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HOP DOMINATION IN CHORDAL BIPARTITE GRAPHS

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Abstract

In a graph G, a vertex is said to 2-step dominate itself and all the vertices which are at distance 2 from it in G. A set D of vertices in G is called a hop dominating set of G if every vertex outside D is 2-step dominated by some vertex of D. Given a graph G and a positive integer k, the hop domination problem is to decide whether G has a hop dominating set of cardinality at most k. The hop domination problem is known to be NP-complete for bipartite graphs. In this paper, we design a linear time algorithm for computing a minimum hop dominating set in chordal bipartite graphs.

Keywords: domination, hop domination, polynomial time algorithm, chordal bipartite graphs.

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1. Introduction

A set $D \subseteq V$ of a graph $G = (V, E)$ is a *dominating set* of G if every vertex in $V \setminus D$ is adjacent to a vertex in D. The *domination number*, $\gamma(G)$, is the minimum cardinality of a dominating set of G. The notion of domination and its variations in graphs has been studied a great deal; a rough estimate says that it occurs in more than 6000 papers to date. We refer the reader to the two so-called domination books by Haynes, Hedetniemi, and Slater [11, 12] for fundamental concepts in domination in graphs. The *distance* between two vertices x and y in a connected graph G, denoted $d_G(x, y)$, is the length of the shortest x, y-path in G. For an integer $k \geq 1$, a vertex in a graph G is said to *k-step dominate* itself and all the vertices that are at distance exactly k apart from it. A set $D \subseteq V$ of a graph $G = (V, E)$ is a *k-step dominating set* of G if every vertex in V is k-step dominated by some vertex of D. The *k-step domination number*, $\gamma_{\rm kstep}(G)$, of G, is the minimum cardinality of a k-step dominating set of G. In 1995 Chartrand *et al.* [6] initiated the concept of 2-step domination in graphs, which was subsequently studied in [4, 9, 15].

The hop domination in graphs is closely related to the 2-step domination number. The concept of hop domination in graphs was introduced by Ayyaswamy and Natarajan [1]. A set $D \subseteq V$ of a graph $G = (V, E)$ is a *hop dominating* set of G if every vertex of $V \setminus D$ is 2-step dominated by some vertex of D. The minimum cardinality of a hop dominating set of a graph G is called the *hop domination number* of G and is denoted by $\gamma_{\rm h}(G)$.

Natarajan and Ayyaswamy [19] studied when the hop domination number is equal to other domination parameters. In [20], they also obtained an upper bound on hop domination number of the subdivision graph of any connected graph G. Ayyaswamy *et al.* [2] established upper and lower bounds on the hop domination number of a tree together with the characterization of extremal trees. Natarajan *et al.* [21] determined the hop domination number in some special family of graphs. Pabilona and Rara [24] characterized the connected hop dominating set in graphs under some binary operations and calculated the connected hop domination number of those graphs. Rakim *et al.* [26] studied the concept of perfect hop domination in graphs and determined the perfect hop domination number in some graph classes. Henning and Rad [13] presented probabilistic upper bounds for the hop domination number of a graph.

Given a graph G and a positive integer k, the hop domination problem is to decide whether G has a hop dominating set of cardinality at most k. Henning and Rad [13] proved that the hop domination problem is NP-complete for planar bipartite graphs and planar chordal graphs. Later, Jalalvand and Rad [16] determined the complexity results on k-step and k-hop dominating sets in graphs. Henning *et al.* [14] presented some hardness results on the hop domination problem and designed a linear time algorithm to compute a minimum hop dominating set in bipartite permutation graphs. Chen and Wang [7] investigated the relationship between the total domination number and the hop domination number in diamond-free graphs. Kundu and Majumder [17] gave a linear time algorithm to compute an optimal k-hop dominating set of a tree for $k \geq 1$.

A *chord* of a cycle is an edge joining two nonconsecutive vertices of the cycle. A bipartite graph G is called a *chordal bipartite* graph if every cycle of length at least 6 has a chord. Most domination problems and their variations are NP-hard for chordal bipartite graphs, as illustrated in Table 1 where we consider fundamental domination type parameters including domination, total domination, independent domination, connected domination, locating-domination, locating-total domination, and paired-domination. In Table 1, we have taken the decision versions of the variations of the domination problems.

Name of the problem	Complexity Status
Domination	NP-complete [18]
Total domination	Polynomial [8, 23]
Locating-domination	NP-complete [10]
Locating-total domination	NP-complete [25]
Connected domination	NP-complete [18]
Independent Domination	NP-complete [8]
Paired-domination	Polynomial [22]

Table 1. Complexities of variations of domination problems in chordal bipartite graphs.

Chordal bipartite graphs are characterized in terms of weak elimination orderings [27] and strong $\mathcal{T}$-elimination orderings [5]. Given a weak elimination ordering of a chordal bipartite graph G, a strong $\mathcal{T}$-elimination ordering of G can be computed in linear time [23]. In this paper, given a weak elimination ordering of a chordal bipartite graph, we present a linear time algorithm to compute a minimum hop dominating set of the chordal bipartite graph.

2. Terminology and Notation

We use the standard notation $[k] = \{1, \ldots, k\}$. Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n(G) = |V(G)|$ and the size of G is $m(G) = |E(G)|$. Two vertices x and y in G are *adjacent* if they are joined by an edge e, that is, if $uv \in E(G)$. Two vertices in a graph G are *independent* if they are not adjacent. A set of pairwise independent vertices in G is an *independent set* of G. The *open neighborhood* of a vertex v in G

is the set $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood* of v is $N_G[v] = \{v\} \cup N_G(v)$. The *degree* of a vertex v is $|N_G(v)|$ and is denoted by $d_G(v)$. We simply use $N(v)$ and $N[v]$ if the context of the graph is clear. A vertex is *isolated* if the degree of the vertex is 0 and is *pendant* if the degree of the vertex is 1. For a set A of vertices in G, the subgraph of G induced by A is denoted by $G[A]$. The *distance* between two vertices x and y in a connected graph G, denoted by $d_G(x, y)$, is the length of the shortest x, y- path in G. For a vertex v in G, we define $SN(v)$ as the set of vertices at distance exactly 2 from v in G, i.e., $SN(v) = \{u \mid d_G(u, v) = 2\}$, and $SN[v] = SN(v) \cup \{v\}$. A vertex u in a graph G is said to 2*-step dominate* itself and all the vertices that are at distance exactly 2 from u.

