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A CHARACTERIZATION OF INTERNALLY 4-CONNECTED $\{P_{10} - \{V_1, V_2\}\}$-MINOR-FREE GRAPHS

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Abstract

Let P_{10} be the Petersen graph. Let $V_8^{--} = P_{10} - \{v_1, v_2\}$, where v_1 and v_2 are the adjacent vertices of P_{10}. In this paper, all internally 4-connected graphs that do not contain V_8^{--} as a minor are charaterized.

Keywords: internally 4-connected, V_8^{--}-minor-free, Petersen graph, 2-connected minor.

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1. Introduction

In this paper, all graphs are considered to be finite and simple. For any two graphs G and H, H is called a *minor* of G, if H can be obtained from G by repeatedly contracting edges, deleting edges and deleting vertices, which is denoted by $H \leq G$. Given a graph G, G is called H*-minor-free*, if no minor of G is isomorphic to H.

Many important problems in graph theroy are about H-minor-free graphs. For instance, there are Hadwiger's Conjecture [8] and Tutte's 4-flow-conjecture [13]. Hadwiger's Conjecture is that every K_n-minor-free graph is $n-1$ colorable, and 4-flow-conjecture states that every bridgeless Petersen-minor-free graph has 4-flow. Up to now, these two conjectures are still unsolved, since the structure of K_n-minor-free ($n \geq 6$) graphs and Petersen-minor-free graphs are unknown.We observe that K_6 and Petersen graph both have fifteen edges. In order to characterize these graphs, we investigate graphs with less than fifteen edges.

Ding [5] characterized all H-minor-free graphs, where H is a 3-connected graph with at most eleven edges. For 3-connected graphs with 12 edges, cube [9], V_8 [11], and octahedron [3] have been determined. In addition, 4-connected Oct^+-minor-free graphs [10] (where Oct^+ is the unique 13-edge graph obtained from the octahedron by adding a nonadjacent edge) and 4-connected $\overline{P}_7$-minor-free graphs [4] ($\overline{P}_7$ denotes the complement of a path on seven vertices) are also solved.

For 2-connected graphs, the characterization problem is solved for the $K_{2,4}$ [6]. 4-connected $K_{2,5}$-minor-free planar graphs [7] are determined by Marshall.

In this article, we focus on P_{10}-minor-free graphs, where P_{10} denotes the Petersen graph. We use V_8^{--} to denote the graph $P_{10} - \{v_1, v_2\}$, where v_1 and v_2 are the adjacent vertices of P_{10}. We investigate all internally 4-connected V_8^{--}-minor-free graphs and our method follows almost the same method as [1]. Obviously, V_8^{--} is also a 2-connected subgraph of V_8, see Figure 1.

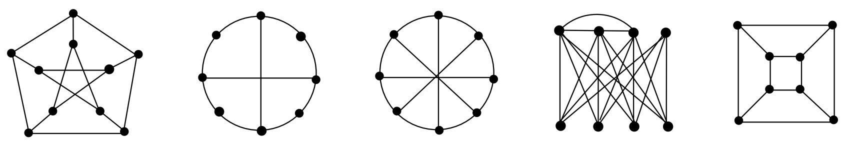

Figure 1. P_{10}, V_8^{--}, V_8, $K_{4,4}^{+3,1}$, and Cube.

Let $k \geq 0$ be an integer. A *k-separation* of a graph G is a pair $\{G_1, G_2\}$ of induced subgraphs of G such that $E(G_1) \cup E(G_2) = E(G)$, $V(G_1) \cup V(G_2) = V(G)$, $V(G_1) - V(G_2) \neq \emptyset$, $V(G_2) - V(G_1) \neq \emptyset$, and $|V(G_1) \cap V(G_2)| = k$. A 3-connected graph G with five or more vertices is said to be *internally* 4*-connected*, if for every 3-separation $\{G_1, G_2\}$ of G, at least one of G_1, G_2 is isomorphic to $K_{1,3}$. According to the above definitions, we can observe that if v is a cubic vertex of an internally 4-connected graph, then the set of vertices adjacent to v is an independent set. This can also be seen in [3]. Let $\mathcal{K}$ be a set of graphs, in which are internally 4-connected minors of $K_{4,5}$ or $K_{4,4}^{+3,1}$ with at least eight vertices, where $K_{4,4}^{+3,1}$ is shown in Figure 1.

