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RELAXED DP-COLORING AND ANOTHER GENERALIZATION OF DP-COLORING ON PLANAR GRAPHS WITHOUT 4-CYCLES AND 7-CYCLES

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Abstract

DP-coloring is generalized via relaxed coloring and variable degeneracy in [P. Sittitrai and K. Nakprasit, *Sufficient conditions on planar graphs to have a relaxed DP-3-coloring*, Graphs Combin. 35 (2019) 837–845], [K.M. Nakprasit and K. Nakprasit, *A generalization of some results on list coloring and DP-coloring*, Graphs Combin. 36 (2020) 1189–1201] and [P. Sittitrai and K. Nakprasit, *An analogue of DP-coloring for variable degeneracy and its applications*, Discuss. Math. Graph Theory]. In this work, we introduce another concept that includes two previous generalizations. We demonstrate its application on planar graphs without 4-cycles and 7-cycles. One implication is that the vertex set of every planar graph without 4-cycles and 7-cycles can be partitioned into three sets in which each of them induces a linear forest and one of them is an independent set. Additionally, we show that every planar graph without 4-cycles and 7-cycles is DP-$(1, 1, 1)$-colorable. This generalizes a result of Lih *et al.* [*A note on list improper coloring planar graphs*, Appl. Math. Lett. 14 (2001) 269–273] that every planar graph without 4-cycles and 7-cycles is $(3, 1)^*$-choosable.

Keywords: relaxed DP-colorings, variable degeneracy, planar graphs, discharging.

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1. Introduction

All considered graphs are finite, simple, undirected, and embedded in the plane. For a graph G, let its vertex set, edge set, face set, and minimum degree be denoted by $V(G), E(G), F(G)$, and $\delta(G)$, respectively. Let $d(x)$ denote the degree of x where $x \in V(G) \cup F(G)$. A *k-vertex* (or *k^+-vertex*) is a vertex of degree k (or at least k). Similar notation is applied to a cycle and a face. A face f is *simple* if its boundary forms a cycle. A face f and a vertex v are *incident* if v is on the boundary of f. We simply say two faces share an edge (or a vertex) instead of the boundary of two faces share an edge (or a vertex). Two faces are *adjacent* if they share at least one edge. If G is a graph and $U \subseteq V(G)$, then $G[U]$ denote the subgraph of G induced by U. A *linear forest* is a forest in which each component is a path.

Vizing [11] in 1976, and independently Erdős, Rubin, and Taylor [5] in 1979, introduced list coloring and choosability. An *assignment* L of a graph G assigns a list $L(v)$ (a set of colors) to each vertex v. A *k-assignment* L is an assignment such that $|L(v)| = k$ for each vertex v. If a graph G admits a proper coloring f where $f(v) \in L(v)$ for each vertex v, then we say G is *L-colorable.* A graph G is *k-choosable* if it is L-colorable for each k-assignment L.

In 1999, Škrekovski [10] and Eaton and Hull [4] independently introduced the concept of relaxed list coloring. A graph G with an assignment L is $(L, d)^*$ choosable if each vertex v of G can be colored with a color $f(v) \in L(v)$ such that at most d neighbors of v receive the color $f(v)$. A graph G is *$(k, d)^*$-choosable* if G is $(L, d)^*$-choosable for each k-assignment L.

Dvořák and Postle [3] introduced a generalization of list coloring which they called *correspondence coloring.* Following Bernshteyn, Kostochka, and Pron [1], we call it a *DP-coloring.* Let L be an assignment of a graph G. We call (H, L) (or simply H) a *cover* of G if it satisfies the following conditions.

- (i) The vertex set of H is $\bigcup_{u \in V(G)}(\{u\} \times L(u)) = \{(u, c) : u \in V(G), c \in L(u)\}$.
- (ii) For each $uv \in E(G)$, the set $E_H(\{u\} \times L(u), \{v\} \times L(v))$ is a matching (the matching may be empty).
- (iii) If $uv \notin E(G)$, then no edges of H connect $\{u\} \times L(u)$ and $\{v\} \times L(v)$.