A *walk* in a graph is a sequence of vertices in which consecutive vertices are adjacent. A *path* is a walk in which all the vertices are different, while a *cycle* is a walk whose first and last vertex are the same and all other vertices are distinct. A *chord* in a cycle is an edge between two nonconsecutive vertices in the cycle. A graph G is *bipartite* if $V(G)$ can be partitioned into two independent sets X and Y such that every edge joins a vertex in X to a vertex in Y. The partition (X, Y) of $V(G)$ is called a *bipartition* of G. A bipartite graph G with bipartition (X, Y) and edge set $E(G)$ is denoted by $G = (X, Y, E)$. A bipartite graph $G = (X, Y, E)$ is a *complete bipartite graph* if every vertex of X is adjacent to every vertex of Y. For a bipartite graph $G = (X, Y, E)$, we use the notation $n_x = |X|$ and $n_y = |Y|$. A graph G is said to be a *chordal bipartite graph* if G is bipartite and every cycle of length at least 6 has a chord. Chordal bipartite graphs form a subclass of bipartite graphs and a superclass of bipartite permutation graphs [3].

A vertex v of a graph G is called a *weak simplicial vertex* if $N_G(v)$ is an independent set of G and for every $u_1, u_2 \in N_G(v)$, either $N_G(u_1) \subseteq N_G(u_2)$ or $N_G(u_2) \subseteq N_G(u_1)$. An ordering $\sigma = (v_1, v_2, \ldots, v_n)$ of the vertices of G is called a *weak elimination ordering* of G if for every $i \in [n]$, v_i is weak simplicial in $G_i = G[\{v_i, v_{i+1}, \ldots, v_n\}]$ and for every $v_j, v_k \in N_{G_i}(v_i)$ with $j < k$, $N_{G_i}(v_j) \subseteq N_{G_i}(v_k)$.

Let $G = (X, Y, E)$ be a bipartite graph, and let $\alpha = (x_1, x_2 \ldots, x_{n_x})$ and $\beta = (y_1, y_2, \ldots, y_{n_y})$ be some orderings of X and Y, respectively. The ordering α and β is called a *strong $\mathcal{T}$-elimination ordering* of G if for each $i \in [n_y]$ and $j, k \in [n_x]$ with $j < k$, where $x_j, x_k \in N_G(y_i)$, we have that $N_{G'}(x_j) \subseteq N_{G'}(x_k)$, where $G' = G\left[\{y_i, y_{i+1}, \ldots, y_{n_y}\} \cup \{x_1, x_2, \ldots, x_{n_x}\}\right]$.

Chordal bipartite graphs are characterized in terms of a weak elimination ordering [27] and are also characterized in terms of a strong $\mathcal{T}$-elimination ordering [5]. Given a chordal bipartite graph $G = (V, E)$, a weak elimination ordering of G can be computed in $O(\min\{m \log n, n^2\})$ time [27].

For notational convenience, for a given set $X = \{x_1, x_2 \ldots, x_{n_x}\}$, we denote X_i as the set $\{x_i, x_{i+1}, \ldots, x_{n_x}\}$ for every $i \in [n_x]$. Similarly given $Y =$

$\{y_1, y_2, \ldots, y_{n_y}\}$, we denote Y_i as the set $\{y_i, y_{i+1}, \ldots, y_{n_y}\}$ for every $i \in [n_y]$. The following relation between a weak elimination ordering and a strong $\mathcal{T}$-elimination ordering of a chordal bipartite graph $G = (X, Y, E)$ is established in [23].

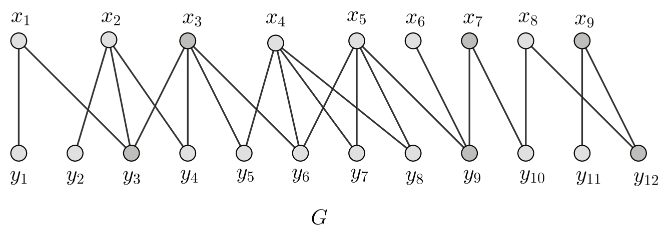

Figure 1. A chordal bipartite graph G.

Theorem 1 [23]**.** *Given a weak elimination ordering σ of a chordal bipartite graph $G = (X, Y, E)$, a strong $\mathcal{T}$-elimination ordering $\sigma_X = (x_1, x_2, \ldots, x_{n_x})$ and $\sigma_Y = (y_1, y_2, \ldots, y_{n_y})$ of G can be obtained in $O(n)$ time such that*

(a) *for each $i \in [n_x]$, we have $N_{G'}(y_j) \subseteq N_{G'}(y_k)$, where $G' = G[X_i \cup Y]$ and $y_j, y_k \in N_{G'}(x_i)$ with $j < k$;*

(b) *for each $i \in [n_y]$, we have $N_{G''}(x_j) \subseteq N_{G''}(x_k)$, where $G'' = G[X \cup Y_i]$ and $x_j, x_k \in N_{G''}(y_i)$ with $j < k$.*

Let $\sigma_X = (x_1, x_2, \ldots, x_{n_x})$ and $\sigma_Y = (y_1, y_2, \ldots, y_{n_y})$ be a strong $\mathcal{T}$-elimination ordering of G and let $y_j \in Y$. Let $i = \max\{k \mid y_j x_k \in E(G)\}$. Then it can be observed that y_j is a pendant vertex of the graph $G' = G[X_i \cup Y]$. Therefore, we have the following lemma.

Lemma 2. *If $\sigma_X = (x_1, x_2, \ldots, x_{n_x})$ and $\sigma_Y = (y_1, y_2, \ldots, y_{n_y})$ is a strong $\mathcal{T}$-elimination ordering of G and $y_j \in Y$, then there exists $i \in [n_x]$ such that y_j is a pendant vertex in $G' = G[X_i \cup Y]$.*

3. Hop Domination in Chordal Bipartite Graphs

In this section, we present a polynomial time algorithm for computing a minimum hop dominating set in chordal bipartite graphs. Given a weak elimination ordering of a chordal bipartite graph $G = (X, Y, E)$ of order n and size m, our algorithm takes $O(n + m)$ time to compute a minimum hop dominating set of G. If G is a disconnected graph having components $G_1, G_2, \ldots, G_r$ where $r \geq 2$, then $\gamma_{\mathrm{h}}(G) = \sum_{i=1}^{r} \gamma_{\mathrm{h}}(G_i)$. Hence it suffices for us to consider only connected chordal bipartite graphs for the purpose of designing our algorithm.