The following is the main theorem of this article.

Theorem 1. *Let G be an internally 4-connected graph. G is V_8^{--}-minor-free if and only if G is either one of the graphs in $\mathcal{K}$ or an internally 4-connected graph of at most seven vertices.*

The rest of this paper is arranged as follows. The next section includes auxiliary results that will be used. Finally, in Section 3, we prove Theorem 1.

2. Auxiliary Results

Let n be a positive integer. A *double-wheel*, denoted by DW_n ($n \geq 3$), is a graph obtained from a cycle C_n ($n \geq 3$) with n vertices by adding two nonadjacent vertices u, v and making both of them adjacent to all vertices on the cycle C_n. An *alternating double-wheel*, denoted by AW_{2n} ($n \geq 3$), is a graph obtained from a cycle C_{2n} by adding two nonadjacent vertices u, v and such that u and v are alternately adjacent to every vertex in C_{2n}. We can observe that DW_n and AW_{2n} are all planar graphs for each n. Let $DW_n^+ = DW_n + uv$, $AW_{2n}^+ = AW_{2n} + uv$, and $\mathcal{W}^+ = \{DW_n^+ : n \geq 3\} \cup \{AW_{2n}^+ : n \geq 3\}$. Note that AW_6 is isomorphic to the cube (see Figure 1), and every graph in $\mathcal{W}^+$ is a nonplanar graph. Let $\mathcal{K}_1$ be a set of graphs such that every graph in $\mathcal{K}_1$ is internally 4-connected and with four vertices incident to all edges.

For any graph G, let $L(G)$ be the *line graph* of G such that $V(L(G)) = E(G)$, and two vertices of $L(G)$ are adjacent if and only if their corresponding edges share a common end vertex in G. The number of vertices and edges in G are denoted by $|G|$ and $||G||$, respectively. Let $e = uv$ be an edge of G. *Contracting* the edge e, denoted by G/e, means deleting the edge e and identifying the vertices u and v to a single vertex w such that w is adjacent to all vertices which are adjacent to u and v.

Theorem 2 [11]. *Every internally 4-connected V_8-minor-free graph G belongs to one of the following five families:*

(1) *G is the graph with seven or fewer vertices;*

(2) *G is isomorphic to $L(K_{3,3})$;*

(3) *G is in* $\mathcal{W}^+$;

(4) *G is in* $\mathcal{K}_1$;

(5) *G is planar.*

Let G be a 3-connected graph and v be a vertex of degree at least four of G. Let $N_G(v)$ denote the set of vertices of G that are adjacent to v, which are also known as *neighbors* of v. Given two sets, A, $B \subseteq N_G(v)$, where $A \cup B = N_G(v)$ and $|A|, |B| \geq 2$, a *vertex split* of v means that the graph G' is obtained from G by replacing the vertex v in G by new vertices a and b such that $N_G(a) = A \cup b$ and $N_G(b) = B \cup a$. If G' is a planar graph, then we call vertex split the *planar split*.

For every integer $n \geq 5$, let C_n^2 be a graph, which is obtained from a cycle C_n by joining every pair of vertices of distance two in the cycle C_n. Note that C_n^2 $(n \geq 5)$ is a 4-connected vertex-transitive graph. The graph *terrahawk* can be found in Figure 4.

The following theorem is a chain theorem by Chun *et al.* in [2].

Theorem 3 [2]**.** *Let G be an internally 4-connected graph such that G is not isomorphic to $K_{3,3}$, terrahawk, C_n^2 $(n \geq 5)$, or AW_{2n} $(n \geq 3)$. Then G has an internally 4-connected minor H with $1 \leq ||G|| - ||H|| \leq 3$.*

Theorem 3 implies that if G is an internally 4-connected graph, then G can be obtained from a series of internally 4-connected graphs $H_0, H_1, H_2, \ldots, H_k$ such that

(1) H_0 is isomorphic to $K_{3,3}$, terrahawk, C_n^2 $(n \geq 5)$, or AW_{2n} $(n \geq 3)$, H_k is isomorphic to G;

(2) H_i $(i = 2, \ldots, k)$ is obtained from H_{i-1} by adding edges or splitting vertices at most three times.

