establishing a suitable bijection, we now show that the recurrence relation established in Lemma 1 holds for the number of digitally convex sets of powers of cycles.

Theorem 2. Let C_n^k be the k^{th} power of the cycle C_n , with $k \geq 1$. Then $n_{\mathcal{D}}(C_i^k) = 2$ for $3 \leq i \leq 2k+1$, $n_{\mathcal{D}}(C_j^k) = 2 + j(j-2k-1)$ for $2k+2 \leq j \leq 2k+4$ and, for $n \geq 2k+5$,

$$n_{\mathcal{D}}(C_n^k) = 2n_{\mathcal{D}}(C_{n-1}^k) - n_{\mathcal{D}}(C_{n-2}^k) + n_{\mathcal{D}}(C_{n-2k-2}^k).$$

Proof. To prove the recurrence, we show a bijection between the digitally convex sets in $\mathcal{D}(C_n^k)$ and the cyclic binary n -bit strings in $\mathcal{B}_{k+1,n}$. If $n < k+1$, then these are the cyclic binary strings with blocks of length exactly n , i.e., $(0 \dots 0)$ and $(1 \dots 1)$. Clearly, since $k+1 \leq 2k+1$, there are exactly two digitally convex sets in C_n^k when $n < k+1$, the sets \emptyset and $V(C_n^k)$. These sets get mapped to $(0 \dots 0)$ and $(1 \dots 1)$, respectively.

Now, suppose $n \geq k+1$. Then $\mathcal{B}_{k+1,n}$ is the set of cyclic binary n -bit strings whose maximal blocks each have length at least $k+1$. Given a digitally convex set $S \in \mathcal{D}(C_n^k)$, we get a corresponding cyclic binary n -bit string S^* in the following way. For each vertex $v_i \in S$, set bits $i, i+1, \dots, i+k$ in S^* to be 1, taking the index mod n if $i+j > n$. After repeating this for each vertex in S , set the remaining bits in S^* to 0. As an example, the digitally convex set $S = \{v_1, v_7\}$ in C_7^2 , shown in Figure 2, corresponds to the cyclic binary string $S^* = (1110001)$.

It is clear from the construction of S^* that each block of 1's must have length at least $k+1$. We show now that each block of 0's in S^* must also have length at least $k+1$. Suppose that there is a block of 0's with length $\ell \leq k$, say bits $i, i+1, \dots, i+\ell-1$. Then, the vertex $v_i \notin S$, since bit i is 0 in S^* and $v_{i+\ell} \in S$, since bit $i+\ell$ is 1 in S^* . In C_n^k , we must have $v_i v_{i+\ell} \in E(C_n^k)$, as $\ell \leq k$, as well as $v_{i+j} v_{i+\ell} \in E(C_n^k)$ for $j = 1, 2, \dots, \ell-1$. In S^* , bit $i-1$ is also 1. So, by the construction of S^* , the vertex $v_{i-k-1} \in S$. This vertex is adjacent to the

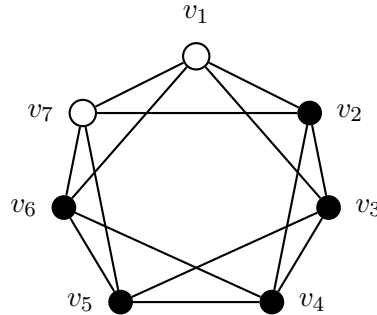


Figure 2. The digitally convex set $S = \{v_1, v_7\}$ of C_7^2 is indicated by white vertices.

vertices $v_{i-k}, v_{i-k+1}, \dots, v_{i-1}$ in C_n^k . Thus, $N[v_i] \subseteq N[\{v_{i+\ell}, v_{i-k-1}\}] \subseteq N[S]$, contradicting the fact that S is digitally convex in C_n^k .