A *transversal* of (H, L) is a vertex set $T \subseteq V(H)$ such that $|T \cap (\{u\} \times L(u))| = 1$ for each vertex u in G. A *DP-coloring* of (H, L) is a transversal T of (H, L) such that T is independent. The *DP-chromatic number of* G is the least number k such that every cover (H, L) of G with k-assignment L has a DP-coloring.

Since names of colors for distinct vertices in DP-coloring are irrelevant, we always assume in this paper that a k-assignment of a graph G has $L(v) = \{1, \ldots, k\}$ for each $v \in V(G)$. In [9], Sittitrai and Nakprasit combined *DP*-coloring and relaxed list coloring as follows. Let (H, L) be a cover of a graph G with a k-assignment L. A transversal T of (H, L) is a *$(t_1, \ldots, t_k)$-coloring* if every $(v, i) \in T$ has degree at most t_i in $H[T]$. If G with a k-assignment L has a $(t_1, \ldots, t_k)$-coloring for every cover (H, L), then we say G is *DP-$(t_1, \ldots, t_k)$-colorable.* One can show that the fact that G is DP-$(t_1, \ldots, t_k)$-colorable where $t_i = d$ $(i \in \{1, \ldots, k\})$ implies G is $(k, d)^*$-choosable.

In this work, we obtain the following result.

Theorem 1. *Every planar graph without* 4*-cycles or* 7*-cycles is DP-*$(1, 1, 1)$*-colorable.*

Theorem 1 generalizes the following result by Lih *et al.* [6].

Theorem 2. *Every planar graph without* 4*-cycles or* 7*-cycles is* $(3, 1)^*$*-choosable.*

Remark that the proof of $(3, 1)^*$-choosability by Lih *et al.* cannot be applied to Theorem 1. For example, Lih *et al.* use the fact that a 3-cycle *abca* is $(L, 1)^*$-colorable if $|L(a)| \geq 2$ and $|L(b)|, |L(c)| \geq 1$. But this fact is not true for DP-coloring. Let $L(a) = \{1, 2\}$, $L(b) = \{1\}$, $L(c) = \{2\}$, and let $(a, 1)(b, 1)$, $(a, 2)(c, 2)$, and $(b, 1)(c, 2)$ be edges of a cover H. One can see that (H, L) has no DP-$(1, 1, 1)$-colorings.

Additionally, we show that every planar graph is DP-$(0, 2, 2)$-colorable. In fact, we present this second main result in a stronger form by using a concept similar to "variable degeneracy" but broader. One immediate consequence of the second main result is that the vertex set of a planar graph without 4-cycles or 7-cycles can be partitioned into three sets such that one set is independent and each of the two remaining sets induces a linear forest.

Some definitions are required to understand the second main result. The concept of variable degeneracy was introduced by Borodin, Kostochka, and Toft [2] as follows. Let f be a function from $V(G)$ to the set of positive integers. A graph G is *strictly f-degenerate* if every subgraph G' has a vertex v with $d_{G'}(v) < f(v)$. Let f_i, where $i \in \{1, \ldots, s\}$, be a function from $V(G)$ to the set of nonnegative integers. An *$(f_1, \ldots, f_s)$-partition* of a graph G is a partition of $V(G)$ into $V_1, \ldots, V_s$ such that the induced subgraph $G[V_i]$ is strictly f_i-degenerate for each $i \in \{1, \ldots, s\}$. Equivalently, the vertices of V_i can be ordered from left to right such that each vertex in V_i has less than $f_i(v)$ neighbors in V_i on the left.

DP-coloring with variable degeneracy was introduced by Nakprasit and Nakprasit [7] and Sittitrai and Nakprasit [8] as follows. Let $F = (f_1, \ldots, f_s)$ and $f_i \in \mathbb{Z}^+ \cup \{0\}$, where $1 \leq i \leq s$. A *DP-F-coloring* T of a cover (H, L) of G is a transversal T of (H, L) in which its vertices can be ordered from left to right so that each element (v, i) in T has less than $f_i(v)$ neighbors on the left. We say that G is *DP-F-colorable* if (G, H) has a DP-F-coloring for every cover H.