3. Main Theorem

The goal of this section is to prove the Theorem 1, which is the main theorem in this article. We first consider the graphs in $\mathcal{K}$. Then, we find the internally 4-connected V_8^{--}-minor-free graphs in V_8-minor-free graphs.

We note that both AW_6 and AW_6^+ are internally 4-connected and are minors of $K_{4,5}$.

Lemma 4. *Both AW_6 and AW_6^+ are V_8^{--}-minor-free.*

Proof. As AW_6 is a subgraph of AW_6^+, it is sufficient to show that AW_6^+ is V_8^{--}-minor-free. Note that V_8^{--} has a 5-cycle. But AW_6^+ contains no 5-cycles. Therefore, AW_6^+ is V_8^{--}-minor-free. ■

Lemma 5. *$K_{4,5}$ is V_8^{--}-minor-free.*

Proof. Suppose with contradiction that $K_{4,5}$ has a V_8^{--}-minor. We denote by $X = \{x_1, x_2, x_3, x_4\}$ the set of four vertices that are incident to all edges of $K_{4,5}$, and let $Y = V(K_{4,5}) - X = \{y_1, y_2, y_3, y_4, y_5\}$, as illustrated in Figure 2.

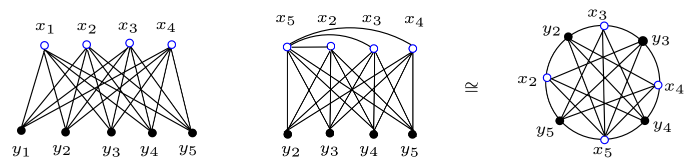

Figure 2. $K_{4,5}$ and $K_{4,4}^{+3,2}$.

Note that $K_{4,5}$ is connected and $|K_{4,5}| = 9 > 8 = |V_8^{--}|$. So the minor V_8^{--} can be obtained from $K_{4,5}$ by two ways.

We first consider that V_8^{--} is obtained from $K_{4,5}$ by contracting an edge, say x_1y_1, without lose of generality, and deleting some edges. Let $K_{4,4}^{+3,2} = K_{4,5}/x_1y_1$ and the new vertex produced be x_5. Then V_8^{--} is a subgraph of $K_{4,4}^{+3,2}$. From the structure of $K_{4,4}^{+3,2}$, the order of the 8-cycle of V_8^{--} must alternate between x_i and y_j $(i, j = 2, 3, \dots, 5)$, see Figure 2. However, $K_{4,4}^{+3,2}$ does not contain V_8^{--} as a subgraph, a contradiction.

Next, suppose V_8^{--} is obtained from $K_{4,5}$ by deleting a vertex and some edges. Without loss of generality, let $H_1 = K_{4,5}\backslash x_1$ and $H_2 = K_{4,5}\backslash y_1$. Both H_1 and H_2 are subgraphs of $K_{4,4}^{+3,2}$. Therefore, H_1 and H_2 are also V_8^{--}-minor-free. A contradiction. ■

Let G be a graph. A *covering* of G is a subset $C(G)$ of $V(G)$ such that every edge of G has at least one end in $C(G)$. A covering $C(G)$ is a *mininum covering* if G has no covering $C^{'}(G)$ with $|C'(G)| < |C(G)|$. If there is no contradiction, we can omit the letter G, use C instead of $C(G)$.

Lemma 6. *Both $K_{4,4}^{+3,1}$ and $K_{4,4}^{+3,2}$ are V_8^{--}-minor-free.*

Proof. According to Lemma 5, $K_{4,4}^{+3,2}$ is V_8^{--}-minor-free, since $K_{4,4}^{+3,2}$ is a minor of $K_{4,5}$.