We now show that there is an injective map from $\mathcal{B}_{k+1,n}$ to the set of digitally convex sets of C_n^k . Let $S^* \in \mathcal{B}_{k+1,n}$. If $S^* = (00\dots 0)$, then let $S = \emptyset$. If $S^* = (11\dots 1)$, then let $S = V(C_n^k)$. Both of these are clearly digitally convex. Otherwise, let B_1, B_2, \dots, B_r be the distinct blocks of at least $k+1$ 1's in S^* . Suppose bits $i, i+1, \dots, i+k+\ell-1$ are the bits of B_1 . Then, let $S_1 = \{v_i, v_{i+1}, \dots, v_{i+\ell-1}\}$. Define S_2, S_3, \dots, S_r similarly. Finally, let $S = S_1 \cup S_2 \cup \dots \cup S_r$. It is clear that S would be mapped to S^* using the above mapping. For example, if $S^* = (1110001)$ and $k = 2$, then bits 7, 1, 2, 3 are the bits of the only block of 1's. This string would be mapped to the set of vertices $\{v_7, v_1\}$, reversing the example of the mapping shown earlier in the proof. We show now that each such set S is a digitally convex set in $\mathcal{D}(C_n^k)$. Suppose otherwise, i.e., that S is not digitally convex. Then, there must be some $v_j \notin S$ such that $N[v_j] = \{v_{j-k}, v_{j-k+1}, \dots, v_j, v_{j+1}, \dots, v_{j+k}\} \subseteq N[S]$.

Since $v_j \notin S$ and $v_{j-k} \in N[S]$, we must have one of the vertices in $N[v_{j-k}] - \{v_j\}$ in S , say v_{j-k+m} for some $m \in \{-k, -k+1, \dots, k-1\}$. Then, by definition of S , the bits $j-k+m, j-k+m+1, \dots, j+m$ are all 1 in S^* . None of these possible vertices v_{j-k+m} is adjacent to v_{j+k} , so one of the vertices in $N[v_{j+k}] - \{v_j\}$ is in S , say v_{j+k+p} , for some $p \in \{-k+1, -k+2, \dots, k\}$. So, again by definition, the bits $j+k+p, j+k+p+1, \dots, j+2k+p$ are each 1 in S^* . In addition, these two vertices can be chosen so that each of the vertices $v_{j-k}, v_{j-k+1}, \dots, v_j, v_{j+1}, \dots, v_{j+k}$ appears in the closed neighbourhood of v_{j-k+m} or v_{j+k+p} . Then, the maximum possible difference between $j-k+m$ and $j+k+p$ is $2k+1$. So the longest block of 0's in S^* between bits $j+m$ and $j+k+p$ has length at most k , contradicting the fact that $S^* \in \mathcal{B}_{k+1,n}$. Therefore, S is digitally convex in C_n^k .

We have now shown a bijection between the digitally convex sets in $\mathcal{D}(C_n^k)$ and the cyclic binary strings in $\mathcal{B}_{k+1,n}$. So they satisfy the same recurrence.

Therefore, $n_{\mathcal{D}}(C_n^k) = 2n_{\mathcal{D}}(C_{n-1}^k) - n_{\mathcal{D}}(C_{n-2}^k) + n_{\mathcal{D}}(C_{n-2k-2}^k)$, with $n_{\mathcal{D}}(C_i^k) = 2$, for $3 \leq i \leq 2k+1$, and $n_{\mathcal{D}}(C_j^k) = 2 + j(j-2k-1)$, for $2k+2 \leq j \leq 2k+4$. ■

Corollary 3. *Let C_n be the cycle of order n . Then $n_{\mathcal{D}}(C_3) = 2$, $n_{\mathcal{D}}(C_4) = 6$, $n_{\mathcal{D}}(C_5) = 12$, $n_{\mathcal{D}}(C_6) = 20$ and, for $n \geq 7$,*

$$n_{\mathcal{D}}(C_n) = 2n_{\mathcal{D}}(C_{n-1}) - n_{\mathcal{D}}(C_{n-2}) + n_{\mathcal{D}}(C_{n-4}).$$

As shown above, the recurrence satisfied by $n_{\mathcal{D}}(C_n)$ is the same recurrence satisfied by the cyclic binary n -bit strings whose blocks each have length at least 2. This recurrence was shown, above and in [11], to have the generating function

$$\frac{2x - 2x^2 + 4x^4}{1 - 2x + x^2 - x^4}.$$

Notice that this expands to

$$\begin{aligned} & 2x + 2x^2 + 2x^3 + 6x^4 + 12x^5 + 20x^6 + \cdots + 74x^9 + 122x^{10} + 200x^{11} + \cdots \\ & + 842x^{14} + 1362x^{15} + \cdots + 9350x^{19} + 15126x^{20} + \cdots + 64080x^{23} + 103684x^{24} \\ & + 167762x^{25} + \cdots + 710646x^{28} + 1149852x^{29} + 1860500x^{30} + \cdots \end{aligned}$$

indicating that C_5 is the smallest cycle with more than ten digitally convex sets, C_{10} is the smallest cycle with more than 100 digitally convex sets, C_{15} is the smallest cycle with more than 1000 digitally convex sets, and C_{20} is the smallest cycle with more than 10 000 digitally convex sets. However, C_{24} , not C_{25} , is the smallest cycle with more than 100 000 digitally convex sets and, similarly, C_{29} is the smallest cycle with more than 1 000 000 digitally convex sets. This pattern suggests that $n_{\mathcal{D}}(C_{5k}) \geq 10^k$, but not that C_{5k} is the smallest cycle satisfying this inequality.