We observe that the restriction in the previous definition is about the number of neighbors on the left of each element in a transversal. We may employ other restrictions as needed to different applications. This observation inspires us to define the following concept. Let B be a condition imposed on ordered vertices. A *DP-B-coloring* of (G, H) is a transversal T with ordered vertices from left to right such that each $(v, c) \in T$ satisfies condition B imposed on each element of H. In this work, we demonstrate the use of this definition by the condition B_A defined as follows. Let T be a transversal of a cover (H, L) of G. We say that T is a *DP-B_A-coloring* if vertices in T can be ordered from left to right such that:

(1) For each $(v, 1) \in T, (v, 1)$ has no neighbor on the left.

(2) For each $(v, c) \in T$ where $c \neq 1$, (v, c) has at most one neighbor on the left and that neighbor (if it exists) is adjacent to at most one vertex on the left of (v, c).

We say that G is *DP-B_A-k-colorable* if every cover (H, L) of a graph G with k-assignment L has a DP-B_A-coloring.

Theorem 3. *Every planar graph without* 4*-cycles or* 7*-cycles is DP-B_A-*3*-colorable.*

Corollary 4. *If G is a planar graph without* 4*-cycles or* 7*-cycles, then*

(i) *G is DP-$(0, 2, 2)$-colorable.*

(ii) *$V(G)$ can be partitioned into three sets in which each of them induces a linear forest and one of them is an independent set.*

Proof. Suppose Theorem 3 holds. Then the first part of the corollary follows immediately from definitions. To obtain the second part, we define edges on H to match exactly the same colors in $L(u)$ and $L(v)$ for each $uv \in E(G)$. One can see that the set of vertices with color 1 is independent and the set of vertices with color i induces a linear forest when $i = 2$ or 3. ■

2. Forbidden Configurations Due to Cycles

Lemma 5. *Let G be a graph without* 4*-cycles and* 7*-cycles. Then the following statements hold.*

(1) *There are no adjacent* 3*-faces.*

(2) *If a* 3*-face is adjacent to a* 5*-face, then they share exactly one edge and two vertices.*

(3) *A* 5*-face is not adjacent to two* 3*-faces.*

(4) *If $\delta(G) \geq 3$, then each* 6*-face is not adjacent to a* 3*-face.*

(5) *If $\delta(G) \geq 3$, then a* 3*-vertex is not incident to a* 3*-face and two* 5*-faces simultaneously.*

Proof. (1) If two 3-faces are adjacent, then G has a 4-cycle, a contradiction.

(2) If a 3-face and a 5-face share three vertices (so they share one or two edges), then G has a 4-cycle, a contradiction.

(3) Suppose to the contrary that a 5-faces C is adjacent to two 3-faces. If those two 3-faces share vertex outside $V(C)$, then G has a 4-cycle, for otherwise G has a 7-cycle, a contradiction. Thus a 5-face is not adjacent to two 3-faces.

(4) Suppose to the contrary that a 6-face f_1 is adjacent to a 3-face f_2. First we suppose f_1 is not a simple face. Then its boundary walk forms two 3-cycles with a common vertex. Thus f_1 adjacent to f_2 yields a 4-cycle, a contradiction. Now we suppose f_1 is a simple face. Since $\delta(G) \geq 3$, f_1 and f_2 share exactly one edge. If f_1 and f_2 share exactly two vertices, then G has a 4-cycle or a 7-cycle, a contradiction. Altogether, f_1 is not adjacent to f_2.

(5) Suppose that $\delta(G) \geq 3$. Observe that if a 5-face is adjacent to a 3-face or another 5-face, then they share exactly one edge and two vertices to avoid a 4-cycle or a 7-cycle. It follows that a 3-vertex incident to a 3-face and two 5-faces yields a 7-cycle. ■

3. Proof of Theorem 1

3.1. Structure of a minimal counterexample

Lemma 6. *Suppose G is a non-DP-$(t_1, \ldots, t_k)$-colorable graph but all of its proper induced subgraphs are DP-$(t_1, \ldots, t_k)$-colorable. Then the following statements hold.*

(1) $\delta(G) \geq k$.