Next, we prove that $K_{4,4}^{+3,1}$ is V_8^{--}-minor-free. Suppose V_8^{--} is a minor of $K_{4,4}^{+3,1}$. As $|V_8^{--}| = |K_{4,4}^{+3,1}|$, V_8^{--} can be obtained from $K_{4,4}^{+3,1}$ by deleting some edges. Let $T = \{t_1, t_2, t_3, t_4\}$ be a set of four vertices that are incident to all edges of $K_{4,4}^{+3,1}$ and let t_4 be the vertex of degree four. We can observe that the minimum covering in $K_{4,4}^{+3,1} - t_4$ is $C = T\backslash\{t_4\}$, and $|C| = 3$. Suppose V_8^{--} is obtained by deleting two edges that are incident with t_4 and deleting other edges in $K_{4,4}^{+3,1}$. Let t be the vertex of degree two in V_8^{--} that corresponds to t_4 in $K_{4,4}^{+3,1}$. Then let C' be the minimum covering in $V_8^{--} - t$. We can observe that $|C'| = 4$. This is a contradiction. Therefore, $K_{4,4}^{+3,1}$ is V_8^{--}-minor-free. ■

Next, we consider graphs in $\mathcal{K}$.

Lemma 7. *The graphs in* $\mathcal{K}$ *are showed in Figure* 3.

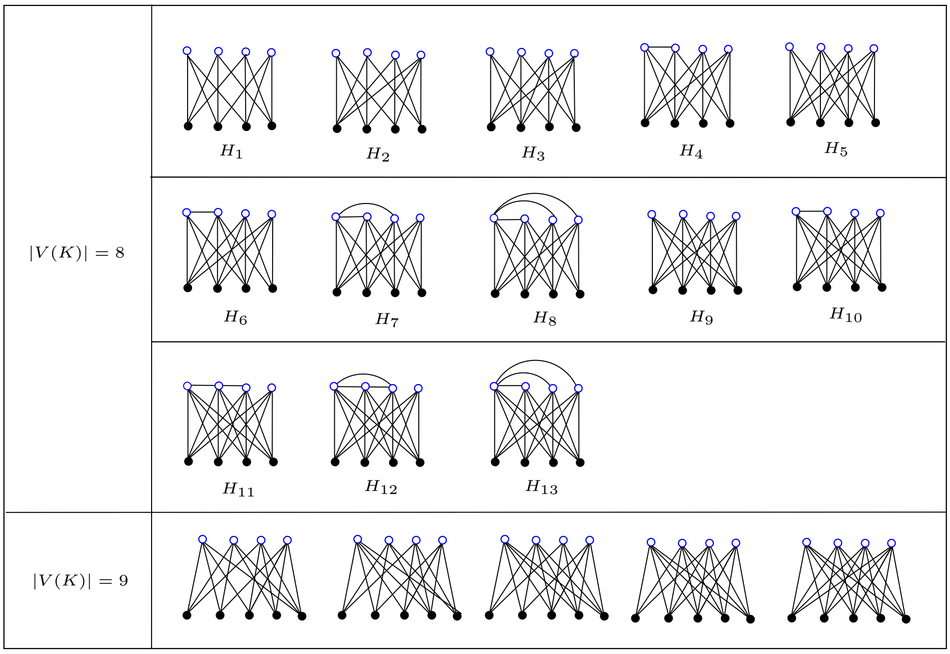

Figure 3. Graphs in $\mathcal{K}$.

Proof. Let $K \in \mathcal{K}$. Then K is an internally 4-connected minor of $K_{4,4}^{+3,1}$ or $K_{4,5}$. Let X be a set of four vertices incident to all edges of K. Let $Y = V(K) - X$ and let Y_3, Y_4 consist of vertices of Y of degrees 3 and 4, respectively. Since K is internally 4-connected, no two vertices in Y_3 have the same neighbors.

If $|V(K)| = 9$, then K is a minor of $K_{4,5}$ that is obtained by deleting some edges. Note that there are only five possible graphs, since $|Y_4|$ can be $1, 2, 3, 4$, or 5), as illustrated in Figure 3. If $|Y_4| = 0$, the resulting graph is not internally 4-connected.