3. DIGITAL CONVEXITY AND CARTESIAN PRODUCTS

A digitally convex set in the Cartesian product $G \square H$ is not necessarily digitally convex when restricted to G or to H . In other words, if $S \in \mathcal{D}(G \square H)$, then the set $S_G = \{x \in V(G) \mid (x, y) \in S\}$ is not necessarily digitally convex in G . As an example, the set $\{(2, 1)\}$, shown in Figure 3, is digitally convex in $K_3 \square K_2$ but $\{2\} \notin \mathcal{D}(K_3)$ and $\{1\} \notin \mathcal{D}(K_2)$, as the only digitally convex sets in a complete graph are the empty set and the entire vertex set. This example shows that, even in small graphs, the number of digitally convex sets of a Cartesian product of graphs G and H cannot be computed from those of G and H in an obvious manner. We begin by examining the number of digitally convex sets in the Cartesian product of complete graphs, $K_n \square K_m$, to show how different this

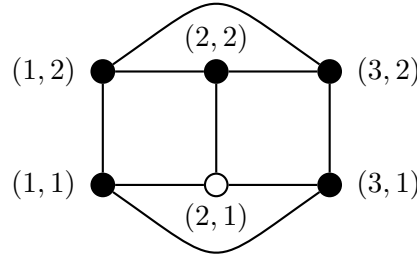


Figure 3. The set $\{(2,1)\} \in \mathcal{D}(K_3 \square K_2)$ is indicated in white.

number is from the number of digitally convex sets in either of the constituent graphs of the product.

Theorem 4. For any $m, n \geq 1$, $n_{\mathcal{D}}(K_n \square K_m) = 2 + (2^n - 2)(2^m - 2)$.

Proof. We begin by denoting the vertices of $K_n \square K_m$ by (v_i, u_j) for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Now, let $\emptyset \neq S_1 \subsetneq \{v_1, v_2, \dots, v_n\}$ and $\emptyset \neq S_2 \subsetneq \{u_1, u_2, \dots, u_m\}$. Then, $S = S_1 \times S_2 = \{(v_i, u_j) \mid v_i \in S_1, u_j \in S_2\}$ is digitally convex in $K_n \square K_m$. Consider $(v_x, u_y) \notin S$. If $v_x \notin S_1$ and $u_y \notin S_2$, then $(v_x, u_y) \notin N[S]$. If $v_x \in S_1$ and $u_y \notin S_2$, then there is some $v_z \notin S_1$ such that $(v_x, u_y)(v_z, u_y) \in E(K_n \square K_m)$ and $(v_z, u_y) \notin N[S]$. If $v_x \notin S_1$ and $u_y \in S_2$, then there is some $u_w \notin S_2$ such that $(v_x, u_y)(v_x, u_w) \in E(K_n \square K_m)$ and $(v_x, u_w) \notin N[S]$. Thus, (v_x, u_y) has a private neighbour with respect to S , and S is digitally convex. There are $(2^n - 2)(2^m - 2)$ such sets S .

Any set of vertices containing a set of type $\{(v_1, u_{i_1}), (v_2, u_{i_2}), \dots, (v_n, u_{i_n})\}$, where each $i_j \in \{1, 2, \dots, m\}$, is a dominating set in $K_n \square K_m$. Similarly, any set of vertices containing a set of type $\{(v_{j_1}, u_1), (v_{j_2}, u_2), \dots, (v_{j_m}, u_m)\}$, where each $j_k \in \{1, 2, \dots, n\}$, is a dominating set. Therefore, the only digitally convex set containing any of these sets of vertices is $V(K_n \square K_m)$.