Let $n_x = |X|$ and $n_y = |Y|$. For $i \in [n_x]$ and for a vertex $x_b \in X$, we use the notation $SN_i(x_b) = X_i \cap \{x_c \in X \mid d_G(x_c, x_b) = 2\}$ and $SN_i[x_b] = SN_i(x_b) \cup \{x_b\}$. Similarly, for $i \in [n_y]$ and for a vertex $y_a \in Y$, we use the notation $SN_i(y_a) = Y_i \cap \{y_c \in Y \mid d_G(y_c, y_a) = 2\}$ and $SN_i[y_a] = SN_i(y_a) \cup \{y_a\}$.

Lemma 3. *Let* $\sigma_X = (x_1, x_2, \ldots, x_{n_x})$ *and* $\sigma_Y = (y_1, y_2, \ldots, y_{n_y})$ *be a strong* $\mathcal{T}$*-elimination ordering of a connected chordal bipartite graph* $G = (X, Y, E)$ *and* $x_p, x_i \in X$ *such that* $d_G(x_p, x_i) = 2$*. If* $a = \max\{k \mid x_i y_k \in E\}$ *and* $b = \max\{k \mid y_a x_k \in E\}$*, then* $SN_i(x_p) \subseteq SN_i[x_b]$*.*

Proof. Let $x_{p'} \in SN_i(x_p)$ be arbitrary. By the definition of b, we have $b \geq i$. If $p' = i$, then it is clear that $x_{p'} = x_i \in SN_i[x_b]$. If $p' = b$, then $x_{p'} = x_b \in SN_i[x_b]$. So assume that $p' \neq i$ and $p' \neq b$. Since $d_G(x_p, x_i) = 2$ and $d_G(x_p, x_{p'}) = 2$, there exist vertices y_q and $y_{q'}$ such that $y_q \in N(x_p) \cap N(x_i)$ and $y_{q'} \in N(x_p) \cap N(x_{p'})$, respectively. If $q' = a$, then $d_G(x_{p'}, x_b) = d_G(x_{p'}, y_{q'}) + d_G(y_{q'}, x_b) = d_G(x_{p'}, y_{q'}) + d_G(y_a, x_b) = 2$. So $x_{p'} \in SN_i[x_b]$ and thus we are done. Hence we may assume that $q' \neq a$. If $q' = q$, then $y_{q'} \in N(x_i)$; thus $q' < a$ by the definition of a. Since σ_X and σ_Y form a strong $\mathcal{T}$-elimination ordering of G, by Theorem 1(a), $x_{p'} \in N_{G'}(y_{q'}) \subseteq N_{G'}(y_a)$, where $G' = G[X_i \cup Y]$. Now $d_G(x_{p'}, x_b) = d_G(x_{p'}, y_a) + d_G(y_a, x_b) = 2$. So $x_{p'} \in SN_i[x_b]$ and thus we are done. Hence we may assume that $q \neq q'$. Let $G' = G[X_i \cup Y]$ and $G'' = G[X_p \cup Y]$.

Assume that $p < i$. We now prove that $x_{p'} \in N(y_a)$. If $q < q'$, then, since $y_q, y_{q'} \in N_{G''}(x_p)$, by Theorem 1(a), $x_i \in N_{G''}(y_q) \subseteq N_{G_p}(y_{q'})$. Now $y_{q'}, y_a \in N_{G'}(x_i)$. By Theorem 1(a), $x_{p'} \in N_{G'}(y_{q'}) \subseteq N_{G'}(y_a)$. If $q' < q$, then by Theorem 1(a), $x_{p'} \in N_{G''}(y_{q'}) \subseteq N_{G''}(y_q)$. Since $y_q, y_a \in N(x_i)$, by Theorem 1(a), $x_{p'} \in N_{G'}(y_q) \subseteq N_{G'}(y_a)$. Now $d_G(x_{p'}, x_b) = d_G(x_{p'}, y_a) + d_G(y_a, x_b) = 2$. This implies that $x_{p'} \in SN_i[x_b]$.

Now assume that $p > i$. If $p = b$, then $d_G(x_{p'}, x_p) = 2 = d_G(x_{p'}, x_b)$ and hence $x_{p'} \in SN_i[x_b]$. So assume that $p \neq b$. Recall that $p' \neq b$ and $p' \neq i$. We now prove that $x_{p'} \in N(y_a)$. If $q < q'$, then, since $y_q, y_a \in N(x_i)$, by Theorem 1(a), $x_p \in N_{G'}(y_q) \subseteq N_{G'}(y_a)$. Now $y_{q'}, y_a \in N(x_p)$, and so by Theorem 1(a), $x_{p'} \in N_{G''}(y_{q'}) \subseteq N_{G''}(y_a)$. This implies that $x_{p'} \in SN_i(x_b)$. If $q' < q$, then $q' < q \leq a$, by definition of a. If $q = a$, then $x_p \in N(y_a)$. If $q < a$, then since $y_q, y_a \in N(x_i)$, by Theorem 1(a), $x_p \in N_{G'}(y_q) \subseteq N_{G'}(y_a)$. Now $y_{q'}, y_a \in N(x_p)$ with $q' < q \leq a$. Thus, by Theorem 1(a), $x_{p'} \in N_{G''}(y_{q'}) \subseteq N_{G''}(y_a)$. Now $d_G(x_{p'}, x_b) = d_G(x_{p'}, y_a) + d_G(y_a, x_b) = 2$. This implies that $x_{p'} \in SN_i[x_b]$. ■

Similar to Lemma 3, the following lemma can also be proved.

Lemma 4. *Let* $\sigma_X = (x_1, x_2, \ldots, x_{n_x})$ *and* $\sigma_Y = (y_1, y_2, \ldots, y_{n_y})$ *be a strong* $\mathcal{T}$*-elimination ordering of a connected chordal bipartite graph* $G = (X, Y, E)$ *and* $y_p, y_i \in Y$ *such that* $d_G(y_p, y_i) = 2$*. If* $a = \max\{k \mid y_i x_k \in E\}$ *and* $b = \max\{k \mid x_a y_k \in E\}$*, then* $SN_i(y_p) \subseteq SN_i[y_b]$*.*

Now we present our algorithm, namely HDS-CBG(G) to compute a minimum hop dominating set in a given connected chordal bipartite graph G. The algorithm processes the vertices $x_1, x_2, \ldots, x_{n_x}$ with respect to a strong $\mathcal{T}$-elimination ordering $\sigma_X = (x_1, x_2, \ldots, x_{n_x})$ and $\sigma_Y = (y_1, y_2, \ldots, y_{n_y})$ of G and at each iteration $i \in [n_x]$, our algorithm selects new vertices to add to the set HD if x_i is not 2-step dominated or there is a pendant vertex $v \in N(x_i)$ in G', where $G' = G[X_i \cup Y]$ such that v is not 2-step dominated by the set HD constructed so far. To achieve this, at each iteration $i \in [n_x]$, our algorithm proceeds as follows.

