

## A CHARACTERIZATION OF INTERNALLY 4-CONNECTED $\{P_{10} - \{V_1, V_2\}\}$ -MINOR-FREE GRAPHS

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### Abstract

Let  $P_{10}$  be the Petersen graph. Let  $V_8^{--} = P_{10} - \{v_1, v_2\}$ , where  $v_1$  and  $v_2$  are the adjacent vertices of  $P_{10}$ . In this paper, all internally 4-connected graphs that do not contain  $V_8^{--}$  as a minor are characterized.

**Keywords:** internally 4-connected,  $V_8^{--}$ -minor-free, Petersen graph, 2-connected minor.

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## 1. INTRODUCTION

In this paper, all graphs are considered to be finite and simple. For any two graphs  $G$  and  $H$ ,  $H$  is called a *minor* of  $G$ , if  $H$  can be obtained from  $G$  by repeatedly contracting edges, deleting edges and deleting vertices, which is denoted by  $H \leq G$ . Given a graph  $G$ ,  $G$  is called  *$H$ -minor-free*, if no minor of  $G$  is isomorphic to  $H$ .

Many important problems in graph theory are about  $H$ -minor-free graphs. For instance, there are Hadwiger's Conjecture [8] and Tutte's 4-flow-conjecture [13]. Hadwiger's Conjecture is that every  $K_n$ -minor-free graph is  $n - 1$  colorable, and 4-flow-conjecture states that every bridgeless Petersen-minor-free graph has 4-flow. Up to now, these two conjectures are still unsolved, since the structure of  $K_n$ -minor-free ( $n \geq 6$ ) graphs and Petersen-minor-free graphs are unknown. We observe that  $K_6$  and Petersen graph both have fifteen edges. In order to characterize these graphs, we investigate graphs with less than fifteen edges.

Ding [5] characterized all  $H$ -minor-free graphs, where  $H$  is a 3-connected graph with at most eleven edges. For 3-connected graphs with 12 edges, cube [9],  $V_8$  [11], and octahedron [3] have been determined. In addition, 4-connected  $Oct^+$ -minor-free graphs [10] (where  $Oct^+$  is the unique 13-edge graph obtained from the octahedron by adding a nonadjacent edge) and 4-connected  $\overline{P}_7$ -minor-free graphs [4] ( $\overline{P}_7$  denotes the complement of a path on seven vertices) are also solved.

For 2-connected graphs, the characterization problem is solved for the  $K_{2,4}$  [6]. 4-connected  $K_{2,5}$ -minor-free planar graphs [7] are determined by Marshall.

In this article, we focus on  $P_{10}$ -minor-free graphs, where  $P_{10}$  denotes the Petersen graph. We use  $V_8^{--}$  to denote the graph  $P_{10} - \{v_1, v_2\}$ , where  $v_1$  and  $v_2$  are the adjacent vertices of  $P_{10}$ . We investigate all internally 4-connected  $V_8^{--}$ -minor-free graphs and our method follows almost the same method as [1]. Obviously,  $V_8^{--}$  is also a 2-connected subgraph of  $V_8$ , see Figure 1.

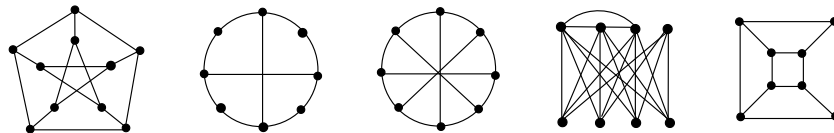


Figure 1.  $P_{10}$ ,  $V_8^{--}$ ,  $V_8$ ,  $K_{4,4}^{+3,1}$ , and Cube.

Let  $k \geq 0$  be an integer. A  $k$ -separation of a graph  $G$  is a pair  $\{G_1, G_2\}$  of induced subgraphs of  $G$  such that  $E(G_1) \cup E(G_2) = E(G)$ ,  $V(G_1) \cup V(G_2) = V(G)$ ,  $V(G_1) - V(G_2) \neq \emptyset$ ,  $V(G_2) - V(G_1) \neq \emptyset$ , and  $|V(G_1) \cap V(G_2)| = k$ . A 3-connected graph  $G$  with five or more vertices is said to be *internally 4-connected*,

if for every 3-separation  $\{G_1, G_2\}$  of  $G$ , at least one of  $G_1, G_2$  is isomorphic to  $K_{1,3}$ . According to the above definitions, we can observe that if  $v$  is a cubic vertex of an internally 4-connected graph, then the set of vertices adjacent to  $v$  is an independent set. This can also be seen in [3]. Let  $\mathcal{K}$  be a set of graphs, in which are internally 4-connected minors of  $K_{4,5}$  or  $K_{4,4}^{+3,1}$  with at least eight vertices, where  $K_{4,4}^{+3,1}$  is shown in Figure 1.

