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MINIMIZING THE NUMBER OF COMPLETE BIPARTITE GRAPHS IN A K_s-SATURATED GRAPH

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Abstract

A graph is F-saturated if it does not contain F as a subgraph but the addition of any edge creates a copy of F. We prove that for $s \geq 3$ and $t \geq 2$,

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the minimum number of copies of $K_{1,t}$ in a K_s-saturated graph is $\Theta(n^{t/2})$. More precise results are obtained in the case of $K_{1,2}$, where determining the minimum number of $K_{1,2}$'s in a K_3-saturated graph is related to the existence of Moore graphs. We prove that for $s \geq 4$ and $t \geq 3$, an n-vertex K_s-saturated graph must have at least $Cn^{t/5+8/5}$ copies of $K_{2,t}$, and we give an upper bound of $O(n^{t/2+3/2})$. These results answer a question of Chakraborti and Loh on extremal K_s-saturated graphs that minimize the number of copies of a fixed graph H. General estimates on the number of $K_{a,b}$'s are also obtained, but finding an asymptotic formula for the number $K_{a,b}$'s in a K_s-saturated graph remains open.

Keywords: extremal graph theory, graph saturation.

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1. Introduction

Let F be a graph with at least one edge. A graph G is *F-free* if G does not contain F as a subgraph. The study of F-free graphs is central to extremal combinatorics. Turán's Theorem, widely considered to be a cornerstone result in graph theory, determines the maximum number of edges in an n-vertex K_s-free graph. An interesting class of F-free graphs are those that are maximal with respect to the addition of edges. We say that a graph G is *F-saturated* if G is F-free and adding any missing edge to G creates a copy of F. The function $\mathrm{sat}(n, F)$ is the *saturation number of F*, and is defined to be the minimum number of edges in an n-vertex F-saturated graph. In some sense, it is dual to the Turán function $\mathrm{ex}(n, F)$ which is the maximum number of edges in an n-vertex F-saturated graph.

One of the first results on graph saturation is a theorem of Erdős, Hajnal, and Moon [11] which determines the saturation number of K_s. They proved that for $2 \leq s \leq n$, there is a unique n-vertex K_s-saturated graph with the minimum number of edges. This graph is the join of a complete graph on $s-2$ vertices and an independent set on $n-s+2$ vertices, denoted $K_{s-2}+\overline{K_{n-s+2}}$. The Erdős-Hajnal-Moon Theorem was proved in the 1960s, and since then, graph saturation has developed into its own area of extremal combinatorics. We recommend the survey of Faudree, Faudree, and Schmitt [7] as a reference for history and significant results in graph saturation.

The function $\mathrm{sat}(n, F)$ concerns the minimum number of edges in an F-saturated graph. However, one can also replace an edge with any graph H, and then ask for the minimum number of copies of H in an n-vertex F-saturated graph. Let us write $\mathrm{sat}(n, H, F)$ for this minimum. This function was introduced in [18] and was motivated by the now well-studied Alon-Shikhelman generalized Turán function [2]. Recalling that the Erdős-Hajnal-Moon Theorem determines

$\text{sat}(n, K_s) = \text{sat}(n, K_2, K_s)$, it is quite natural to study the generalized function $\text{sat}(n, K_r, K_s)$, where $2 \le r < s$. Answering a question of Kritschgau *et al.* [18], Chakraborti and Loh [5] proved that for every $2 \le r < s$, there is an n_r such that for all $n \ge n_r$,

$$\text{sat}(n, K_r, K_s) = (n - s + 2)\binom{s-2}{r-1} + \binom{s-2}{r}.$$

Furthermore, they showed that $K_{s-2} + \overline{K_{n-s+2}}$ is the unique graph that minimizes the number of copies of K_r among all n-vertex K_s-saturated graphs for $n \ge n_r$. They proved a similar result for cycles (some assumptions are needed on the length of the cycle in relation to s) where the critical point is that $K_{s-2} + \overline{K_{n-s+2}}$ is the unique extremal graph. We refer the reader to [5] for the specific statements of these theorems. Chakraborti and Loh then asked the following question (Problem 10.5 in [5]).

Question 1. *Is there a graph H for which $K_{s-2} + \overline{K_{n-s+2}}$ does not (uniquely) minimize the number of copies of H among all n-vertex K_s-saturated graphs for all large enough n?*

Here we answer this question and show that there are graphs H for which $K_{s-2} + \overline{K_{n-s+2}}$ is not the unique extremal graph. We begin by stating our first two result, Theorems 2 and 3, where $H = K_{1,t}$. Together, they demonstrate a change in behaviour between the cases $H = K_{1,2}$ and $H = K_{1,t}$ with $t > 2$.