(2) *If $t_i = d \geq 1$ for each $i \in \{1, \ldots, k\}$, then every neighbor of a k-vertex has degree at least $k + d$.*

Proof. (1) Suppose to the contrary that G has a vertex v of degree at most $k-1$. Let L be a k-assignment of G and let (H, L) be a cover of G that does not have a DP-$(t_1, \ldots, t_k)$-coloring. By our assumption, $G' = G - v$ has a DP-$(t_1, \ldots, t_k)$-coloring T'. Since $d(v) \leq k - 1$, there exists $(v, i) \in V(H)$ that does not have a neighbor in T'. So, we add (v, i) to T' to obtain a desired coloring, a contradiction.

(2) Suppose to the contrary that u and v are adjacent vertices where $d(u) = k$ and $d(v) \leq k + d - 1$. Let L be a k-assignment of G and let (H, L) be a cover of G that does not have a DP-$(t_1, \ldots, t_k)$-coloring. By assumption, $G' = G - \{u, v\}$ has a DP-$(t_1, \ldots, t_k)$-coloring T'. Then there is $(u, b) \in V(H)$ that does not have a neighbor in T'. Suppose (v, c) is adjacent to (u, b) in H. If (v, c) has at most $d-1$ neighbors in T', then we add (u, b) and (v, c) in T' to obtain a desired coloring, a contradiction. Suppose (v, c) has at least d neighbors in T'. Then there exists $(v, i) \in V(H)$ that does not have a neighbor in T'. So, we add (u, b) and (v, i) to T' to obtain a desired coloring, a contradiction. This completes the proof. ■

The next result immediately follows.

Corollary 7. *Suppose G is a non-DP-$(1,1,1)$-colorable graph but all of its proper induced subgraphs are DP-$(1,1,1)$-colorable. Then the following statements hold.*

(1) $\delta(G) \geq 3$.

(2) *There are no adjacent 3-vertices.*

Lemma 8. *Suppose G is a counterexample to Theorem 1 but all of its proper induced subgraphs are DP-$(1,1,1)$-colorable. If f is a face of G, then the number of its incident 3-vertices plus the number of its adjacent 3-faces is at most $d(f)$.*

Proof. Let f be a face with a boundary walk $v_1, v_2, \ldots, v_k$. Let f_i be a face sharing an edge $v_i v_{i+1}$ with f where subscripts are taken modulo k. We claim that if $d(f_i) = d(v_i) = 3$, then $d(f_{i-1}) \geq 4$ and $d(v_{i-1}) \geq 4$. Suppose that $d(f_i) = d(v_i) = 3$. It follows from Corollary 7(2) that $d(v_{i-1}) \geq 4$. If $d(f_{i-1}) = 3$, then there are adjacent 3-cycles, a contradiction. So, the claim holds. It follows from the claim that the average number of v_i and f_i with degree 3 for each i is at most 1. This implies the lemma. ∎

3.2. Discharging procedure

Suppose G is a counterexample to Theorem 1 but all of its proper induced subgraphs are DP-$(1,1,1)$-colorable. Let $\mu(x) = d(x) - 4$ be the initial charge of a vertex or a face x and let $\mu^*(x)$ denote the final charge of x after the discharging process. By the Euler's formula, $\sum_{v \in V(G)} \mu(v) + \sum_{f \in F(G)} \mu(f) = \sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) = -8$. We define discharging rules as follows.

Discharging Rules.

(R1) Each 5^+-face gives $\frac{1}{3}$ to each adjacent 3-face.

(R2) Each 5-face gives $\frac{1}{3}$ to each incident 3-vertex.

(R3) Each 6^+-face gives $\frac{2}{3}$ to each incident 3-vertex.

We aim to show that the final charge $\mu^*(x)$ for each $x \in V(G) \cup F(G)$ is nonnegative. Since the total of charge is not changed by the rules, we obtain a contradiction and prove the main result.