The following is the main theorem of this article.

**Theorem 1.** *Let  $G$  be an internally 4-connected graph.  $G$  is  $V_8^{--}$ -minor-free if and only if  $G$  is either one of the graphs in  $\mathcal{K}$  or an internally 4-connected graph of at most seven vertices.*

The rest of this paper is arranged as follows. The next section includes auxiliary results that will be used. Finally, in Section 3, we prove Theorem 1.

## 2. AUXILIARY RESULTS

Let  $n$  be a positive integer. A *double-wheel*, denoted by  $DW_n$  ( $n \geq 3$ ), is a graph obtained from a cycle  $C_n$  ( $n \geq 3$ ) with  $n$  vertices by adding two nonadjacent vertices  $u, v$  and making both of them adjacent to all vertices on the cycle  $C_n$ . An *alternating double-wheel*, denoted by  $AW_{2n}$  ( $n \geq 3$ ), is a graph obtained from a cycle  $C_{2n}$  by adding two nonadjacent vertices  $u, v$  and such that  $u$  and  $v$  are alternately adjacent to every vertex in  $C_{2n}$ . We can observe that  $DW_n$  and  $AW_{2n}$  are all planar graphs for each  $n$ . Let  $DW_n^+ = DW_n + uv$ ,  $AW_{2n}^+ = AW_{2n} + uv$ , and  $\mathcal{W}^+ = \{DW_n^+ : n \geq 3\} \cup \{AW_{2n}^+ : n \geq 3\}$ . Note that  $AW_6$  is isomorphic to the cube (see Figure 1), and every graph in  $\mathcal{W}^+$  is a nonplanar graph. Let  $\mathcal{K}_1$  be a set of graphs such that every graph in  $\mathcal{K}_1$  is internally 4-connected and with four vertices incident to all edges.

For any graph  $G$ , let  $L(G)$  be the *line graph* of  $G$  such that  $V(L(G)) = E(G)$ , and two vertices of  $L(G)$  are adjacent if and only if their corresponding edges share a common end vertex in  $G$ . The number of vertices and edges in  $G$  are denoted by  $|G|$  and  $||G||$ , respectively. Let  $e = uv$  be an edge of  $G$ . *Contracting* the edge  $e$ , denoted by  $G/e$ , means deleting the edge  $e$  and identifying the vertices  $u$  and  $v$  to a single vertex  $w$  such that  $w$  is adjacent to all vertices which are adjacent to  $u$  and  $v$ .

**Theorem 2** [11]. *Every internally 4-connected  $V_8$ -minor-free graph  $G$  belongs to one of the following five families:*

- (1)  $G$  is the graph with seven or fewer vertices;
- (2)  $G$  is isomorphic to  $L(K_{3,3})$ ;

- (3)  $G$  is in  $\mathcal{W}^+$ ;
- (4)  $G$  is in  $\mathcal{K}_1$ ;
- (5)  $G$  is planar.

Let  $G$  be a 3-connected graph and  $v$  be a vertex of degree at least four of  $G$ . Let  $N_G(v)$  denote the set of vertices of  $G$  that are adjacent to  $v$ , which are also known as *neighbors* of  $v$ . Given two sets,  $A, B \subseteq N_G(v)$ , where  $A \cup B = N_G(v)$  and  $|A|, |B| \geq 2$ , a *vertex split* of  $v$  means that the graph  $G'$  is obtained from  $G$  by replacing the vertex  $v$  in  $G$  by new vertices  $a$  and  $b$  such that  $N_{G'}(a) = A \cup b$  and  $N_{G'}(b) = B \cup a$ . If  $G'$  is a planar graph, then we call vertex split the *planar split*.

For every integer  $n \geq 5$ , let  $C_n^2$  be a graph, which is obtained from a cycle  $C_n$  by joining every pair of vertices of distance two in the cycle  $C_n$ . Note that  $C_n^2$  ( $n \geq 5$ ) is a 4-connected vertex-transitive graph. The graph *terrahawk* can be found in Figure 4.

The following theorem is a chain theorem by Chun *et al.* in [2].