Theorem 2. (i) *For $n \ge 3$,*

$$\binom{n}{2} - \frac{n^{3/2}}{2} \le \text{sat}(n, K_{1,2}, K_3) \le \binom{n-1}{2}.$$

(ii) *For $n \ge s \ge 4$,*

$$\text{sat}(n, K_{1,2}, K_s) = (s-2)\binom{n-1}{2} + (n-s+2)\binom{s-2}{2}.$$

Furthermore, $K_{s-2} + \overline{K_{n-s+2}}$ is the unique n-vertex K_s-saturated graph with the minimum number of copies of $K_{1,2}$.

Theorem 3. *For integers $n \ge s \ge 3$ and $t \ge 3$,*

$$\text{sat}(n, K_{1,t}, K_s) = \Theta\left(n^{t/2+1}\right).$$

A consequence of Theorem 3 is that for $s \ge 4$ and large enough n in terms of t, $K_{s-2} + \overline{K_{n-s+2}}$ does not minimize the number of copies of $K_{1,t}$ among all

n-vertex K_s-saturated graphs. Indeed, $K_{s-2} + \overline{K_{n-s+2}}$ has $\Theta(n^t)$ copies of $K_{1,t}$. Interestingly, the special case of determining $\mathrm{sat}(n, K_{1,2}, K_3)$ is related to the existence of Moore graphs. This is discussed further in the Concluding Remarks section, but when a Moore graph of diameter 2 and girth 5 exists, this graph will have fewer copies of $K_{1,2}$ than $K_1 + \overline{K_{n-1}} = K_{1,n-1}$. Thus, any potential result that determines $\mathrm{sat}(n, K_{1,2}, K_3)$ exactly would have to take this into account.

The graph used to prove the upper bound of Theorem 3 is a K_s-saturated graph with maximum degree at most $c_s n^{1/2}$. This graph was constructed by Alon, Erdős, Holzman, and Krivelevich [1] and is structurally very different from $K_{s-2} + \overline{K_{n-s+2}}$. Using this graph, one can prove a more general upper bound that applies to any connected bipartite graph. This will be stated in Theorem 5 below.

Next we turn our attention to counting copies of $K_{2,t}$ ($t \geq 2$) in K_s-saturated graphs. The graph $K_1 + \overline{K_{n-1}}$ is K_3-saturated and $K_{2,t}$-free. Thus,

$$\mathrm{sat}(n, K_{2,t}, K_3) = 0$$

for all $t \geq 2$. For $t = 2$, Chakraborti and Loh [5] proved that for every $s \geq 4$,

$$\mathrm{sat}(n, K_{2,2}, K_s) = (1 + o(1))\binom{s-2}{2}\binom{n}{2}. \tag{1}$$

Observe that the graph $K_{s-2} + \overline{K_{n-s+2}}$ has

$$\binom{s-2}{2}\binom{n-s+2}{2} + \binom{s-2}{3}(n-s+2) + \binom{s-2}{4}$$

copies of $K_{2,2}$ and this gives the upper bound of (1). Now the focus of [5] was on counting complete graphs and counting cycles, so here the above result is stated in terms of $K_{2,2}$, but of course $K_{2,2} = C_4$. However, it is important and relevant to this work to mention that Chakraborti and Loh [5] proved the following theorem which shows that $K_{s-2} + \overline{K_{n-s+2}}$ does minimize the number of cycles in certain cases.

Theorem 4 (Chakraborti and Loh). *Let $s \geq 4$ and $r \geq 7$ if r odd, and $r \geq 4\sqrt{s-2}$ if r is even. There is an $n_{r,s}$ such that for all $n \geq n_{r,s}$, the graph $K_{s-2} + \overline{K_{n-s+2}}$ minimizes the number of copies of C_r over all n-vertex K_s-saturated graphs. Moreover, when $r \leq 2s - 4$, this is the unique extremal graph.*

It is conjectured in [5] that $K_{s-2} + \overline{K_{n-s+2}}$ is the unique graph that minimizes the number of copies of C_r among all K_s-saturated graphs. Currently, it is known that $K_{s-2} + \overline{K_{n-s+2}}$ minimizes the number of copies of K_r (Erdős-Hajnal-Moon for $r = 2$ and [5] for $r > 2$), and minimizes the number of cycles under certain assumptions. Theorem 3 shows $K_{s-2} + \overline{K_{n-s+2}}$ *does not* minimize the number of copies of $K_{1,t}$ since it has about $\binom{n-s+2}{t} = \Theta(n^t)$ copies of $K_{1,t}$. We extend this to $K_{a,b}$ with $1 \leq a+1 < b$ in the next theorem which is based on the graph constructed by Alon, Erdős, Holzman, and Krivelevich [1].