Two vertices (v_x, u_y) and (v_w, u_z) dominate the neighbourhoods of both of the vertices (v_x, u_z) and (v_w, u_y) . So any digitally convex set containing the former pair of vertices must also contain the latter pair. Therefore, every nonempty digitally convex set in $K_n \square K_m$ must be $V(K_n \square K_m)$ or must take on the form $S_1 \times S_2$, where S_1 and S_2 are defined as above.

Therefore, along with the empty set, the graph $K_n \square K_m$ has a total of $2 + (2^n - 2)(2^m - 2)$ digitally convex sets. ■

We turn now to the Cartesian product of paths, beginning with $P_n \square P_2$. As with the Cartesian product of complete graphs, many of the digitally convex sets in the product of paths are no longer digitally convex when restricted to one of the

constituent paths. Thus, there is no obvious method of using the digitally convex sets of the constituent graphs to generate those of the product. We can, however, use the digitally convex sets of the graphs $P_{n-1} \square P_2$, $P_{n-2} \square P_2$ and $P_{n-3} \square P_2$ to determine those of $P_n \square P_2$.

Theorem 5. *Let P_n be the path of order n . Then $n_{\mathcal{D}}(P_1 \square P_2) = 2$, $n_{\mathcal{D}}(P_2 \square P_2) = 6$ and $n_{\mathcal{D}}(P_3 \square P_2) = 16$ and, for $n \geq 4$,*

$$n_{\mathcal{D}}(P_n \square P_2) = n_{\mathcal{D}}(P_{n-1} \square P_2) + 3n_{\mathcal{D}}(P_{n-2} \square P_2) + 2n_{\mathcal{D}}(P_{n-3} \square P_2).$$

Proof. We begin by denoting the vertices of $P_n \square P_2$ by $v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n$, with $v_i v_{i+1} \in E(P_n \square P_2)$ and $u_i u_{i+1} \in E(P_n \square P_2)$, for $i = 1, 2, \dots, n-1$, and $v_j u_j \in E(P_n \square P_2)$ for $j = 1, 2, \dots, n$.

Now, we prove the initial conditions. Since $P_1 \square P_2 \cong K_2$, we have that $n_{\mathcal{D}}(P_1 \square P_2) = n_{\mathcal{D}}(K_2) = 2$. Similarly, $P_2 \square P_2 \cong C_4$, so $n_{\mathcal{D}}(P_2 \square P_2) = n_{\mathcal{D}}(C_4) = 6$. Finally, the 16 digitally convex sets of $P_3 \square P_2$ are $\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{u_1\}, \{u_2\}, \{u_3\}, \{v_1, v_3\}, \{v_1, u_1\}, \{v_3, u_3\}, \{u_1, u_3\}, \{v_1, v_3, u_1\}, \{v_1, u_1, u_2\}, \{v_2, v_3, u_3\}, \{v_3, u_2, u_3\}$, and $V(P_3 \square P_2)$. So $n_{\mathcal{D}}(P_3 \square P_2) = 16$.

Suppose $n \geq 4$. We begin by showing $n_{\mathcal{D}}(P_n \square P_2) \geq n_{\mathcal{D}}(P_{n-1} \square P_2) + 3n_{\mathcal{D}}(P_{n-2} \square P_2) + 2n_{\mathcal{D}}(P_{n-3} \square P_2)$. We now construct three pairwise disjoint families \mathcal{D}_i , $i = 1, 2, 3$, of digitally convex sets in $\mathcal{D}(P_n \square P_2)$ such that $|\mathcal{D}_i| = c_i n_{\mathcal{D}}(P_{n-i} \square P_2)$, where $c_1 = 1, c_2 = 3, c_3 = 2$.

To construct \mathcal{D}_1 , let $S \in \mathcal{D}(P_{n-1} \square P_2)$. If $v_{n-1}, u_{n-1} \notin S$, then S is digitally convex in $P_n \square P_2$, because the vertices v_n and u_n are each a private neighbour for themselves with respect to S . Then, we add S to \mathcal{D}_1 . If $v_{n-1} \in S$ or $u_{n-1} \in S$, then $S \cup \{v_n, u_n\}$ is digitally convex in $P_n \square P_2$, because each vertex in $V(P_{n-1} \square P_2) - S$ must have a private neighbour with respect to S in $V(P_{n-1} \square P_2) - \{v_{n-1}, u_{n-1}\}$, which is also a private neighbour with respect to $S \cup \{v_n, u_n\}$ in $P_n \square P_2$. Then, we add $S \cup \{v_n, u_n\}$ to \mathcal{D}_1 . Note that $|\mathcal{D}_1| = n_{\mathcal{D}}(P_{n-1} \square P_2)$, as desired.