**Theorem 3** [2]. *Let  $G$  be an internally 4-connected graph such that  $G$  is not isomorphic to  $K_{3,3}$ , terrahawk,  $C_n^2$  ( $n \geq 5$ ), or  $AW_{2n}$  ( $n \geq 3$ ). Then  $G$  has an internally 4-connected minor  $H$  with  $1 \leq ||G|| - ||H|| \leq 3$ .*

Theorem 3 implies that if  $G$  is an internally 4-connected graph, then  $G$  can be obtained from a series of internally 4-connected graphs  $H_0, H_1, H_2, \dots, H_k$  such that

- (1)  $H_0$  is isomorphic to  $K_{3,3}$ , terrahawk,  $C_n^2$  ( $n \geq 5$ ), or  $AW_{2n}$  ( $n \geq 3$ ),  $H_k$  is isomorphic to  $G$ ;
- (2)  $H_i$  ( $i = 2, \dots, k$ ) is obtained from  $H_{i-1}$  by adding edges or splitting vertices at most three times.

### 3. MAIN THEOREM

The goal of this section is to prove the Theorem 1, which is the main theorem in this article. We first consider the graphs in  $\mathcal{K}$ . Then, we find the internally 4-connected  $V_8^{--}$ -minor-free graphs in  $V_8$ -minor-free graphs.

We note that both  $AW_6$  and  $AW_6^+$  are internally 4-connected and are minors of  $K_{4,5}$ .

**Lemma 4.** *Both  $AW_6$  and  $AW_6^+$  are  $V_8^{--}$ -minor-free.*

**Proof.** As  $AW_6$  is a subgraph of  $AW_6^+$ , it is sufficient to show that  $AW_6^+$  is  $V_8^{--}$ -minor-free. Note that  $V_8^{--}$  has a 5-cycle. But  $AW_6^+$  contains no 5-cycles. Therefore,  $AW_6^+$  is  $V_8^{--}$ -minor-free. ■

**Lemma 5.**  $K_{4,5}$  is  $V_8^{--}$ -minor-free.

**Proof.** Suppose with contradiction that  $K_{4,5}$  has a  $V_8^{--}$ -minor. We denote by  $X = \{x_1, x_2, x_3, x_4\}$  the set of four vertices that are incident to all edges of  $K_{4,5}$ , and let  $Y = V(K_{4,5}) - X = \{y_1, y_2, y_3, y_4, y_5\}$ , as illustrated in Figure 2.

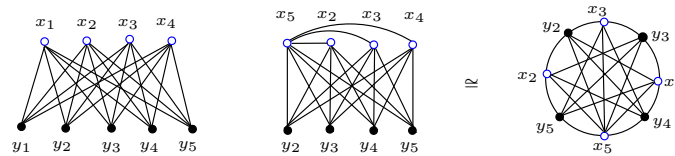


Figure 2.  $K_{4,5}$  and  $K_{4,4}^{+3,2}$ .

Note that  $K_{4,5}$  is connected and  $|K_{4,5}| = 9 > 8 = |V_8^{--}|$ . So the minor  $V_8^{--}$  can be obtained from  $K_{4,5}$  by two ways.

We first consider that  $V_8^{--}$  is obtained from  $K_{4,5}$  by contracting an edge, say  $x_1y_1$ , without loss of generality, and deleting some edges. Let  $K_{4,4}^{+3,2} = K_{4,5}/x_1y_1$  and the new vertex produced be  $x_5$ . Then  $V_8^{--}$  is a subgraph of  $K_{4,4}^{+3,2}$ . From the structure of  $K_{4,4}^{+3,2}$ , the order of the 8-cycle of  $V_8^{--}$  must alternate between  $x_i$  and  $y_j$  ( $i, j = 2, 3, \dots, 5$ ), see Figure 2. However,  $K_{4,4}^{+3,2}$  does not contain  $V_8^{--}$  as a subgraph, a contradiction.

Next, suppose  $V_8^{--}$  is obtained from  $K_{4,5}$  by deleting a vertex and some edges. Without loss of generality, let  $H_1 = K_{4,5} \setminus x_1$  and  $H_2 = K_{4,5} \setminus y_1$ . Both  $H_1$  and  $H_2$  are subgraphs of  $K_{4,4}^{+3,2}$ . Therefore,  $H_1$  and  $H_2$  are also  $V_8^{--}$ -minor-free. A contradiction. ■

Let  $G$  be a graph. A *covering* of  $G$  is a subset  $C(G)$  of  $V(G)$  such that every edge of  $G$  has at least one end in  $C(G)$ . A covering  $C(G)$  is a *minimum covering* if  $G$  has no covering  $C'(G)$  with  $|C'(G)| < |C(G)|$ . If there is no contradiction, we can omit the letter  $G$ , use  $C$  instead of  $C(G)$ .

**Lemma 6.** Both  $K_{4,4}^{+3,1}$  and  $K_{4,4}^{+3,2}$  are  $V_8^{--}$ -minor-free.

**Proof.** According to Lemma 5,  $K_{4,4}^{+3,2}$  is  $V_8^{--}$ -minor-free, since  $K_{4,4}^{+3,2}$  is a minor of  $K_{4,5}$ .