Proof. By Corollary 7(1), every vertex v is a 3^+-vertex. If v is a 4^+-vertex, then it does not involve in a discharging process and thus $\mu^*(v) = \mu(v) \geq 0$.

Consider a 3-vertex v. If v is not incident to a 3-face, then $\mu^*(v) \geq \mu(v) + 3 \times \frac{1}{3} = 0$ by (R2) and (R3). If v is incident to a 3-face, then it is incident to two 5^+-faces and one of which is a 6^+-face by Lemmas 5(1) and 5(5). Thus $\mu^*(v) \geq \mu(v) + \frac{1}{3} + \frac{2}{3} = 0$ by (R2) and (R3).

Consider a 3-face f. It follows from Lemma 5 that every face adjacent to f is a 5^+-face. Thus $\mu^*(f) = \mu(f) + 3 \times \frac{1}{3} = 0$ by (R1).

If f is a 4-face, then its charge is not affected by the discharging procedure and thus $\mu^*(f) = \mu(f) = 0$.

Consider a 5-face f. Then f is incident to at most two 3-vertices by Corollary 7(2) and is adjacent to at most one 3-face by Lemma 5(3). Thus $\mu^*(f) \geq \mu(f) - 3 \times \frac{1}{3} = 0$ by (R1) and (R2).

Consider a 6-face f. Then f is incident to at most three 3-vertices by Corollary 7(2) and is not adjacent to a 3-face by Lemma 5(4). Thus $\mu^*(f) \geq \mu(f) - 3 \times \frac{2}{3} = 0$ by (R3).

If a 7-face is a simple face, then G has a 7-cycle, for otherwise G has a 4-cycle. Thus G does not contain a 7-face.

Consider a k-face f where $k \geq 8$. Suppose that f has r incident 3-vertices and s adjacent 3-faces. We have that $\mu^*(f) = \mu(f) - r \times \frac{1}{3} - s \times \frac{2}{3}$ by (R1) and (R3). Since $r + s \leq k$ by Lemma 8 and $r \leq k/2$ by Corollary 7(2), we have $r \times \frac{2}{3} + s \times \frac{1}{3} = (r+s) \times \frac{1}{3} + r \times \frac{1}{3} \leq k \times \frac{1}{3} + \frac{k}{2} \times \frac{1}{3} = \frac{k}{2}$. Thus $\mu^*(f) \geq \mu(f) - \frac{k}{2} = \frac{k}{2} - 4 \geq 0$. ■

4. PROOF OF THEOREM 3

4.1. Structure of a minimal counterexample

First, we introduce a concept used in the next two lemmas. Let G be a graph with a vertex v and a cover H. Let T' be a DP-B_A-coloring of $G - v$ with an appropriate order R. *Adding* (v, i) *to the right of* T' is the process to have the transversal $T' \cup \{(v, i)\}$ of G with an order such that vertices in T' are ordered first according to the order R and then we put (v, i) at the farthest right. If (v, i) according to such order satisfies the condition of DP-B_A-coloring, then $T' \cup \{(v, i)\}$ is a DP-B_A-coloring of G since all remaining vertices in T satisfy the condition by the order R already.

Lemma 9. *If G is a non-DP-B_A-3-colorable graph but all of its proper induced subgraphs are DP-B_A-3-colorable, then $\delta(G) \geq 3$.*

Proof. Suppose to the contrary that G has a vertex v with degree at most 2. Let L be a 3-assignment of G and let (H, L) be a cover of G that does not have a DP-B_A-coloring. By minimality, $G' = G - v$ has a DP-B_A-coloring T'. Since $d(v) \leq 2$, there exists $(v, i) \in V(H)$ that does not have a neighbor in T'. We add (v, i) to the right of T'. Since (v, i) does not have a neighbor in T', we obtain a desired coloring. This contradiction completes the proof. ■

Lemma 10. *Suppose G is a non-DP-B_A-3-colorable graph but all of its proper induced subgraphs are DP-B_A-3-colorable. If a 3-vertex u in G is adjacent to a*