Theorem 5. *Let F be a connected bipartite graph with part sizes a and b with $1 \leq a+1 < b$. If $s \geq 3$ is an integer, then*

$$\operatorname{sat}(n, F, K_s) = \begin{cases} 0 & \text{if } a > s-2, \\ O\left(n^{\frac{1}{2}(a+b+1)}\right) & \text{if } a \leq s-2, \end{cases}$$

where the implicit constant can depend on a, b, and s.

Theorem 5 naturally suggests the following question: how many copies of $K_{2,t}$ must there be in a K_s-saturated graph? In this direction we prove the following.

Theorem 6. *Let $s \geq 4$ and $t \geq 3$ be integers. If G is an n-vertex K_s-saturated graph, then G contains at least*

$$Cn^{t/5+8/5}$$

copies of $K_{2,t}$ for some constant $C > 0$.

Saturation problems with restrictions on the degrees have also been well-studied. Duffus and Hanson [9] investigated triangle saturated graphs with minimum degree 2 and 3. Day [8] resolved a 20 year old conjecture of Bollobás [15] which asked for a lower bound on the number of edges in K_s-saturated graphs with minimum degree t. Gould and Schmitt [14] studied K_2^t (the complete t-partite graph with parts of size 2) saturated graphs with a given minimum degree. Furthermore, K_s-saturated graphs with restrictions on the maximum degree were studied in [1, 13, 16, 20]. Turning to generalized saturation numbers, as a step towards generalizing Day's Theorem, Cole *et al.* [6] proved bounds on the number of triangles a K_s-saturated graph with minimum degree t. Motivated by these results, we prove a lower bound on the number of $K_{a,b}$ in K_s-saturated graphs in terms of the minimum degree.

Theorem 7. *Let $s \geq 4$ and $2 \leq a < b$ be integers with $a \leq s-2$. If G is an n-vertex K_s-saturated graph with minimum degree $\delta(G)$, then G contains at least*

$$c\left(\frac{n-\delta(G)-1}{\delta(G)^{a-1}}\right)^{b/2}$$

copies of $K_{a,b}$ for some constant $c = c(s,a,b) > 0$.

Theorem 7 shows that if $0 \le \alpha < \frac{1}{a-1}$ and $\delta(G) \le \kappa n^{\alpha}$ for some $\kappa > 0$, then G contains at least $cn^{b/2(1-\alpha(a-1))}$ copies of $K_{a,b}$. In particular, when G has constant minimum degree, we obtain $\Omega(n^{t/2})$ copies of $K_{2,t}$. This improves the lower bound of Theorem 6, but comes at the cost of a minimum degree assumption that rules out the Alon *et al.* graph which has minimum degree $\Theta_s(n^{1/2})$ and $\Theta(n^{t/2+3/2})$ copies of $K_{2,t}$.

In the next subsection we give the notation that will be used in our proofs. Section 2 contains the proofs of Theorems 2 and 3. Section 3 contains the proofs of Theorems 5, 6, and 7.

1.1. Notation

For graphs H_1 and H_2, we write $\mathcal{N}(H_1, H_2)$ for the number of copies of H_1 in H_2. For a graph G and $x, y \in V(G)$, write $N(x)$ for the neighborhood of x, and $N(x, y)$ for $N(x) \cap N(y)$. More generally, if $X \subset V(G)$ and $v \in V(G)$, then $N(v, X)$ is the set of vertices adjacent to all vertices in $\{v\} \cup X$. We write $d(v) = |N(v)|$, $d(X) = |N(X)|$, and $d(v, X) = |N(v, X)|$. The set $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v. For a hypergraph $\mathcal{H}$, $d_{\mathcal{H}}(v)$ is the number of hyperedges containing v. Similarly, $d_{\mathcal{H}}(X)$ and $d_{\mathcal{H}}(v, X)$ is the number of hyperedges containing X and $\{v\} \cup X$, respectively.