- An array D is maintained on the vertices of G to track whether a vertex v of G is 2-step dominated or not by the set HD constructed so far. In particular, $D[v] = 1$ if v is 2-step dominated by HD; otherwise $D[v] = 0$. Initially, $D[v] = 0$ for all $v \in V(G)$.
- In the Lines 6–9, the algorithm checks whether the vertices of the set $N[x_i]$ in G' are 2-step dominated or not by the set HD constructed so far.
- In the Lines 10-18, the algorithm checks the two conditions (a) $D[x_i] = 0$ or not, and (b) $N_{G'}(x_i)$ has a pendant vertex v such that $D[v] = 0$, and adds new vertices to the set HD following some rules. All details are described in the algorithm.

For every $i \in [n_x]$, we define $\ell(i)$ and $\rho(i)$ as follows.

- $\ell(i) = \max\{k \mid x_i y_k \in E(G)\}$ and $\rho(i) = \max\{k \mid x_k y_{\ell(i)} \in E(G)\}$.

In Table 2, we explain the execution of the algorithm HDS-CBG(G) on the chordal bipartite graph G shown in Figure 1. Note that in Table 2, we have considered those iterations of the algorithm in which some vertices are selected. In Table 2, we have a column, namely "**The set** HD". Initially, HD $= \emptyset$. Then the updated set HD is described for different iterations.

For each $i \in [n_x]$, let HD_i be the set HD computed at the end of the i-th iteration of the algorithm. The following lemmas can be observed from the algorithm HDS-CBG(G).

Lemma 5. *If x_i is the considered vertex for $i \in [n_x]$ at some point of the algorithm* HDS-CBG(G)*, then the following are true.*

(a) $D[v] = 1$ *for all* $v \in \bigcup_{s \in [i-1]} N[x_s]$.

(b) *If $x_i \notin \mathrm{HD}_{i-1}$ and x_i has a neighbor v such that $N(v) \cap \mathrm{HD}_{i-1} \neq \emptyset$, then x_i is 2-step dominated by* HD_{i-1}.

(c) *If $N(x_i) \cap \mathrm{HD}_{i-1} \neq \emptyset$, then every $v \in N(x_i)$ is 2-step dominated by* HD_{i-1}.

Notice that $\mathrm{HD}_0 = \emptyset$. At the termination of the algorithm HDS-CBG(G), by Lemma 5, HD_{n_x} is a hop dominating set of G. Therefore, to prove that HD_{n_x} is a minimum hop dominating set of G, it is sufficient to prove that there is a minimum hop dominating set S^* of G such that $\mathrm{HD}_{n_x} \subseteq S^*$. To prove this, we

Algorithm 1: HDS-CBG(G)

Input: A connected chordal bipartite graph $G = (X, Y, E)$, where $|X| = n_x$ with a weak elimination ordering of G;

Output: A hop dominating set HD of G;

Compute $\sigma_X = (x_1, x_2, \ldots, x_{n_x})$ and $\sigma_Y = (y_1, y_2, \ldots, y_{n_y})$, a strong $\mathcal{T}$-elimination ordering of G;

$D[v] = 0$ for all $v \in X \cup Y$;

Initialize $\mathrm{HD} = \emptyset$;

for ($i = 1$ **to** n_x) **do**

  Let $G' = G[X_i \cup Y]$;

  if ($x_i \notin \mathrm{HD}$ *and* x_i *has a neighbor* v *such that* $N(v) \cap \mathrm{HD} \neq \emptyset$) **then** `/* Case 1 */`

    $D[x_i] = 1$;

  if ($N(x_i) \cap \mathrm{HD} \neq \emptyset$) **then** `/* Case 2 */`

    $D[u] = 1$ for all $u \in N(x_i)$;

  if ($D[x_i] = 0$ *and* $N_{G'}(x_i)$ *has a pendant vertex* v *such that* $D[v] = 0$) **then** `/* Case 3 */`

    $\mathrm{HD} = \mathrm{HD} \cup \{x_{\rho(i)}, y_{\ell(i)}\}$;

    $D[u] = 1$ for all $u \in N[x_i]$ and $D[x_{\rho(i)}] = 1$;

  else if ($D[x_i] = 0$ *and* $N_{G'}(x_i)$ *has no pendant vertex* v *such that* $D[v] = 0$) **then** `/* Case 4 */`

    $\mathrm{HD} = \mathrm{HD} \cup \{x_{\rho(i)}\}$;

    $D[x_i] = 1$ and $D[x_{\rho(i)}] = 1$;

  else if ($D[x_i] \neq 0$ *and* $N_{G'}(x_i)$ *has a pendant vertex* v *such that* $D[v] = 0$) **then** `/* Case 5 */`

    $\mathrm{HD} = \mathrm{HD} \cup \{y_{\ell(i)}\}$;

    $D[u] = 1$ for all $u \in N(x_i)$;

return HD;

use induction on i, $i \in [n_x] \cup \{0\}$, and prove that each HD_i, $i \in [n_x] \cup \{0\}$, is contained in some minimum hop dominating set of G. Since $\mathrm{HD}_0 = \emptyset$, the base case is true. Assume that $i \geq 1$ and that the set HD_{i-1} is contained in some minimum hop dominating set S' of G. We now show that HD_i is contained in some minimum hop dominating set of G. For this purpose, we proceed with a series of lemmas. In each lemma, we construct a minimum hop dominating set of G containing HD_i from the minimum hop dominating set S' of G. We recall our earlier notation $\ell(i) = \max\{k \mid x_i y_k \in E(G)\}$ and $\rho(i) = \max\{k \mid x_k y_{\ell(i)} \in E(G)\}$ which will be used in the following lemmas. In each of the lemma, we assume that $\sigma_X = (x_1, x_2, \ldots, x_{n_x})$ and $\sigma_Y = (y_1, y_2, \ldots, y_{n_y})$ is a strong $\mathcal{T}$-elimination ordering of the graph $G = (X, Y, E)$.

Lemma 6. *Let S' be a minimum hop dominating set of G such that $\mathrm{HD}_{i-1} \subseteq S'$ and $G' = G[X_i \cup Y]$. If $D[x_i] = 0$ and G' has a pendant vertex $y_j \in N(x_i)$ such that $D[y_j] = 0$, then there is a minimum hop dominating set of G containing $\mathrm{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}$.*

Proof. Let $x_a \in S'$ be the vertex that 2-step dominates x_i and $y_b \in S'$ be the vertex that 2-step dominates y_j. Since $D[x_i] = 0$ and $D[y_j] = 0$, we note that $x_a, y_b \notin \mathrm{HD}_{i-1}$. We proceed further with proving the following claims.