Next, we prove that  $K_{4,4}^{+3,1}$  is  $V_8^{--}$ -minor-free. Suppose  $V_8^{--}$  is a minor of  $K_{4,4}^{+3,1}$ . As  $|V_8^{--}| = |K_{4,4}^{+3,1}|$ ,  $V_8^{--}$  can be obtained from  $K_{4,4}^{+3,1}$  by deleting some edges. Let  $T = \{t_1, t_2, t_3, t_4\}$  be a set of four vertices that are incident to all edges of  $K_{4,4}^{+3,1}$  and let  $t_4$  be the vertex of degree four. We can observe that the minimum covering in  $K_{4,4}^{+3,1} - t_4$  is  $C = T \setminus \{t_4\}$ , and  $|C| = 3$ . Suppose  $V_8^{--}$  is obtained by deleting two edges that are incident with  $t_4$  and deleting other edges

in  $K_{4,4}^{+3,1}$ . Let  $t$  be the vertex of degree two in  $V_8^{--}$  that corresponds to  $t_4$  in  $K_{4,4}^{+3,1}$ . Then let  $C'$  be the minimum covering in  $V_8^{--} - t$ . We can observe that  $|C'| = 4$ . This is a contradiction. Therefore,  $K_{4,4}^{+3,1}$  is  $V_8^{--}$ -minor-free. ■

Next, we consider graphs in  $\mathcal{K}$ .

**Lemma 7.** *The graphs in  $\mathcal{K}$  are showed in Figure 3.*

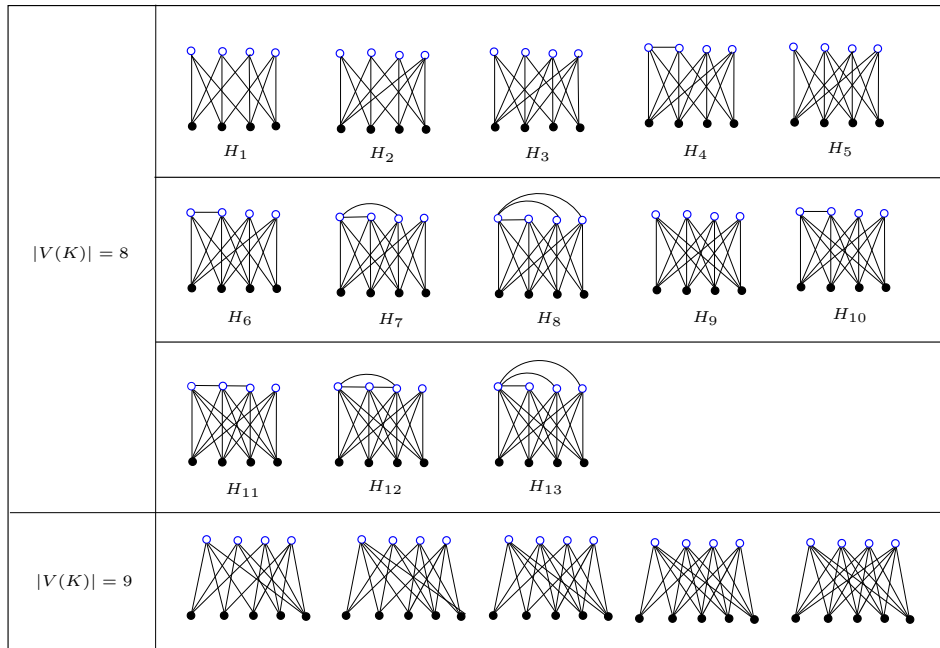


Figure 3. Graphs in  $\mathcal{K}$ .

**Proof.** Let  $K \in \mathcal{K}$ . Then  $K$  is an internally 4-connected minor of  $K_{4,4}^{+3,1}$  or  $K_{4,5}$ . Let  $X$  be a set of four vertices incident to all edges of  $K$ . Let  $Y = V(K) - X$  and let  $Y_3, Y_4$  consist of vertices of  $Y$  of degrees 3 and 4, respectively. Since  $K$  is internally 4-connected, no two vertices in  $Y_3$  have the same neighbors.

If  $|V(K)| = 9$ , then  $K$  is a minor of  $K_{4,5}$  that is obtained by deleting some edges. Note that there are only five possible graphs, since  $|Y_4|$  can be 1, 2, 3, 4, or 5), as illustrated in Figure 3. If  $|Y_4| = 0$ , the resulting graph is not internally 4-connected.