2. Bounds on $\mathrm{sat}(n, K_{1,t}, K_s)$

2.1. Proof of Theorem 2

We have the upper bound

$$\mathrm{sat}(n, K_{1,2}, K_s) \le (s-2)\binom{n-1}{2} + (n-s+2)\binom{s-2}{2} \tag{2}$$

coming from the number of copies of $K_{1,2}$ in the graph $K_{s-2} + \overline{K_{n-s+2}}$.

When $s = 3$, a convexity argument can give a matching lower bound up to an error term of order $O(n^{3/2})$. Let G be an n-vertex K_3-saturated graph. If $e(G) \ge \frac{\sqrt{n-1}n}{2}$, then

$$\mathcal{N}(K_{1,2}, G) = \sum_{v \in V(G)} \binom{d(v)}{2} \ge n\binom{2e(G)/2}{2} = \frac{2e(G)^2}{n} - e(G) \ge \binom{n}{2} - \frac{n^{3/2}}{2}.$$

Now assume that $e(G) < \frac{\sqrt{n-1}n}{2}$. If x and y are not adjacent, then they must be joined by a path of length 2. Hence,

$$\mathcal{N}(K_{1,2}, G) \ge e(\overline{G}) = \left(\binom{n}{2} - e(G)\right) \ge \binom{n}{2} - \frac{n^{3/2}}{2}.$$

This inequality completes the proof of Theorem 2 part (i).

Next we prove part (ii) by showing

$$\operatorname{sat}(n, K_{1,2}, K_s) \geq (s-2)\binom{n-1}{2} + (n-s+2)\binom{s-2}{2}$$

for $s \geq 4$. Let G be an n-vertex K_s-saturated graph with $n \geq s \geq 4$. Kim, Kim, Kostochka and O [17] proved that

$$\sum_{v \in V(G)} (d(v)+1)(d(v)+2-s) \geq (s-2)n(n-s+1).$$

It is easy to check that

$$\sum_{v \in V(G)} (d(v)+1)(d(v)+2-s) = \sum_{v \in V(G)} (d(v)-1)d(v) + (4-s) \sum_{v \in V(G)} d(v) + (2-s)n.$$

Therefore,

$$\sum_{v \in V(G)} (d(v)-1)d(v) \geq (s-2)n(n-s+1) + (s-4)2e(G) + (s-2)n. \tag{3}$$

By the Erdős-Hajnal-Moon Theorem

$$\operatorname{sat}(n, K_s) = (s-2)(n-s+2) + \binom{s-2}{2},$$

and $K_{s-2} + \overline{K_{n-s+2}}$ is the unique n-vertex K_s-saturated with $\operatorname{sat}(n, K_s)$ edges. Thus,

$$2e(G) \geq 2(s-2)(n-s+2) + 2\binom{s-2}{2} = (s-2)(2n-s+1).$$

Plugging this into (3) we get that if $s \geq 4$,

$$\sum_{v \in V(G)} (d(v)-1)d(v) \geq (s-2)n(n-s+1) + (s-4)(s-2)(2n-s+1) + (s-2)n.$$

Dividing through by 2 and simplifying the right hand side yields

$$\sum_{v \in V(G)} \binom{d(v)}{2} \geq (s-2)\binom{n-1}{2} + (n-s+2)\binom{s-2}{2}$$

where equality will hold only if G is $K_{s-2} + \overline{K_{n-s+2}}$. This completes the proof of part (ii) of Theorem 2.

2.2. Proof of Theorem 3

Using another inequality of Kim, Kim, Kostochka, and O [17] and the Power Means Inequality, we prove a lower bound on the number of copies of $K_{1,t}$ in a K_s-saturated graph that gives the correct order of magnitude for $t \geq 3$.

Proposition 8. *Let $n \geq s \geq 3$ and $t \geq 3$ be integers. If G is an n-vertex K_s-saturated graph, then*

$$\begin{aligned}\mathcal{N}(K_{1,t}, G) &\geq \frac{((n-1)^2(s-2) + (s-2)^2(n-s+2))^{t/2}}{t^t n^{t/2-1}} \\ &= \left(\frac{\sqrt{s-2}}{t}\right)^t n^{t/2+1} + O_{s,t}(n^{t/2}).\end{aligned}$$