To construct \mathcal{D}_2 , let $S \in \mathcal{D}(P_{n-2} \square P_2)$. If $v_{n-2}, u_{n-2} \in S$, then S is digitally convex in $P_n \square P_2$, because the vertices u_n and v_n are private neighbours for themselves, as well as for v_{n-1} and u_{n-1} , with respect to S in $P_n \square P_2$. Then, we add S to \mathcal{D}_2 . This set is not digitally convex in $P_{n-1} \square P_2$, as the vertices v_{n-1} and u_{n-1} have no private neighbours with respect to S . The set $S \cup \{v_{n-1}\}$ is also digitally convex in $P_n \square P_2$, because the vertex u_n is a private neighbour for itself, v_n and u_{n-1} with respect to $S \cup v_{n-1}$ in $P_n \square P_2$. Then, we add $S \cup \{v_{n-1}\}$ to \mathcal{D}_2 . Similarly, $S \cup \{u_{n-1}\}$ is digitally convex in $P_n \square P_2$, so we add it to \mathcal{D}_2 .

If $v_{n-2} \in S$ and $u_{n-2} \notin S$, then $S \cup \{v_n\}$ is digitally convex in $P_n \square P_2$, because the vertex u_{n-1} is a private neighbour for itself, v_{n-1} and u_n with respect to $S \cup \{v_n\}$ in $P_n \square P_2$. Then, we add $S \cup \{v_n\}$ to \mathcal{D}_2 . The set $S \cup \{v_{n-1}\}$ is also digitally convex in $P_n \square P_2$, because the vertex u_n is a private neighbour for itself,

v_n and u_{n-1} with respect to $S \cup \{v_{n-1}\}$. Then, we add $S \cup \{v_{n-1}\}$ to \mathcal{D}_2 . Similarly, $S \cup \{u_{n-1}\}$ is digitally convex in $P_n \square P_2$, so we add it to \mathcal{D}_2 .

If $v_{n-2} \notin S$ and $u_{n-2} \in S$ then, by the same argument as above, $S \cup \{v_{n-1}\}$, $S \cup \{u_{n-1}\}$ and $S \cup \{u_n\}$ are all digitally convex in $P_n \square P_2$. We add them all to \mathcal{D}_2 .

If $v_{n-2}, u_{n-2} \notin S$, then $S \cup \{v_n\}$ is digitally convex in $P_n \square P_2$ because the vertex u_{n-1} is a private neighbour for itself, u_n and v_{n-1} with respect to $S \cup \{v_n\}$ in $P_n \square P_2$. Then, we add $S \cup \{v_n\}$ to \mathcal{D}_2 . Similarly, $S \cup \{u_n\}$ is digitally convex in $P_n \square P_2$, so we add it to \mathcal{D}_2 . If, in addition, $v_{n-3}, u_{n-3} \notin S$, then $S \cup \{v_n, u_n\}$ is digitally convex in $P_n \square P_2$, because both $u_{n-2}, v_{n-2} \notin N[S \cup \{v_n, u_n\}]$ in $P_n \square P_2$. So u_{n-2} and v_{n-2} are private neighbours for u_{n-1} and v_{n-1} with respect to $S \cup \{v_n, u_n\}$. Then, we add $S \cup \{v_n, u_n\}$ to \mathcal{D}_2 . If $v_{n-3} \in S$ and $u_{n-3} \notin S$, then $S \cup \{v_{n-1}\}$ is digitally convex in $P_n \square P_2$, because the vertex u_{n-2} is a private neighbour for itself and for v_{n-2} , and the vertex u_n is a private neighbour for itself, v_n and u_{n-1} with respect to $S \cup \{v_{n-1}\}$ in $P_n \square P_2$. Then, we add $S \cup \{v_{n-1}\}$ to \mathcal{D}_2 . Similarly, if $v_{n-3} \notin S$ and $u_{n-3} \in S$, then $S \cup \{u_{n-1}\}$ is digitally convex in $P_n \square P_2$. So we add it to \mathcal{D}_2 . Now, we have $|\mathcal{D}_2| = 3n_{\mathcal{D}}(P_{n-2} \square P_2)$, as desired.