Suppose  $|V(K)| = 8$ . Then  $K$  is a minor of  $K_{4,4}^{+3,1}$  by deleting edges. Note that for  $H_i$  ( $i = 1, 2, \dots, 13$ ), adding any edge will lead the resulting graph not to be internally 4-connected. If  $|Y_3| = 4$ ,  $K$  must be  $H_1$ . If  $|Y_3| = 3$ ,  $K$  is the  $H_2$ . Suppose  $|Y_3| = 2$ , then  $K$  is either  $H_3$  or  $H_4$ . If  $|Y_3| = 1$ ,  $K$  is  $H_5, H_6, H_7$

or  $H_8$ . If  $|Y_3| = 0$ ,  $K$  is  $H_9$ ,  $H_{10}$ ,  $H_{11}$ ,  $H_{12}$  or  $H_{13}$ . Therefore, there are thirteen graphs on eight vertices and five graphs on nine vertices in  $\mathcal{K}$ . ■

**Lemma 8.** *Every graph in  $\mathcal{K}$  is  $V_8^{--}$ -minor-free.*

**Proof.** Let  $K \in \mathcal{K}$ . Then  $K$  is an internally 4-connected minor of  $K_{4,4}^{+3,1}$  or  $K_{4,5}$ . According to Lemma 5 and Lemma 6, both  $K_{4,4}^{+3,1}$  and  $K_{4,5}$  are  $V_8^{--}$ -minor-free. Therefore,  $K$  is also  $V_8^{--}$ -minor-free. ■

**Lemma 9.** *Let  $G$  be an internally 4-connected  $V_8^{--}$ -minor-free planar graph. Then  $G$  is isomorphic to  $C_6^2$ ,  $DW_5$  or  $AW_6$ .*

**Proof.** Let  $G$  be an internally 4-connected  $V_8^{--}$ -minor-free planar graph. Suppose with contradiction that  $G$  is not isomorphic to  $C_6^2$ ,  $DW_5$  or  $AW_6$ . According to Theorem 3, there is a series of internally 4-connected graphs  $H_0, H_1, H_2, \dots, H_k$  such that  $H_0$  is isomorphic to  $K_{3,3}$ , terrahawk,  $C_n^2$  ( $n \geq 5$ ), or  $AW_{2n}$  ( $n \geq 3$ ),  $H_k$  is isomorphic to  $G$  and  $H_i$  ( $i = 2, \dots, k$ ) is obtained from  $H_{i-1}$  by adding edges and splitting vertices at most three times.

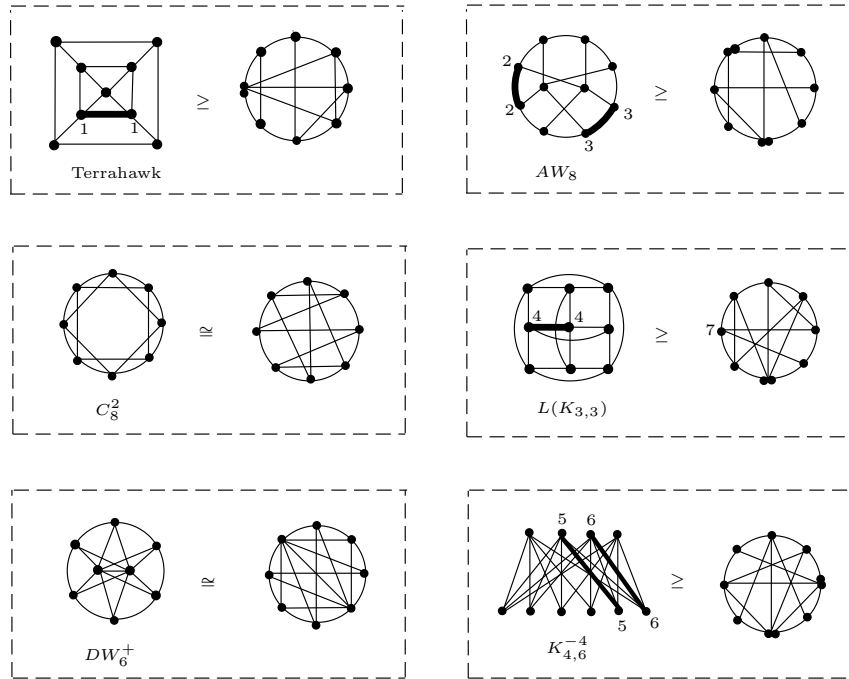
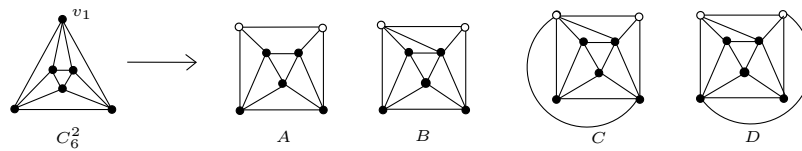
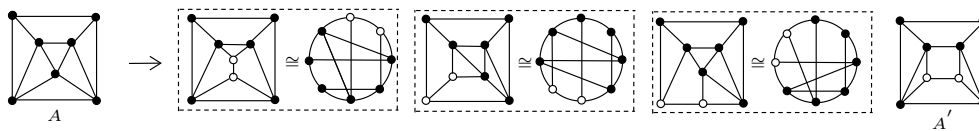
As illustrated in Figure 4, note that terrahawk contains a  $V_8^{--}$ -minor (by contracting the thick edge labeled 1), and so does  $AW_8$  (by contracting the thick edges labeled 2 and 3). And  $C_8^2$  also contains a  $V_8^{--}$ -minor, see Figure 4.