Suppose $|V(K)| = 8$. Then K is a minor of $K_{4,4}^{+3,1}$ by deleting edges. Note that for H_i $(i = 1, 2, \ldots, 13)$, adding any edge will lead the resulting graph not to be internally 4-connected. If $|Y_3| = 4$, K must be H_1. If $|Y_3| = 3$, K is the H_2. Suppose $|Y_3| = 2$, then K is either H_3 or H_4. If $|Y_3| = 1$, K is H_5, H_6, H_7

or H_8. If $|Y_3| = 0$, K is H_9, H_{10}, H_{11}, H_{12} or H_{13}. Therefore, there are thirteen graphs on eight vertices and five graphs on nine vertices in $\mathcal{K}$. ■

Lemma 8. *Every graph in $\mathcal{K}$ is V_8^{--}-minor-free.*

Proof. Let $K \in \mathcal{K}$. Then K is an internally 4-connected minor of $K_{4,4}^{+3,1}$ or $K_{4,5}$. According to Lemma 5 and Lemma 6, both $K_{4,4}^{+3,1}$ and $K_{4,5}$ are V_8^{--}-minor-free. Therefore, K is also V_8^{--}-minor-free. ■

Lemma 9. *Let G be an internally 4-connected V_8^{--}-minor-free planar graph. Then G is isomorphic to C_6^2, DW_5 or AW_6.*

Proof. Let G be an internally 4-connected V_8^{--}-minor-free planar graph. Suppose with contradiction that G is not isomorphic to C_6^2, DW_5 or AW_6. According to Theorem 3, there is a series of internally 4-connected graphs $H_0, H_1, H_2, \ldots, H_k$ such that H_0 is isomorphic to $K_{3,3}$, terrahawk, C_n^2 $(n \geq 5)$, or AW_{2n} $(n \geq 3)$, H_k is isomorphic to G and H_i $(i = 2, \ldots, k)$ is obtained from H_{i-1} by adding edges and splitting vertices at most three times.

As illustrated in Figure 4, note that terrahawk contains a V_8^{--}-minor (by contracting the thick edge labeled 1), and so does AW_8 (by contracting the thick edges labeled 2 and 3). And C_8^2 also contains a V_8^{--}-minor, see Figure 4.

Both AW_{2n} $(n \geq 4)$ and C_{2n}^2 $(n \geq 4)$ contain a V_8^{--}-minor, since they contain AW_8 and C_8^2 as a minor, respectively. Neither C_{2n+1}^2 $(n \geq 2)$ nor $K_{3,3}$ is a planar graph, because they contain K_5 and $K_{3,3}$ as a minor, respectively. Therefore, we only need to consider that H_0 is isomorphic to C_6^2 or AW_6.

Case 1. H_0 is isomorphic to C_6^2. Then H_1 is obtained from C_6^2 by splitting vertices and adding edges. Since adding any nonadjacent edge to C_6^2 will generate a nonplanar graph, we only consider the planar splits of C_6^2. (The process can also be seen in [1]). Note that C_6^2 is a vertex-transitive graph. Without loss of generality, we assume that the first vertex we split in C_6^2 is v_1. Up to symmetry, there are only four planar splits A, B, C and D, as illustrated in Figure 5. And D is isomorphic to DW_5, which is the only internally 4-connected planar graph with seven vertices [11]. Therefore, H_1 is constructed from A, B, C or D by splitting vertices at least once and adding edges.

Case 1.1. H_1 is constructed from A. We only consider the planar splits of A. Firstly, considering the cases that both the two new vertices have degree three, because other planar splits contain these special splits. Up to symmetry, there are four such splits, the first three cases contain a V_8^{--}-minor and the last one A' is V_8^{--}-minor-free, as illustrated in Figure 6.