Claim 7. *If $b \neq j$, then there is a minimum hop dominating set containing $\mathrm{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}$.*

Proof. Since $b \neq j$, we have $d_G(y_j, y_b) = 2$. By Lemma 4, $SN_j(y_b) \subseteq SN_j[y_{\ell(i)}]$. By Lemma 5, the vertices from $\{x_1, x_2 \ldots, x_{i-1}\} \cup \{y_1, y_2, \ldots, y_{j-1}\}$ are 2-step dominated by HD_{i-1}.

If $a = i$ and $\rho(i) = i$, then $(S' \setminus \{y_b\}) \cup \{y_{\ell(i)}\}$ is a minimum hop dominating set of G containing $\mathrm{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}$. If $a = i$ and $\rho(i) \neq i$, then $d_G(x_i, x_{\rho(i)}) = 2$ and by Lemma 3, $SN_i(x_i) \subseteq SN_i[x_{\rho(i)}]$. Thus, $(S' \setminus \{x_a, y_b\}) \cup \{x_{\rho(i)}, y_{\ell(i)}\}$ is a minimum hop dominating set of G containing $\mathrm{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}$.

If $a \neq i$, then $d_G(x_i, x_a) = 2$. By Lemma 3, we have $SN_i(x_a) \subseteq SN_i[x_{\rho(i)}]$, i.e., the vertices of the set $\{x_i, x_{i+1}, \ldots, x_{n_x}\}$ that are 2-step dominated by x_a are 2-step dominated by $x_{\rho(i)}$. If $a < i$, then by Lemma 5, x_a is 2-step dominated by HD_{i-1}. If $a > i$, then let $x_i y_d x_a$ be a shortest path between x_i and x_a. Then $y_c, y_{\ell(i)} \in N_{G'}(x_i)$ with $c \leq \ell(i)$. By Theorem 1(a), $x_a \in N_{G'}(y_c) \subseteq N_{G'}(y_{\ell(i)})$. This implies that $x_a \in SN_i[x_{\rho(i)}]$. Hence, $(S' \setminus \{x_a, y_b\}) \cup \{x_{\rho(i)}, y_{\ell(i)}\}$ is a minimum hop dominating set of G containing $\mathrm{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}$. This completes the proof of Claim 7. □

Claim 8. *If $b = j$, then there is a minimum hop dominating set containing $\mathrm{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}$.*

Itera-tion i	**Consider-ed vertex**	**Conditions**	$x_{\rho(i)}, y_{\ell(i)}$	**Applied Case**	**The set** HD	**Update**
1	x_1	$D[x_1] = 0$ and $N_{G'}(x_1)$ has a pendant vertex y_1 with $D[y_1] = 0$	$x_{\rho(1)} = x_3$ $y_{\ell(1)} = y_3$	**Case 3**	HD = HD $\cup \{x_3, y_3\}$	$D[x_1] = 1$, $D[y_1] = 1$, $D[x_3] = 1$, $D[y_3] = 1$
2	x_2	(i) $x_2 \notin$ HD and x_2 has a neighbor y_3 with $N(y_3) \cap$ HD $= \{x_3\}$ (ii) $N(x_2) \cap$ HD $= \{y_3\}$		(i)**Case 1** (ii)**Case 2**	(i) HD = HD (ii) HD = HD	(i)$D[x_2] = 1$ (ii)$D[y_2] = 1$, $D[y_4] = 1$
3	x_3	$N(x_3) \cap$ HD $= \{y_3\}$		**Case 2**	HD = HD	$D[y_5] = 1$, $D[y_6] = 1$
4	x_4	$x_4 \notin$ HD and x_4 has a neighbor y_5 with $N(y_5) \cap$ HD $= \{x_3\}$		**Case 1**	HD = HD	$D[x_4] = 1$
5	x_5	(i) $x_5 \notin$ HD and x_5 has a neighbor y_6 with $N(y_6) \cap$ HD $= \{x_3\}$ (ii) $D[x_5] = 1$ and $N_{G'}(x_5)$ has a pendant vertex y_7 with $D[y_7] = 0$	(ii)$y_{\ell(5)} = y_9$	(i)**Case 1** (ii)**Case 5**	(i) HD = HD (ii) HD = HD $\cup \{y_9\}$	(i)$D[x_5] = 1$ (ii)$D[y_7] = 1$, $D[y_8] = 1$, $D[y_9] = 1$
6	x_6	$D[x_6] = 0$ and $N_{G'}(x_6)$ has no pendant vertex v with $D[v] = 0$	$x_{\rho(6)} = x_7$	**Case 4**	HD = HD $\cup \{x_7\}$	$D[x_6] = 1$, $D[x_7] = 1$
7	x_7	$N(x_7) \cap$ HD $= \{y_9\}$		**Case 2**	HD = HD	$D[y_{10}] = 1$
8	x_8	$x_8 \notin$ HD and x_8 has a neighbor y_{10} with $N(y_{10}) \cap$ HD $= \{x_7\}$		**Case 1**	HD = HD	$D[x_8] = 1$
9	x_9	$D[x_9] = 0$ and $N_{G'}(x_9)$ has a pendant vertex y_{11} with $D[y_{11}] = 0$	$x_{\rho(9)} = x_9$ $y_{\ell(9)} = y_{12}$	**Case 3**	HD = HD $\cup \{x_9, y_{12}\}$	$D[x_9] = 1$, $D[y_{11}] = 1$, $D[y_{12}] = 1$

Table 2. Illustration of the algorithm HDS-CBG(G) on the graph G shown in Figure 1.

Proof. If $\ell(i) \neq j$, then by Lemma 4, $SN_j(y_b) \subseteq SN_j[y_{\ell(i)}]$. Thus, $(S' \setminus \{y_b\}) \cup \{y_{\ell(i)}\}$ is a minimum hop dominating set of G containing $\mathrm{HD}_{i-1} \cup \{y_{\ell(i)}\}$ in which y_j is 2-step dominated by $y_{\ell(i)}$. Hence, by Claim 7, we obtain a minimum hop dominating set of G containing $\mathrm{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}$. So we may assume that $\ell(i) = j$, implying that $b = \ell(i) = j$. If $a = i$ and $\rho(i) = i$, then S' is the minimum hop dominating set of G containing $\mathrm{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}$. If $a = i$ and $\rho(i) \neq i$, then $d_G(x_i, x_{\rho(i)}) = 2$ and by Lemma 3, $SN_i(x_i) \subseteq SN_i[x_{\rho(i)}]$. So $(S' \setminus \{x_a\}) \cup \{x_{\rho(i)}\}$ is a minimum hop dominating set of G containing $\mathrm{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}$.