Both  $AW_{2n}$  ( $n \geq 4$ ) and  $C_{2n}^2$  ( $n \geq 4$ ) contain a  $V_8^{--}$ -minor, since they contain  $AW_8$  and  $C_8^2$  as a minor, respectively. Neither  $C_{2n+1}^2$  ( $n \geq 2$ ) nor  $K_{3,3}$  is a planar graph, because they contain  $K_5$  and  $K_{3,3}$  as a minor, respectively. Therefore, we only need to consider that  $H_0$  is isomorphic to  $C_6^2$  or  $AW_6$ .

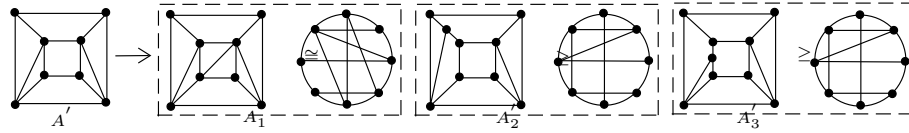
*Case 1.*  $H_0$  is isomorphic to  $C_6^2$ . Then  $H_1$  is obtained from  $C_6^2$  by splitting vertices and adding edges. Since adding any nonadjacent edge to  $C_6^2$  will generate a nonplanar graph, we only consider the planar splits of  $C_6^2$ . (The process can also be seen in [1]). Note that  $C_6^2$  is a vertex-transitive graph. Without loss of generality, we assume that the first vertex we split in  $C_6^2$  is  $v_1$ . Up to symmetry, there are only four planar splits  $A$ ,  $B$ ,  $C$  and  $D$ , as illustrated in Figure 5. And  $D$  is isomorphic to  $DW_5$ , which is the only internally 4-connected planar graph with seven vertices [11]. Therefore,  $H_1$  is constructed from  $A$ ,  $B$ ,  $C$  or  $D$  by splitting vertices at least once and adding edges.

*Case 1.1.*  $H_1$  is constructed from  $A$ . We only consider the planar splits of  $A$ . Firstly, considering the cases that both the two new vertices have degree three, because other planar splits contain these special splits. Up to symmetry, there are four such splits, the first three cases contain a  $V_8^{--}$ -minor and the last one  $A'$  is  $V_8^{--}$ -minor-free, as illustrated in Figure 6.

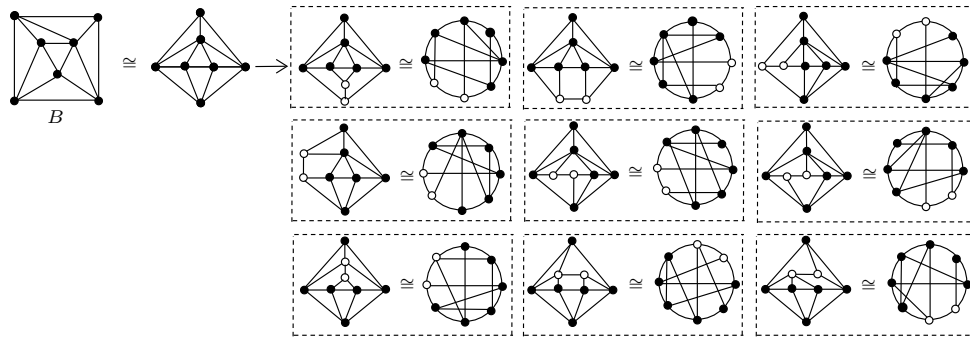
If we add any edge to  $A'$ , the resulting graph  $A'_1$  will contain a  $V_8^{--}$ -minor, see Figure 7. And every planar split of  $A'$  contains a  $V_8^{--}$ -minor since it contains

Figure 4. Graphs which have a  $V_8^{--}$ -minor.Figure 5. Four planar splits of  $C_6^2$ .Figure 6. Four planar splits of  $A$ .