If we add any edge to A', the resulting graph A'_1 will contain a V_8^{--}-minor, see Figure 7. And every planar split of A' contains a V_8^{--}-minor since it contains

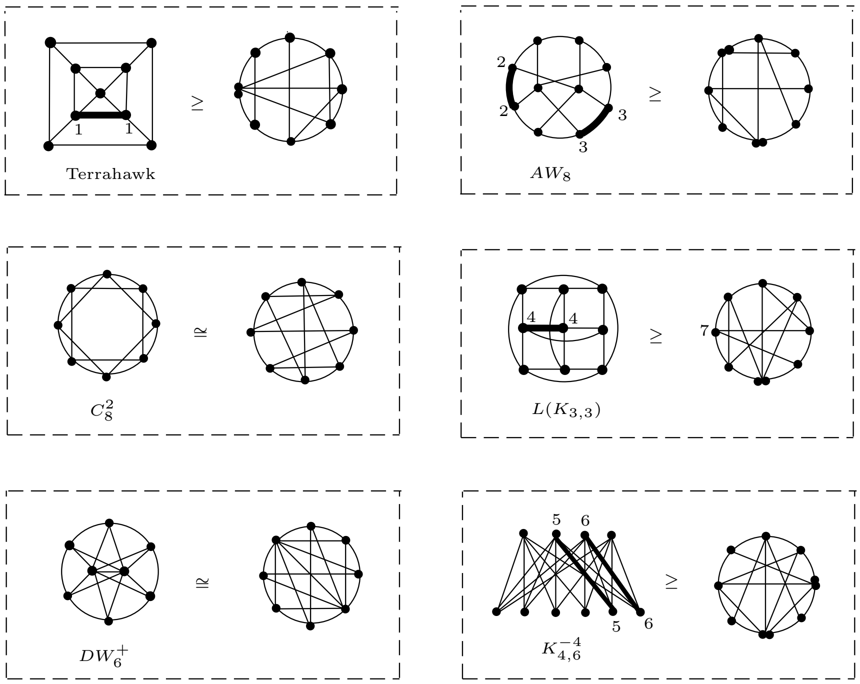

Figure 4. Graphs which have a V_8^{--}-minor.

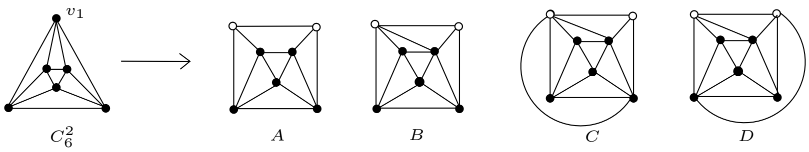

Figure 5. Four planar splits of C_6^2.

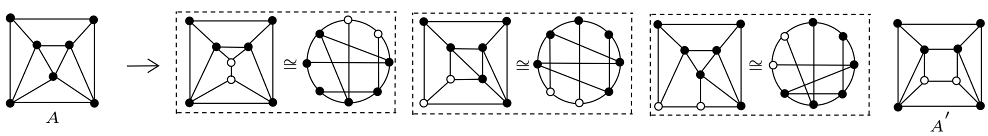

Figure 6. Four planar splits of A.

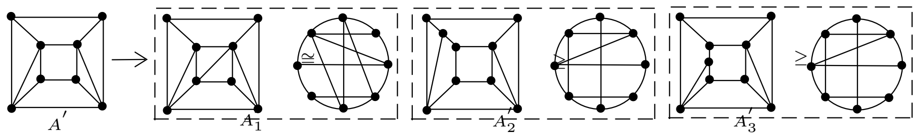

Figure 7. Graphs generated from $A^{'}$.

planar split A_2' or A_3' as a minor that both new vertices have degree three, as shown in Figure 7.

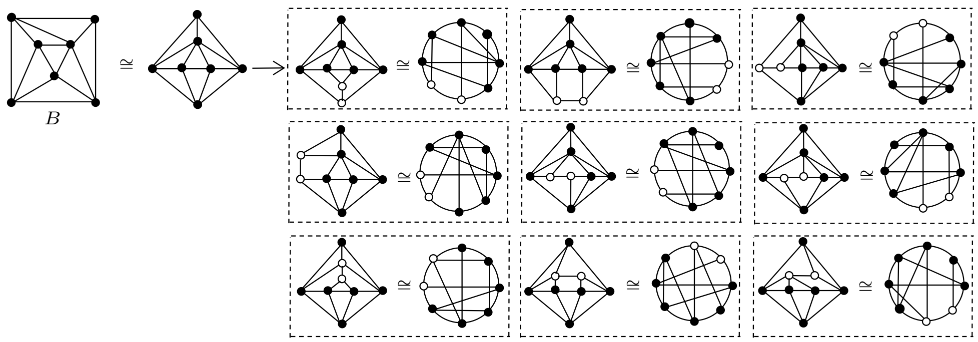

Figure 8. Nine planar splits of B.