Finally, to construct \mathcal{D}_3 , let $S \in \mathcal{D}(P_{n-3} \square P_2)$. If $v_{n-3}, u_{n-3} \notin S$, then $S \cup \{v_{n-1}\}$ is digitally convex in $P_n \square P_2$, because the vertex u_{n-2} is a private neighbour for itself, v_{n-2} and u_{n-1} , and the vertex u_n is a private neighbour for itself and v_n with respect to $S \cup \{v_{n-1}\}$. Then, we add $S \cup \{v_{n-1}\}$ to \mathcal{D}_3 . Similarly, $S \cup \{u_{n-1}\}$ is digitally convex in $P_n \square P_2$, so we add it to \mathcal{D}_3 .

If $v_{n-3} \in S$ or $u_{n-3} \in S$, then $S \cup \{v_{n-2}, v_n\}$ is digitally convex in $P_n \square P_2$, because the vertex u_{n-1} is a private neighbour for itself, u_{n-2} , v_{n-1} and u_n with respect to $S \cup \{v_{n-2}, v_n\}$ in $P_n \square P_2$. Then, we add $S \cup \{v_{n-2}, v_n\}$ to \mathcal{D}_3 . Similarly, $S \cup \{u_{n-2}, u_n\}$ is digitally convex in $P_n \square P_2$, so we add it to \mathcal{D}_3 . Now, we have $|\mathcal{D}_3| = 2n_{\mathcal{D}}(P_{n-3} \square P_2)$, as desired.

Now, we have $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$ for $i \neq j$, and each \mathcal{D}_i , $i = 1, 2, 3$, is a subset of $\mathcal{D}(P_n \square P_2)$. Thus $n_{\mathcal{D}}(P_n \square P_2) \geq |\mathcal{D}_1| + |\mathcal{D}_2| + |\mathcal{D}_3| = n_{\mathcal{D}}(P_{n-1} \square P_2) + 3n_{\mathcal{D}}(P_{n-2} \square P_2) + 2n_{\mathcal{D}}(P_{n-3} \square P_2)$.

Now, to show the reverse inequality, let $S \in \mathcal{D}(P_n \square P_2)$.

(a) Suppose $v_n, u_n \in S$. If $v_{n-1} \in S$ or $u_{n-1} \in S$, then each $x \notin S$ has a private neighbour with respect to S in $V(P_{n-1} \square P_2)$. Thus, $S - \{v_n, u_n\}$ is digitally convex in $P_{n-1} \square P_2$. If $v_{n-1}, u_{n-1} \notin S$, then $v_{n-2}, u_{n-2} \notin N[S]$. Thus, $v_{n-3}, u_{n-3} \notin S$ and $S - \{v_n, u_n\}$ is digitally convex in $P_{n-2} \square P_2$.

(b) Suppose $v_n \in S$ and $u_n \notin S$. Then, $u_{n-1} \notin N[S]$, so $v_{n-1}, u_{n-1}, u_{n-2} \notin S$. If $v_{n-2} \in S$ and $v_{n-3}, u_{n-3} \notin S$, then it must be the case that $u_{n-3} \notin N[S]$. So $S - \{v_n\}$ is digitally convex in $P_{n-2} \square P_2$. If $v_{n-2} \in S$ and $v_{n-3} \in S$ or $u_{n-3} \in S$, then $S - \{v_n, v_{n-2}\}$ is digitally convex in $P_{n-3} \square P_2$. If $v_{n-2} \notin S$, then at most one of v_{n-3} and u_{n-3} can be in S . So either $v_{n-2} \notin N[S]$ or $u_{n-2} \notin N[S]$. Then, $S - \{v_n\}$ is digitally convex in $P_{n-2} \square P_2$.

(c) Suppose $v_n \notin S$ and $u_n \in S$, then $v_{n-1} \notin N[S]$, so $v_{n-1}, u_{n-1}, v_{n-2} \notin S$. If $u_{n-2} \in S$ and $v_{n-3}, u_{n-3} \notin S$, then $S - \{u_n\}$ is digitally convex in $P_{n-2} \square P_2$. If $u_{n-2} \in S$ and $v_{n-3} \in S$ or $u_{n-3} \in S$, then $S - \{u_n, u_{n-2}\}$ is digitally convex in $P_{n-3} \square P_2$. If $u_{n-2} \notin S$, then $S - \{u_n\}$ is digitally convex in $P_{n-2} \square P_2$.