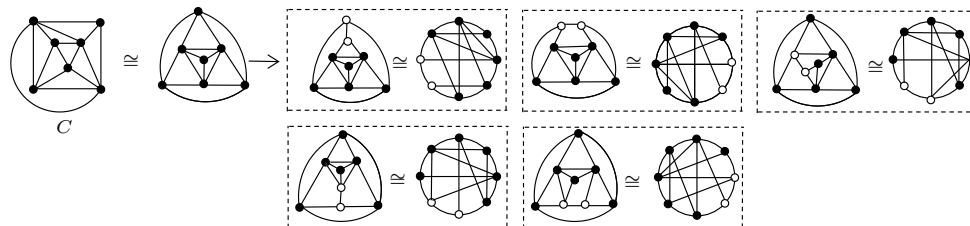


Figure 7. Graphs generated from  $A'$ .

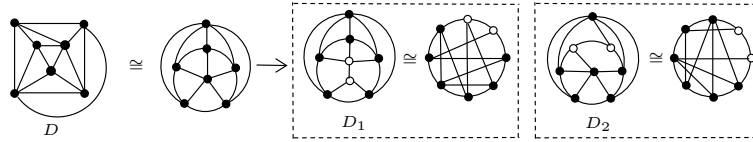
planar split  $A'_2$  or  $A'_3$  as a minor that both new vertices have degree three, as shown in Figure 7.

Figure 8. Nine planar splits of  $B$ .

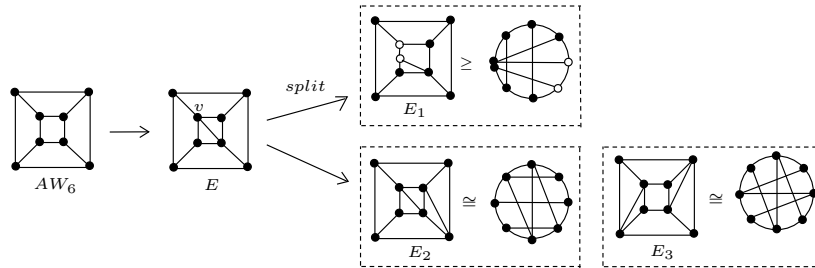
*Case 1.2.*  $H_1$  is constructed from  $B$ . We consider the planar splits of  $B$ . Every planar split of  $B$  contains a  $V_8^{--}$ -minor, since it contains one of the nine graphs as a minor (see Figure 8), which contains a  $V_8^{--}$ -minor.

Figure 9. Five planar splits of  $C$ .

*Case 1.3.*  $H_1$  is constructed from  $C$ . Similarly, for graph  $C$ , every planar split of  $C$  contains a  $V_8^{--}$ -minor since it contains a planar split as shown in Figure 9, which contains a  $V_8^{--}$ -minor.

Figure 10. Two planar splits of  $D$ .

*Case 1.4.*  $H_1$  is constructed from  $D$ . For graph  $D$ , we first consider splitting a degree-5 vertex of  $D$ . Suppose one new vertex of degree three and the other vertex of degree four. Up to symmetry, there is exactly one such planar split  $D_1$ , which contains a  $V_8^{--}$ -minor, as illustrated in Figure 10. Then every graph obtained from  $D$  by splitting a degree-5 vertex will have a  $V_8^{--}$ -minor, since it contains  $D_1$  as a minor. Next, having a planar split of a vertex with degree four in  $D$ . Then every such planar split of  $D$  contains a  $V_8^{--}$ -minor since it contains the planar split  $D_2$  as a minor, that both new vertices have degree three, as shown in Figure 10.

Figure 11. The graphs generated from  $AW_6$ .

*Case 2.*  $H_0$  is isomorphic to  $AW_6$ . Since  $AW_6$  is a cubic graph, the vertex in  $AW_6$  cannot be split. Note that  $AW_6$  is also a vertex-transitive and edge-transitive graph. Let  $E$  be the graph obtained from  $AW_6$  by adding a nonadjacent edge  $e_1$ , as illustrated in Figure 11. Note that  $E$  is not an internally 4-connected graph and is  $V_8^{--}$ -minor-free. Then having a planar split of  $v$ , a vertex of degree four in  $E$ , and every graph will contain a  $V_8^{--}$ -minor since it contains a planar split  $E_1$  as a minor, that both new vertices have degree three, as shown in Figure 11. If we continue adding edges to  $E$  and splitting vertices at least once, then the resulting graph will also contain  $E_1$  as a minor. Therefore, the resulting graph has a  $V_8^{--}$ -minor. Next, we only consider adding edges to  $E$ . The graph generated from  $E$  either is not an internally 4-connected graph, or contains  $E_2$  or  $E_3$  as a minor, see Figure 11.