3-*vertex, then* u *has two* 5^+-*neighbors. Moreover, if* x *is a* 5-*neighbor of* u, *then* x *has a* 4^+-*neighbor.*

Proof. Let a 3-vertex u be adjacent to x, y and a 3-vertex v. By minimality, $G - \{u, v\}$ has a DP-B_A-coloring T. Choose $(u, c_u) \in V(H)$ such that (u, c_u) is not adjacent to vertices in T and choose (v, c_v) similarly. If $c_u \neq 1$, or $c_u = c_v = 1$ and $(u, 1)$ is not adjacent to $(v, 1)$, then we add (v, c_v) and then (u, c_u) to the right of T. Since (v, c_v) is not adjacent to any vertices in T and it is the only vertex that may adjacent to (u, c_u), it follows that $T \cup \{(u, c_u), (v, c_v)\}$ is a desired coloring.

By symmetry, it remains to consider the case that $c_u = c_v = 1$ and $(u, 1)$ and $(v, 1)$ are adjacent, and we call this case *unfavorable situation*. Note that $(u, 2)$ has exactly one neighbor, say (x, x_2) in T, otherwise we can choose 2 or 3 to be c_u and we can avoid unfavorable situation. If (x, x_2) has at most one neighbor in T, then we add $(u, 2)$ and subsequently $(v, 1)$ to the right of T. By assumption, $(u, 2)$ satisfies the condition of a DP-B_A-coloring. Moreover, $(v, 1)$ has no neighbors in $T \cup \{(u, 2), (v, 1)\}$, and thus we have a desired coloring. This contradiction yields that (x, x_2) has at least two neighbors in T.

We aim to show that x is a 5^+-vertex. If we can add (x, x_1) or (x, x_3) where $\{x_1, x_2, x_3\} = \{1, 2, 3\}$ to the right of $T - \{(x, x_2)\}$ to get a DP-B_A-coloring T' of $G - \{u, v\}$, then $(u, 2)$ has no neighbors in T'. Consequently, we can avoid unfavorable situation by having $c_u = 2$ and then obtain a desired coloring which is a contradiction. Thus we cannot add (x, x_1) or (x, x_3) to the right of $T - \{(x, x_2)\}$ to get a DP-B_A-coloring of $G - \{u, v\}$. It follows that each of (x, x_1) and (x, x_3) have neighbors in T. Recall that (x, x_2) has at least two neighbors in T. Altogether, x in G has at least five neighbors including u. By symmetry, y is also a 5^+-vertex.

Next we show that a 5-vertex x has a 4^+-neighbor. Suppose x is a 5-vertex. By the above argument, (x, x_2) has exactly two neighbors, (x, x_1) has exactly one neighbor, and (x, x_3) has exactly one neighbor in T. By symmetry, assume $x_3 \neq 1$ and (x, x_3) is adjacent to only (z, c_z) in T. If we can add (z, c'_z) to the right of $T - \{(x, x_2), (z, c_z)\}$ where $c'_z \neq c_z$ to obtain a DP-B_A-coloring T'' of $G - \{x, u, v\}$, then we can add (x, x_3) that has no neighbors in T'' to the right of T'' to obtain a DP-B_A-coloring of $G - \{u, v\}$. Recall that $(u, 2)$ is adjacent to only (x, x_2) in T. Consequently, $(u, 2)$ has no neighbors in $T'' \cup \{(x, x_3)\}$. It follows that we can avoid unfavorable situation by having $T'' \cup \{(x, 3)\}$ as a DP-B_A-coloring of $G - \{u, v\}$ and choosing $c_u = 2$. Thus we assume that we cannot add (z, c'_z) to the right of $T - \{(x, x_2), (z, c_z)\}$ to obtain a DP-B_A-coloring of $G - \{x, u, v\}$. One can use a similar argument for the vertex x to prove that z is a 4^+-vertex. Thus x is a 5-vertex with a 4^+-neighbor or a 6^+-vertex, and so is y by symmetry. This completes the proof. ■