If $a \neq i$, then $d_G(x_i, x_a) = 2$. By Lemma 3, we have $SN_i(x_a) \subseteq SN_i[x_{\rho(i)}]$, i.e., the vertices of the set $\{x_i, x_{i+1}, \ldots, x_{n_x}\}$ that are 2-step dominated by x_a are 2-step dominated by $x_{\rho(i)}$. If $a < i$, then by Lemma 5, x_a is 2-step dominated by HD_{i-1}. If $a > i$, then let $x_i y_d x_a$ be a shortest path between x_i and x_a. Then, $y_c, y_{\ell(i)} \in N_{G'}(x_i)$ with $c \leq \ell(i)$. By Theorem 1(a), $x_a \in N_{G'}(y_c) \subseteq N_{G'}(y_{\ell(i)})$. This implies that $x_a \in SN_i[x_{\rho(i)}]$. Therefore, $(S' \setminus \{x_a\}) \cup \{x_{\rho(i)}\}$ is a minimum hop dominating set of G containing $\mathrm{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}$. This completes the proof of Claim 8. □

We now return to the proof of Lemma 6. If $b \neq j$, then by Claim 7, we obtain a minimum hop dominating set of G containing $\mathrm{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}$. If $b = j$, then by Claim 8, we obtain a minimum hop dominating set S'' of G containing $\mathrm{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}$. This completes the proof of Lemma 6. ■

Lemma 9. *Let S' be a minimum hop dominating set of G such that $\mathrm{HD}_{i-1} \subseteq S'$ and $G' = G[X_i \cup Y]$. If $D[x_i] = 0$ and $N_{G'}(x_i)$ has no pendant vertex v such that $D[v] = 0$, then there is a minimum hop dominating set of G containing $\mathrm{HD}_{i-1} \cup \{x_{\rho(i)}\}$.*

Proof. Let $x_a \in S'$ be a vertex that 2-step dominates x_i. Since $D[x_i] = 0$, we have $x_a \notin \mathrm{HD}_{i-1}$. Clearly, by definition of $\rho(i)$, we have $\rho(i) \geq i$.

First assume that $a = i$. If $\rho(i) = i$, then $a = i = \rho(i)$ and hence S' is a minimum hop dominating set of G containing $\mathrm{HD}_{i-1} \cup \{x_{\rho(i)}\}$. If $\rho(i) \neq i$, then $d_G(x_i, x_{\rho(i)}) = 2$ and by Lemma 3, $SN_i(x_a) \subseteq SN_i[x_{\rho(i)}]$. Since by Lemma 5, every vertex of the set $\{x_1, x_2, \ldots, x_{i-1}\} \cup \{y_1, y_2, \ldots, y_{j-1}\}$ is 2-step dominated by HD_{i-1}, $(S' \setminus \{x_a\}) \cup \{x_{\rho(i)}\}$ is a minimum hop dominating set of G containing $\mathrm{HD}_{i-1} \cup \{x_{\rho(i)}\}$.

Now assume $a \neq i$. Then, $d_G(x_i, x_a) = 2$ and by Lemma 3, $SN_i(x_a) \subseteq SN_i[x_{\rho(i)}]$, i.e., the vertices of the set $\{x_i, x_{i+1}, \ldots, x_{n_x}\}$ that are 2-step dominated by x_a are 2-step dominated by $x_{\rho(i)}$. Also by Lemma 5, the vertices from $\{x_1, x_2 \ldots, x_{i-1}\} \cup \{y_1, y_2, \ldots, y_{k-1}\}$ are 2-step dominated by HD_{i-1}, where $k = \min\{d \mid y_d \in N(x_i) \text{ and } D[y_d] = 0\}$. If $a < i$, then by Lemma 5, x_a is 2-step dominated by HD_{i-1}. If $a > i$, then let $x_i y_d x_a$ be a shortest path between x_i and x_a. Then $y_c, y_{\ell(i)} \in N_{G'}(x_i)$ with $c \leq \ell(i)$. By Theorem 1(a), $x_a \in N_{G'}(y_c) \subseteq N_{G'}(y_{\ell(i)})$. This implies that $x_a \in SN_i[x_{\rho(i)}]$. Hence, $(S' \setminus \{x_a\}) \cup \{x_{\rho(i)}\}$ is a hop dominating set of G containing $\mathrm{HD}_{i-1} \cup \{x_{\rho(i)}\}$. ■

Lemma 10. *Let S' be a minimum hop dominating set of G such that $\mathrm{HD}_{i-1} \subseteq S'$ and $G' = G[X_i \cup Y]$. If $D[x_i] \neq 0$ and G' has a pendant vertex $y_j \in N(x_i)$ such that $D[y_j] = 0$, then there is a minimum hop dominating set of G containing $\mathrm{HD}_{i-1} \cup \{y_{\ell(i)}\}$.*

Proof. Let $y_b \in S'$ be a vertex that 2-step dominates y_j. Since $D[y_j] = 0$, we have $y_b \notin \mathrm{HD}_{i-1}$. If $b = j$, then by Claim 8 of Lemma 6, we get a minimum hop dominating set of G containing $\mathrm{HD}_{i-1} \cup \{y_{\ell(i)}\}$. Hence we may assume that $b \neq j$. Thus, $d_G(y_j, y_b) = 2$. By Lemma 4, we have $SN_j(y_b) \subseteq SN_j[y_{\ell(i)}]$, i.e., the vertices of the set $\{y_j, y_{j+1}, \ldots, y_{n_y}\}$ that are 2-step dominated by y_b are now 2-step dominated by $y_{\ell(i)}$. By Lemma 5, all the vertices from $\{x_1, x_2 \ldots, x_{i-1}\} \cup \{y_1, y_2, \ldots, y_{j-1}\}$ are 2-step dominated by HD_{i-1}. If $b < j$, then by Lemma 5, y_b is 2-step dominated by HD_{i-1}. If $b > j$, then, since y_j is a pendant neighbor of x_i, the path $y_j x_i y_b$ is a shortest path between y_j and y_b. Then $y_b, y_{\ell(i)} \in N_{G'}(x_i)$ with $b \leq \ell(i)$. This implies that $y_b \in SN_j[y_{\ell(i)}]$. Again since $D[x_i] \neq 0$, $x_i \in \mathrm{HD}_{i-1}$ or x_i is also 2-step dominated by HD_{i-1}. Hence, $(S' \setminus \{y_b\}) \cup \{y_{\ell(i)}\}$ is a hop dominating set of G containing $\mathrm{HD}_{i-1} \cup \{y_{\ell(i)}\}$. ■