(d) Suppose $v_n, u_n \notin S$. Then, at most one of v_{n-1} and u_{n-1} can be in S . If $v_{n-1} \in S$ and at least one of v_{n-2} and u_{n-2} is in S , then $S - \{v_{n-1}\}$ is digitally convex in $P_{n-2} \square P_2$. If $v_{n-1} \in S$ and $v_{n-2}, u_{n-2} \notin S$, then it must be the case that $u_{n-2} \notin N[S]$. So $u_{n-3} \notin S$. If $v_{n-3} \in S$, then $S - \{v_{n-1}\}$ is digitally convex in $P_{n-2} \square P_2$. If $v_{n-3} \notin S$, then $S - \{v_{n-1}\}$ is digitally convex in $P_{n-3} \square P_2$. Similarly, if $u_{n-1} \in S$ and at least one of v_{n-2} and u_{n-2} is in S , then $S - \{u_{n-1}\}$ is digitally convex in $P_{n-2} \square P_2$. If $u_{n-1} \in S$, $v_{n-2}, u_{n-2} \notin S$ and $u_{n-3} \in S$, then $S - \{u_{n-1}\}$ is digitally convex in $P_{n-2} \square P_2$. If $u_{n-1} \in S$, $v_{n-2}, u_{n-2} \notin S$ and $u_{n-3} \notin S$, then $S - \{u_{n-1}\}$ is digitally convex in $P_{n-3} \square P_2$.

Each digitally convex set in $P_{n-1} \square P_2$ has been counted here at most once, each digitally convex set in $P_{n-2} \square P_2$ at most three times, and each digitally convex set in $P_{n-3} \square P_2$ at most twice. Refer to Table 1 for a summary of which digitally convex sets in $P_{n-2} \square P_2$ and $P_{n-3} \square P_2$ are counted in each part of the above argument. Therefore, $n_{\mathcal{D}}(P_n \square P_2) \leq n_{\mathcal{D}}(P_{n-1} \square P_2) + 3n_{\mathcal{D}}(P_{n-2} \square P_2) + 2n_{\mathcal{D}}(P_{n-3} \square P_2)$. ■

Note that, in addition to proving the given recurrence, this proof of Theorem 5 provides a method of generating the collection of digitally convex sets of $P_n \square P_2$ from those of $P_{n-1} \square P_2$, $P_{n-2} \square P_2$ and $P_{n-3} \square P_2$.

Given a set \mathcal{A} of $n \times m$ binary arrays, we let \mathcal{A}^* be the set of arrays obtained as follows. For each array A in \mathcal{A} , construct a new array A^* by taking the minimum value of corresponding elements of A and their horizontal and vertical neighbours. In other words, each element of A^* is the minimum value over the closed neighbourhood of the corresponding element of A .

As an example, consider the following array A .

$$\begin{array}{ccc} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}$$

Taking the minimum value over the closed neighbourhood of each element in the array produces the array A^* .

$$\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}$$

Note that in this process, two distinct arrays A_1 and A_2 can produce the same array A^* . We now establish a bijection between these $n \times m$ binary arrays and the digitally convex sets in $\mathcal{D}(P_n \square P_m)$.

$P_{n-2} \square P_2$	
	(a), (b), and (c)
	(b), and twice in (d)
	(c), and twice in (d)
	(b), (c), and (d)
$P_{n-3} \square P_2$	
	(b) and (c)
	twice in (d)

 Table 1. Summary of the counting argument in Theorem 5, with white vertices in S .

Theorem 6. Let $\mathcal{A}_{n,m}$ be the set of all $n \times m$ binary arrays. Then, $n_{\mathcal{D}}(P_n \square P_m) = |\mathcal{A}_{n,m}^*|$.

Proof. First, we label the vertices of the product $P_n \square P_m$. Let the vertices of P_n be u_1, u_2, \dots, u_n , with $u_i u_{i+1} \in E(P_n)$ for $i = 1, 2, \dots, n-1$, and let the vertices of P_m be v_1, v_2, \dots, v_m , with $v_j v_{j+1} \in E(P_m)$ for $j = 1, 2, \dots, m-1$. Then, the vertices of $P_n \square P_m$ have the form (u_i, v_j) .

Now we show a bijection between the digitally convex sets in $\mathcal{D}(P_n \square P_m)$ and the arrays in $\mathcal{A}_{n,m}^*$. Let $A^* \in \mathcal{A}_{n,m}^*$ and consider the set $S = \{(u_i, v_j) \mid a_{i,j}^* = 1\}$. Each vertex $(u_x, v_y) \notin S$ corresponds to an entry $a_{x,y}^*$ that has value 0 in A^* .