Case 1.2. H_1 is constructed from B. We consider the planar splits of B. Every planar split of B contains a V_8^{--}-minor, since it contains one of the nine graphs as a minor (see Figure 8), which contains a V_8^{--}-minor.

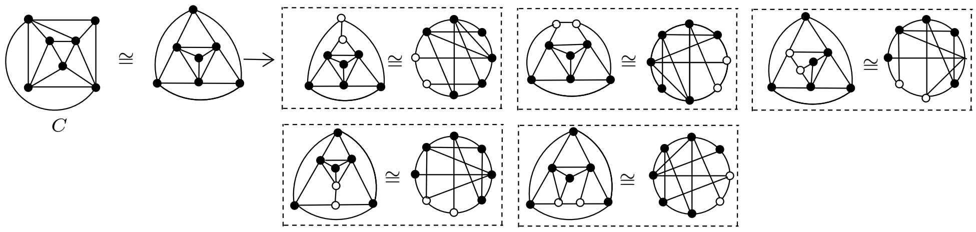

Figure 9. Five planar splits of C.

Case 1.3. H_1 is constructed from C. Similarly, for graph C, every planar split of C contains a V_8^{--}-minor since it contains a planar split as shown in Figure 9, which contains a V_8^{--}-minor.

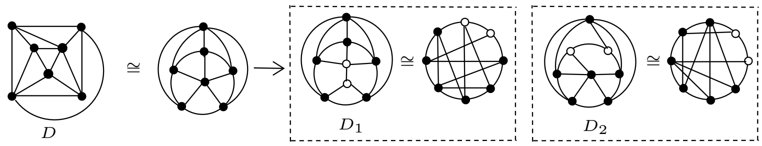

Figure 10. Two planar splits of D.

Case 1.4. H_1 is constructed from D. For graph D, we first consider splitting a degree-5 vertex of D. Suppose one new vertex of degree three and the other vertex of degree four. Up to symmetry, there is exactly one such planar split D_1, which contains a V_8^{--}-minor, as illustrated in Figure 10. Then every graph obtained from D by splitting a degree-5 vertex will have a V_8^{--}-minor, since it contains D_1 as a minor. Next, having a planar split of a vertex with degree four in D. Then every such planar split of D contains a V_8^{--}-minor since it contains the planar split D_2 as a minor, that both new vertices have degree three, as shown in Figure 10.

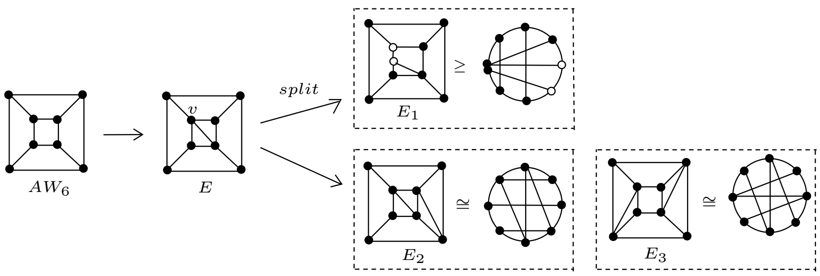

Figure 11. The graphs generated from AW_6.

Case 2. H_0 is isomorphic to AW_6. Since AW_6 is a cubic graph, the vertex in AW_6 cannot be split. Note that AW_6 is also a vertex-transitive and edge-transitive graph. Let E be the graph obtained from AW_6 by adding a nonadjacent edge e_1, as illustrated in Figure 11. Note that E is not an internally 4-connected graph and is V_8^{--}-minor-free. Then having a planar split of v, a vertex of degree four in E, and every graph will contain a V_8^{--}-minor since it contains a planar split E_1 as a minor, that both new vertices have degree three, as shown in Figure 11. If we continue adding edges to E and splitting vertices at least once, then the resulting graph will also contain E_1 as a minor. Therefore, the resulting graph has a V_8^{--}-minor. Next, we only consider adding edges to E. The graph generated from E either is not an internally 4-connected graph, or contains E_2 or E_3 as a minor, see Figure 11.