4.2. Discharging procedure

Suppose G is a counterexample to Theorem 3 but all of its proper induced subgraphs are DP-B_A-3-colorable. Let $\mu(x) = d(x) - 4$ be the initial charge of a vertex or a face x and let $\mu^*(x)$ denote the final charge of x after the discharging process. By the Euler's formula, $\sum_{v\in V(G)} \mu(v) + \sum_{f\in F(G)} \mu(f) = \sum_{v\in V(G)}(d(v) - 4) + \sum_{f\in F(G)}(d(f) - 4) = -8$. We call a 3-vertex v a *bad* 3-vertex if v is adjacent to another 3-vertex, otherwise we call it a *good* 3-vertex. We define discharging rules as follows.

Discharging Rules.

(R0) Each 5^+-vertex gives $\frac{1}{4}$ to each adjacent bad 3-vertex.

(R1) Each 5^+-face gives $\frac{1}{3}$ to each adjacent 3-face.

(R2) Each 5-face gives $\frac{1}{6}$ to each incident bad 3-vertex and $\frac{1}{3}$ to each incident good 3-vertex.

(R3) Each 6^+-face gives $\frac{1}{3}$ to each incident bad 3-vertex and $\frac{2}{3}$ to each incident good 3-vertex.

We aim to show that the final charge $\mu^*(x)$ for each $x \in V(G) \cup F(G)$ is nonnegative. Since the total of charge is not changed by the rules, we obtain a contradiction and prove the main result.

Proof. By Lemma 9, every vertex v is a 3^+-vertex.

Consider a good 3-vertex v. If v is not incident to a 3-face, then $\mu^*(v) \geq \mu(v) + 3 \times \frac{1}{3} = 0$ by (R2) and (R3). If v is incident to a 3-face, then it is incident to two 5^+-faces and one of which is a 6^+-face by Lemmas 5(1) and 5(5). Thus $\mu^*(v) \geq \mu(v) + \frac{1}{3} + \frac{2}{3} = 0$ by (R2) and (R3).

Consider a bad 3-vertex v. By Lemma 10, v is adjacent to two 5^+-vertices. If v is not incident to a 3-face, then $\mu^*(v) \geq \mu(v)+2\times\frac{1}{4}+3\times\frac{1}{6} = 0$ by (R0), (R2), and (R3). If v is incident to a 3-face, then it is incident to two 5^+-faces one of which is a 6^+-face by Lemmas 5(1) and 5(1)(5). Thus $\mu^*(v) \geq \mu(v) + 2 \times \frac{1}{4} + \frac{1}{6} + \frac{1}{3} = 0$ by (R0), (R2), and (R3).

If v is a 4-vertex, then it does not involve in a discharging process and thus $\mu^*(v) = \mu(v) = 0$.

Consider a 5-vertex v. If v is adjacent to a bad 3-vertex, say u, then v has a 4^+-neighbor by Lemma 10. Consequently, v is adjacent to at most four bad 3-vertices. Thus $\mu^*(v) \geq \mu(v) - 4 \times \frac{1}{4} = 0$ by (R0).

Consider a k-vertex v where $k \geq 6$. Then $\mu^*(v) \geq \mu(v) - k \times \frac{1}{4} = (k - 4) - k \times \frac{1}{4} > 0$ by (R0).

Consider a 3-face f. It follows from Lemma 5(1) that every face adjacent to f is a 5^+-face. Thus $\mu^*(f) = \mu(f) + 3 \times \frac{1}{3} = 0$ by (R1).

If f is a 4-face, then its charge is not affected by the discharging procedure and thus $\mu^*(f) = \mu(f) = 0$.

Consider a 5-face f. From Lemma 5(3), f is adjacent to at most one 3-face. If f is incident to at most two 3-vertices, then $\mu^*(f) \geq \mu(f) - \frac{1}{3} - 2 \times \frac{1}{3} = 0$ by (R1) and (R2). If f is incident to at least three 3-vertices, then f is incident to exactly three 3-vertices in which two of them are bad 3-vertices by Lemma 10. It follows that $\mu^*(f) \geq \mu(f) - \frac{1}{3} - 2 \times \frac{1}{6} - \frac{1}{3} = 0$ by (R1) and (R2).