We now return to the proof of the statement that HD_{n_x} is a minimum hop dominating set of the chordal bipartite graph G. Recall that by the induction hypothesis, for $i \geq 1$, HD_{i-1} is contained in a minimum hop dominating set S' of G. Notice that the algorithm considers the vertex x_i and its neighbors at the i-th iteration of the algorithm. Further, the algorithm first updates whether any vertex $N(x_i)$ can be 2-step dominated or not. For this, it checks two conditions. The first condition is whether $N(x_i) \cap \mathrm{HD}_{i-1} \neq \emptyset$, and the second condition is whether x_i has a neighbor v such that $N(v) \cap \mathrm{HD}_{i-1} \neq \emptyset$. If $N(x_i) \cap \mathrm{HD}_{i-1} \neq \emptyset$, say $v \in N(x_i) \cap \mathrm{HD}_{i-1}$, then every vertex of $N(x_i) \setminus \{v\}$ is 2-step dominated by v. So in this case, $\mathrm{HD}_i = \mathrm{HD}_{i-1}$, and hence HD_i is contained in S'. Similarly if x_i has a neighbor v such that $N(v) \cap \mathrm{HD}_{i-1} \neq \emptyset$, then the vertex x_i is 2-step dominated by v. Thus in this case $\mathrm{HD}_i = \mathrm{HD}_{i-1}$, and hence HD_i is contained in S'.

After this, the algorithm checks whether any vertex of $N[x_i]$ is not 2-step dominated by HD_{i-1}. If x_i is not 2-step dominated by HD_{i-1} (i.e., if $D[x_i] = 0$) and $N_{G'}(x)$ has a pendant neighbor v that is not 2-step dominated by HD_{i-1}, then the algorithm selects $x_{\rho(i)}$ and $y_{\ell(i)}$. Thus, $\mathrm{HD}_i = \mathrm{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}$. By Lemma 6, there is a minimum hop dominating set of G containing $\mathrm{HD}_i = \mathrm{HD}_{i-1} \cup \{x_{\rho(i)}, y_{\ell(i)}\}$. If x_i is not 2-step dominated by HD_{i-1} (i.e., if $D[x_i] = 0$) and $N_{G'}(x)$ has no pendant neighbor v that is not 2-step dominated by HD_{i-1}, then the algorithm selects $x_{\rho(i)}$. Thus, $\mathrm{HD}_i = \mathrm{HD}_{i-1} \cup \{x_{\rho(i)}\}$. By Lemma 9, there is a minimum hop dominating set of G containing $\mathrm{HD}_i = \mathrm{HD}_{i-1} \cup \{x_{\rho(i)}\}$. If x_i is 2-step dominated by HD_{i-1} (i.e., if $D[x_i] \neq 0$) and $N_{G'}(x)$ has a pendant neighbor v that is not 2-step dominated by HD_{i-1}, then the algorithm selects $y_{\ell(i)}$. Hence, $\mathrm{HD}_i = \mathrm{HD}_{i-1} \cup \{y_{\ell(i)}\}$. By Lemma 10, there is a minimum hop dominating set of G containing $\mathrm{HD}_i = \mathrm{HD}_{i-1} \cup \{y_{\ell(i)}\}$. Therefore, by induction HD_{n_x} is a minimum hop dominating set of G. We record this formally in the following lemma.

Lemma 11. HD_{n_x} *is a minimum hop dominating set of the given chordal bipartite graph G.*

We now discuss how the algorithm HDS-CBG(G) can be implemented in $O(n+m)$ time for a given chordal bipartite graph G having n vertices and m edges. Given a chordal bipartite graph with a weak elimination ordering, by Theorem 1, a strong $\mathcal{T}$-elimination ordering of G can be computed in $O(n)$ time. We maintain an array D to track whether a vertex is 2-step dominated or not by the hop dominating set constructed thus far. Initially, $D[v]=0$ for all $v \in V(G)$. At the i-th iteration, the algorithm checks the set $\{v \in N(x_i) \mid D[v]=0\}$ and the D-label of the vertex x_i which can be done in $O(|N[x_i]|)$ time, i.e., in $O(d_G(x_i)+1)=O(d_G(x_i))$ time. Once new vertices are selected by the algorithm, the vertices in $N(x_i)$ are updated to be 2-step dominated, i.e., the D-label is made 1 for the selected vertices and the 2-step dominated vertices of $N[x_i]$.

To select new vertices, the algorithm checks the conditions as mentioned in Case 1–5 of the algorithm. In Case 1 of the algorithm at the i-iteration it checks the condition "$x_i \notin \mathrm{HD}$ and x_i has neighbor v such that $N(v) \cap \mathrm{HD} \neq \emptyset$". To do this, we maintain two arrays $\mathcal{L}$ and $\mathcal{A}$ on the vertices of G. Initially, $\mathcal{L}[v]=\mathcal{A}[v]=0$ for every $v \in V(G)$. Once a vertex v is selected to the set HD, $\mathcal{L}[v]$ is made 1 and $\mathcal{A}[u]$ is made 1 for every $u \in N(v)$. Hence we can conclude that if $\mathcal{L}[v]=1$ at any iteration of the algorithm, then v belongs to HD and if $\mathcal{A}[u]=1$, then a neighbor of u is already present in HD. Throughout the algorithm, the arrays $\mathcal{L}$ and $\mathcal{A}$ can be maintained in at most

$$\sum_{v \in V(G)} O(d_G(v)) = O(n+m)$$

time. In other cases, the algorithm looks for a pendant neighbor of x_i in $G' = G[X_i \cup Y]$. For this, we maintain an array $\mathcal{B}$ on the vertices of Y. Initially, $\mathcal{B}[y]=d_G(y)$ for every $y \in Y$. For every $i \in [n_x]$, after the end of the i-th iteration $\mathcal{B}[y]$ is decremented by 1 for every $y \in N(x_i)$. Hence, at the beginning of the i-th iteration if $\mathcal{B}[y']=1$ for some $y' \in N(x_i)$, then y' is a pendant neighbor of x_i in G'. Throughout the algorithm, the array $\mathcal{B}$ can therefore be maintained in at most $\sum_{v \in V(G)} O(d_G(v)) = O(n+m)$ time. Moreover, at the i-th iteration, the arrays D, $\mathcal{L}$, $\mathcal{A}$, and $\mathcal{B}$ can be maintained in at most $O(d_G(x_i))$ time. As before, all other conditions at the i-th iteration can be checked in at most $O(d_G(x_i))$. Thus the degrees of the vertices of the graph are scanned a constant number of times throughout the algorithm. Therefore, the algorithm in total takes at most $O(n+m)$ time.