According to all of the above, we know that H_1 contains a V_8^{--}-minor, and so does G. A contradiction. Therefore, G is isomorphic to C_6^2, DW_5 or AW_6. ■

***Proof of Theorem* 1.** Let G be an internally 4-connected graph. We can observe that every G with seven or fewer vertices is V_8^{--}-minor-free. According to Lemma 8, if $G \in \mathcal{K}$, G is also V_8^{--}-minor-free.

For the necessity, V_8 contains V_8^{--} as a subgraph, so all internally 4-connected V_8^{--}-minor-free graphs must be V_8-minor-free graphs as described in Theorem 2. We must decide which of those graphs are actually V_8^{--}-minor-free. Let G be an internally 4-connected V_8^{--}-minor-free graph. If $|G| \leq 7$, then G is V_8^{--}-minor-free obviously. Next, suppose that $|G| \geq 8$.

Case 1. G is isomorphic to $L(K_{3,3})$. Then as shown in Figure 4, G has a V_8^{--}-minor by contracting the thick edge labeled 4.

Case 2. G is in $\mathcal{W}^+$. We note that AW_8 is a subgragh of AW_8^+, and it has a V_8^{--}-minor by contracting the thick edges labeled 2 and 3, as illustrated in Figure 4. Therefore, AW_8^+ contains a V_8^{--}-minor. And DW_6^+ also has a V_8^{--}-minor, see Figure 4. Since $\left\{DW_n^+ : n \geq 6\right\}$ and $\left\{AW_{2n}^+ : n \geq 4\right\}$ have DW_6^+ and AW_8^+ as a minor, respectively, they all have V_8^{--}-minor. Note that AW_6^+ belongs to $\mathcal{K}$ and is V_8^{--}-minor-free according to Lemma 4.

Case 3. G is in $\mathcal{K}_1$. We claim that $G \in \mathcal{K}_1$ is V_8^{--}-minor-free if and only if $G \in \mathcal{K}$. If $G \in \mathcal{K}$, then $G \in \mathcal{K}_1$ and is V_8^{--}-minor-free according to Lemma 7 and Lemma 8. Suppose $G \in \mathcal{K}_1$.

If $|G| \geq 10$, then G contains $K_{4,6}^{-4}$ as a minor, since $K_{4,6}^{-4}$ is the minimal graph on ten vertices in $\mathcal{K}_1$. Note that $K_{4,6}^{-4}$ contains a V_8^{--}-minor by contracting the thick edges labeled 5 and 6, see Figure 4. Hence, G contains a V_8^{--}-minor.

For $|G| = 9$, we consider the graph $K_{4,5}$. Adding any edge to the color class of size five in $K_{4,5}$ will lead the graph not belong to $\mathcal{K}_1$, which is a contradiction. If add any edge to the color class of size four in $K_{4,5}$ will lead a V_8^{--}-minor. According to Lemma 5, $K_{4,5}$ is V_8^{--}-minor-free. Therefore, $K_{4,5}$ is the maximal V_8^{--}-minor-free graph with nine vertices in $\mathcal{K}_1$. Then G is a minor of $K_{4,5}$ and is also V_8^{--}-minor-free. So $G \in \mathcal{K}$.

Suppose $|G| = 8$. According to Lemma 6 and the analysis of above, we can similarly prove that $K_{4,4}^{+3,1}$ and $K_{4,4}^{+3,2}$, which belong to $\mathcal{K}_1$, are maximal V_8^{--}-minor-free graphs with eight vertices. Therefore, G is a minor of $K_{4,4}^{+3,1}$ or $K_{4,4}^{+3,2}$ and is also V_8^{--}-minor-free. So G with eight vertices belongs to $\mathcal{K}$.

Case 4. G is planar. According to Lemma 9, G is isomorphic to AW_6, which belongs to $\mathcal{K}$. ■

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