Then, either the corresponding entry in A also has value 0, or it has value 1 and has a horizontal or vertical neighbour with value 0. In the first case, every entry in the closed neighbourhood of $a_{x,y}^*$ also has value 0 in A^* . In $P_n \square P_m$, this means that none of the vertices in $N[(u_x, v_y)]$ is in S , so (u_x, v_y) is its own private neighbour. In the second case, there is an entry $a_{w,z}$ in the closed neighbourhood of $a_{x,y}$ which has value 0 in A . Then, in A^* , every entry in the closed neighbourhood of $a_{w,z}^*$ has value 0, including $a_{x,y}^*$. In $P_n \square P_m$, this means that none of the vertices in $N[(u_w, v_z)]$ is in S and $(u_w, v_z)(u_x, v_y) \in E(P_n \square P_m)$, so the vertex (u_w, v_z) is a private neighbour for (u_x, v_y) with respect to S . Therefore, S is digitally convex in $P_n \square P_m$.

It is clear from the construction of S that this mapping from the binary arrays $\mathcal{A}_{n,m}^*$ to the sets in $\mathcal{D}(P_n \square P_m)$ is injective. It remains to be shown that the mapping is surjective. Consider $S \in \mathcal{D}(P_n \square P_m)$ and let B be the $n \times m$ array with $b_{i,j} = 1$ if $(u_i, v_j) \in S$ and $b_{i,j} = 0$ otherwise. Then, let C be the $n \times m$ array whose entries are the maximum over the closed neighbourhood of the corresponding entry in B . In other words, $c_{i,j} = 1$ if any of the entries in the closed neighbourhood of $b_{i,j}$ has value 1, and $c_{i,j} = 0$ otherwise. Clearly, $C \in \mathcal{A}$ and now we show that $C^* = B$. By construction of C , each entry of C^* whose corresponding entry in B has value 1 also has value 1 in C^* . So if $C^* \neq B$, then there is some i, j with $c_{i,j}^* = 1$ and $b_{i,j} = 0$. This means that, in C , each entry in the closed neighbourhood of $c_{i,j}$ has value 1. However, the entries in C are defined to be 1 because their corresponding entry in B has a 1 in its closed neighbourhood. In other words, the entries in the closed neighbourhood of $b_{i,j}$ each either have value 1 or have a horizontal or vertical neighbour with value 1. In terms of the set S , this corresponds to a vertex (u_i, v_j) with every vertex in $N[(u_i, v_j)]$ in $N[S]$, i.e., (u_i, v_j) has no private neighbour with respect to S in $P_n \square P_m$. This contradicts S being digitally convex and thus $C^* = B$.

It is clear that B gets mapped to the digitally convex set S , using the mapping described above. Therefore, $n_{\mathcal{D}}(P_n \square P_m) = |\mathcal{A}_{n,m}^*|$. ■

The number of digitally convex sets of $P_n \square P_m$ follows the OEIS sequence A217637 [16]. The OEIS notes an observation from Andrew Howroyd that this sequence also enumerates the maximal independent sets in the graph $P_n \square P_m \square P_2$. Euler, Oleksik and Skupień [8] prove this equivalence for $m = 2$ and for $m = 3$. However, the correspondence between the digitally convex sets in $P_n \square P_m$ and the maximal independent sets in $P_n \square P_m \square P_2$ is not clear, even for very small values of n and m .

4. CONCLUSION

In this paper, we established a linear recurrence that is satisfied by the number of cyclic binary n -bit strings whose blocks each have length at least k , for some

$k \geq 2$. We then showed that these cyclic binary strings can be used to enumerate the digitally convex sets of powers of cycles. It is possible that cyclic binary strings with other properties can be used to enumerate the digitally convex sets of circulant graphs or other graphs that can be constructed by adding additional edges to a cycle, using the method of manipulating generating function as in the proof of Lemma 1.

We provided a closed formula for the number of digitally convex sets in the graph $K_n \square K_m$. We established a recurrence relation for $n_{\mathcal{D}}(P_n \square P_2)$ and defined a class of $n \times m$ binary arrays for which there is a one-to-one correspondence with the digitally convex sets of $P_n \square P_m$. The problem of enumerating the digitally convex sets for other graph products, such as the strong product, the direct product and the categorical product remains open.

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