Consider a 6-face f. From Lemma 5(4), f is not adjacent to a 3-face. If f is incident to at most three 3-vertices, then $\mu^*(f) \geq \mu(f) - 3 \times \frac{2}{3} = 0$ by (R3). If f is incident to at least four 3-vertices, then f is incident to exactly four 3-vertices in which all of them are bad 3-vertices by Lemma 10. It follows that $\mu^*(f) = \mu(f) - 4 \times \frac{1}{3} > 0$ by (R3).

If a 7-face is a simple face, then G has a 7-cycle, otherwise G has a 4-cycle. Thus G does not contain a 7-face.

Finally, consider a k-face f where $k \geq 8$. Assume that all subscripts are taken modulo k. Let $v_1, v_2, \ldots, v_k$ be the vertices on the boundary of f, and let f_i be a face sharing an edge v_iv_{i+1} with f. We construct a new discharging rule for f such that each of its incident 3-vertices and adjacent 3-faces gains charge by the new rule not less than it gains by the original rules.

First, let f send $\frac{1}{2}$ to each v_i. If f_i is a 3-face, then let $\alpha(i) = 1$, otherwise $\alpha(i) = 0$. If v_i is a 3-vertex, then let $\beta(i) = 1$, otherwise $\beta(i) = 0$. Let v_i send charge $\frac{\alpha(i)}{6}$ to f_i and $\beta(i+1)(\frac{1}{4} - \frac{\alpha(i)}{6})$ to v_{i+1}. Similarly, let v_i send charge $\frac{\alpha(i-1)}{6}$ to f_{i-1} and $\beta(i-1)(\frac{1}{4} - \frac{\alpha(i-1)}{6})$ to v_{i-1}. Then each 3-face f_i gains $2 \times \frac{1}{6}$ from v_i and v_{i+1}, and each 4^+-vertex gains a nonnegative charge by the new rule.

Consider a good 3-vertex v_i. Note that at most one of f_{i-1} and f_i is a 3-face to avoid a 4-cycle. By symmetry, assume f_{i-1} is not a 3-face. Then v_i receives $\frac{1}{2}$ from f, receives $\frac{1}{4}$ from v_{i-1}, receives at least $\frac{1}{4} - \frac{1}{6} = \frac{1}{12}$ from v_{i+1}, and sends at most $\frac{1}{6}$ to f_i. Thus v_i gains charge at least $\frac{1}{2} + \frac{1}{4} + \frac{1}{12} - \frac{1}{6} = \frac{2}{3}$ by the new rule.

Consider bad 3-vertices v_i and v_{i+1}. By Lemma 10, v_{i-1} and v_{i+2} are 5^+-vertices. Since charge sent from v_i to v_{i+1} and charge sent from v_{i+1} to v_i are the same, we ignore this distribution in the calculation. Note that if f_i is a 3-face, then none of f_{i-1} and f_{i+1} are 3-faces. Assume f_i is a 3-face. Then v_i receives $\frac{1}{2}$ from f, receives $\frac{1}{4}$ from v_{i-1}, and sends $\frac{1}{6}$ to f_i. Thus v_i gains $\frac{1}{2} + \frac{1}{4} - \frac{1}{6} = \frac{7}{12} > \frac{1}{3}$ by the new rule. Assume f_i is not a 3-face. Then v_i receives $\frac{1}{2}$ from f, receives at least $\frac{1}{4} - \frac{1}{6} = \frac{1}{12}$ from v_{i-1}, and sends at most $\frac{1}{6}$ to f_{i-1}. Thus v_i gains at least $\frac{1}{2} + \frac{1}{12} - \frac{1}{6} = \frac{5}{12} > \frac{1}{3}$ by the new rule.

Altogether, let f send charge at most $\frac{k}{2}$ with a distribution to its incident 3-faces and adjacent 3-faces that satisfies the original rules. Thus $\mu^*(f) \geq \mu(f) - \frac{k}{2} = \frac{k}{2} - 4 \geq 0$.

This completes the proof. ∎

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