According to all of the above, we know that  $H_1$  contains a  $V_8^{--}$ -minor, and so does  $G$ . A contradiction. Therefore,  $G$  is isomorphic to  $C_6^2$ ,  $DW_5$  or  $AW_6$ . ■

**Proof of Theorem 1.** Let  $G$  be an internally 4-connected graph. We can observe that every  $G$  with seven or fewer vertices is  $V_8^{--}$ -minor-free. According to Lemma 8, if  $G \in \mathcal{K}$ ,  $G$  is also  $V_8^{--}$ -minor-free.

For the necessity,  $V_8$  contains  $V_8^{--}$  as a subgraph, so all internally 4-connected  $V_8^{--}$ -minor-free graphs must be  $V_8$ -minor-free graphs as described in Theorem 2. We must decide which of those graphs are actually  $V_8^{--}$ -minor-free. Let  $G$  be an internally 4-connected  $V_8^{--}$ -minor-free graph. If  $|G| \leq 7$ , then  $G$  is  $V_8^{--}$ -minor-free obviously. Next, suppose that  $|G| \geq 8$ .

*Case 1.*  $G$  is isomorphic to  $L(K_{3,3})$ . Then as shown in Figure 4,  $G$  has a  $V_8^{--}$ -minor by contracting the thick edge labeled 4.

*Case 2.*  $G$  is in  $\mathcal{W}^+$ . We note that  $AW_8$  is a subgraph of  $AW_8^+$ , and it has a  $V_8^{--}$ -minor by contracting the thick edges labeled 2 and 3, as illustrated in Figure 4. Therefore,  $AW_8^+$  contains a  $V_8^{--}$ -minor. And  $DW_6^+$  also has a  $V_8^{--}$ -minor, see Figure 4. Since  $\{DW_n^+ : n \geq 6\}$  and  $\{AW_{2n}^+ : n \geq 4\}$  have  $DW_6^+$  and  $AW_8^+$  as a minor, respectively, they all have  $V_8^{--}$ -minor. Note that  $AW_6^+$  belongs to  $\mathcal{K}$  and is  $V_8^{--}$ -minor-free according to Lemma 4.

*Case 3.*  $G$  is in  $\mathcal{K}_1$ . We claim that  $G \in \mathcal{K}_1$  is  $V_8^{--}$ -minor-free if and only if  $G \in \mathcal{K}$ . If  $G \in \mathcal{K}$ , then  $G \in \mathcal{K}_1$  and is  $V_8^{--}$ -minor-free according to Lemma 7 and Lemma 8. Suppose  $G \in \mathcal{K}_1$ .

If  $|G| \geq 10$ , then  $G$  contains  $K_{4,6}^{-4}$  as a minor, since  $K_{4,6}^{-4}$  is the minimal graph on ten vertices in  $\mathcal{K}_1$ . Note that  $K_{4,6}^{-4}$  contains a  $V_8^{--}$ -minor by contracting the thick edges labeled 5 and 6, see Figure 4. Hence,  $G$  contains a  $V_8^{--}$ -minor.

For  $|G| = 9$ , we consider the graph  $K_{4,5}$ . Adding any edge to the color class of size five in  $K_{4,5}$  will lead the graph not belong to  $\mathcal{K}_1$ , which is a contradiction. If add any edge to the color class of size four in  $K_{4,5}$  will lead a  $V_8^{--}$ -minor. According to Lemma 5,  $K_{4,5}$  is  $V_8^{--}$ -minor-free. Therefore,  $K_{4,5}$  is the maximal  $V_8^{--}$ -minor-free graph with nine vertices in  $\mathcal{K}_1$ . Then  $G$  is a minor of  $K_{4,5}$  and is also  $V_8^{--}$ -minor-free. So  $G \in \mathcal{K}$ .

Suppose  $|G| = 8$ . According to Lemma 6 and the analysis of above, we can similarly prove that  $K_{4,4}^{+3,1}$  and  $K_{4,4}^{+3,2}$ , which belong to  $\mathcal{K}_1$ , are maximal  $V_8^{--}$ -minor-free graphs with eight vertices. Therefore,  $G$  is a minor of  $K_{4,4}^{+3,1}$  or  $K_{4,4}^{+3,2}$  and is also  $V_8^{--}$ -minor-free. So  $G$  with eight vertices belongs to  $\mathcal{K}$ .

*Case 4.*  $G$  is planar. According to Lemma 9,  $G$  is isomorphic to  $AW_6$ , which belongs to  $\mathcal{K}$ . ■

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