Due to the above discussion and Lemma 11, we have the following theorem.

Theorem 12. *Given a weak elimination ordering of a chordal bipartite graph G having n vertices and m edges, a minimum hop dominating set of G can be computed in $O(n+m)$ time.*

4. Conclusion

The hop domination problem is known to be NP-complete for planar bipartite graphs, planar chordal graphs, and perfect elimination bipartite graphs. In this paper, we present a linear time algorithm for computing a minimum hop dominating set in chordal bipartite graphs if a weak elimination ordering of the graph is given. Since the hop domination problem is NP-complete for chordal graphs, it would be very interesting to decide the complexity of the minimum hop domination problem in subclasses of chordal graphs such as block graphs and interval graphs.

References

[1] S.K. Ayyaswamy and C. Natarajan, *Hop domination in graphs*, manuscript.

[2] S.K. Ayyaswamy, B. Krishnakumari, C. Natarajan and Y.B. Venkatakrishnan, *Bounds on the hop domination number of a tree*, Proc. Indian Acad. Sci. Math. Sci. **125** (2015) 449–455.
https://doi.org/10.1007/s12044-015-0251-6

[3] A. Brandstädt, V.B. Le and J.P. Spinrad, Graph Classes: A Survey (SIAM, Philadelphia, PA, USA, 1999).
https://doi.org/10.1137/1.9780898719796

[4] Y. Caro, A. Lev and Y. Roditty, *Some results in step domination of graphs*, Ars Combin. **68** (2003) 105–114.

[5] M.S. Chang, *Algorithms for maximum matching and minimum fill-in on chordal bipartite graphs*, 7th ISAAC, Lecture Notes in Comput. Sci. **1178** (1996) 146–155.
https://doi.org/10.1007/BFb0009490

[6] G. Chartrand, F. Harary, M. Hossain and K. Schultz, *Exact* 2*-step domination in graphs*, Math. Bohem. **120** (1995) 125–134.
https://doi.org/10.21136/MB.1995.126228

[7] X. Chen and Y. Wang, *On total domination and hop domination in diamond-free graphs*, Bull. Malays. Math. Sci. Soc. **43** (2020) 1885–1891.
https://doi.org/10.1007/s40840-019-00778-w

[8] P. Damaschke, H. Müller and D. Kratsch, *Domination in convex and chordal bipartite graphs*, Inform. Process. Lett. **36** (1990) 231–236.
https://doi.org/10.1016/0020-0190(90)90147-P

[9] G. Dror, A. Lev and Y. Roditty, *A note: some results in step domination of trees*, Discrete Math. **289** (2004) 137–144.
https://doi.org/10.1016/j.disc.2004.08.007

[10] F. Foucaud, *Decision and approximation complexity for identifying codes and locating-dominating sets in restricted graph classes*, J. Discrete Algorithms **31** (2015) 48–68.
https://doi.org/10.1016/j.jda.2014.08.004

[11] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker Inc., New York, 1998).

[12] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in Graphs: Advanced Topics (Marcel Dekker Inc., New York, 1998).

[13] M.A. Henning and N. Jafari Rad, *On 2-step and hop dominating sets in graphs*, Graphs Combin. **33** (2017) 913–927.
https://doi.org/10.1007/s00373-017-1789-0

[14] M.A. Henning, S. Pal and D. Pradhan, *Algorithm and hardness results on hop domination in graphs*, Inform. Process. Lett. **153** (2020) 105872.
https://doi.org/10.1016/j.ipl.2019.105872

[15] P. Hersh, *On exact n-step domination*, Discrete Math. **205** (1999) 235–239.
https://doi.org/10.1016/S0012-365X(99)00024-2

[16] M. Farhadi Jalalvand and N. Jafari Rad, *On the complexity of k-step and k-hop dominating sets in graphs*, Math. Montisnigri **40** (2017) 36–41.

[17] S. Kundu and S. Majumder, *A linear time algorithm for optimal k-hop dominating set of a tree*, Inform. Process. Lett. **116** (2016) 197–202.
https://doi.org/10.1016/j.ipl.2015.07.014

[18] H. Müller and A. Brandstädt, *The NP-completeness of* Steiner Tree *and* Dominating Set *for chordal bipartite graphs*, Theoret. Comput. Sci. **53** (1987) 257–265.
https://doi.org/10.1016/0304-3975(87)90067-3

[19] C. Natarajan and S.K. Ayyaswamy, *Hop domination in graphs*-II, An. Stiint. Univ. "Ovidius" Constanta Ser. Mat. **23** (2015) 187–199.
https://doi.org/10.1515/auom-2015-0036

[20] C. Natarajan and S.K. Ayyaswamy, *A note on the hop domination number of a subdivision graph*, Int. J. Appl. Math. **32** (2019) 381–390.
https://doi.org/10.12732/ijam.v32i3.2

[21] C. Natarajan, S.K. Ayyaswamy and G. Sathiamoorthy, *A note on hop domination number of some special families of graphs*, Int. J. Pure Appl. Math. **119** (2018) 14165–14171.

[22] B.S. Panda and D. Pradhan, *Minimum paired-dominating set in chordal bipartite graphs and perfect elimination bipartite graphs*, J. Comb. Optim. **26** (2013) 770–785.
https://doi.org/10.1007/s10878-012-9483-x

[23] D. Pradhan, *Complexity of certain functional variants of total domination in chordal bipartite graphs*, Discrete Math. Algorithms Appl. **4(3)** (2012) 1250045.
https://doi.org/10.1142/S1793830912500450

[24] Y.M. Pabilona and H.M. Rara, *Connected hop domination in graphs under some binary operations*, Asian-Eur. J. Math. **11** (2018) 1850075.
https://doi.org/10.1142/S1793557118500754

[25] H. Rahbani, N. Jafari Rad and M.R. Sadeghi, *A note on the complexity of locating-total domination in graphs*, Theoret. Comput. Sci. **799** (2019) 32–39.
https://doi.org/10.1016/j.tcs.2019.09.039

[26] R.C. Rakim, C.J.C. Saromines and H.M. Rara, *Perfect hop domination in graphs*, Appl. Math. Sci. **12** (2018) 635–649.
https://doi.org/10.12988/ams.2018.8576

[27] R. Uehara, *Linear time algorithms on chordal bipartite and strongly chordal graphs*, in: Automata, Languages and Programming, Lecture Notes in Comput. Sci. **2380** (2002) 993–1004.
https://doi.org/10.1007/3-540-45465-9_85

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