Proof. Let G be an n-vertex K_s-saturated graph. Kim, Kim, Kostochka and O [17] proved that

$$\sum_{v \in V(G)} d(v)^2 \geq (n-1)^2(s-2) + (s-2)^2(n-s+2) \tag{4}$$

and that equality holds if and only if G is $K_{s-2} + \overline{K_{n-s+2}}$, except for in the case that $s = 3$ where equality holds if and and only if G is $K_1 + \overline{K_{n-1}}$ or a Moore graph. By the Power Means Inequality,

$$\sum_{v \in V(G)} d(v)^2 \leq n^{1-2/t} \left(\sum_{v \in V(G)} d(v)^t \right)^{t/2}. \tag{5}$$

Combining (4) and (5) with the inequality $\sum_{v \in V(G)} d(v)^t \leq t^t \sum_{v \in V(G)} \binom{d(v)}{t}$ and rearranging gives

$$\mathcal{N}(K_{1,t}, G) = \sum_{v \in V(G)} \binom{d(v)}{t} \geq \frac{((n-1)^2(s-2) + (s-2)^2(n-s+2))^{t/2}}{t^t n^{t/2-1}}.$$

This completes the proof of Proposition 8. ■

Proposition 9. *Let $s \geq 3$ and $t \geq 3$ be integers. For sufficiently large n,*

$$\mathrm{sat}(n, K_{1,t}, K_s) \leq \frac{c_s^t n^{t/2+1}}{t!}$$

where c_s is a constant depending only on s.

Proof. By a result of Alon, Erdős, Holzman, and Krivelevich, for each $s \geq 3$ and sufficiently large n, there is a K_s-saturated graph G with maximum degree $c_s\sqrt{n}$ (the constant c_s satisfies $c_s \to 2s$ as $s \to \infty$). The number of $K_{1,t}$'s in G is then

$$\sum_{v \in V(G)} \binom{d(v)}{t} \leq n\binom{\Delta(G)}{t} \leq \frac{c_s^t n^{t/2+1}}{t!}.$$

■

***Proof of Theorem* 3.** The proof of Theorem 3 follows from Propositions 8 and 9. ■

3. Bounds on $\mathrm{sat}(n, K_{2,t}, K_s)$ with $s \geq 4$ and $t \geq 3$

3.1. Upper bound on $\mathrm{sat}(n, K_{2,t}, K_s)$

We begin this section by stating two simple lemmas that can be easily proved by counting embeddings.

Lemma 10. *Let F be a connected bipartite graph with parts of size a and b. If G is an n-vertex graph with maximum degree Δ, then*

$$\mathcal{N}(F, G) \leq n\Delta^{a+b-1}.$$

Lemma 11. *Let F be a connected bipartite graph with parts of size a and b. For any n-vertex graph G,*

$$\mathcal{N}(K_{a,b}, G) \leq \mathcal{N}(F, G).$$

Having stated these two lemmas, we are ready to prove Theorem 5.

***Proof of Theorem* 5.** If $a > s-2$, then $K_{s-2} + \overline{K_{n-s+2}}$ is K_s-saturated with no copies of F. Indeed, a copy of F would need at least a vertices from the K_{s-2}, but $a > s-2$.

Now assume $a \leq s-2$. Let G_q^s be the K_s-saturated graph constructed in [1] where n (and thus q) is chosen large enough so that $b < \frac{q+1}{2}$. There is a constant $c_s > 0$ such that $\Delta(G_q^s) \leq c_s\sqrt{n}$. By Lemma 10, the number of copies of F in G_q^s is at most $nc_s^{a+b-1}n^{(1/2)(a+b-1)} = c_s^{a+b+1}n^{(1/2)(a+b+1)}$. ■

We conclude this subsection by showing that the graph G_q^s used in the proof of Theorem 5 cannot be used to further improve upon the upper bound of $O\left(n^{\frac{1}{2}(a+b+1)}\right)$ when $F = K_{a,b}$. Since we are showing that G_q^s cannot be used to improve the upper bound, we will be brief in our argument. We will use the same terminology as in [1], but one point at which we differ is the notation we use for a vertex. A vertex in G_q^s is determined by its level, place, type, and copy. A vertex at level i, place j, type t, and copy c will be written as

$$((i-1)q+j,t,c).$$

First, take n large enough so that $b<\frac{q+1}{2}$. Choose a sequence $i_1,i_2,\dots,i_a$ of levels with $1\le i_1<i_2<\dots<i_a\le\frac{q+1}{2}$. Likewise, choose a sequence of b levels $\frac{q+1}{2}\le i_{a+1}<i_{a+2}<\dots<i_{a+b}\le q+1$. This can be done in $\binom{\frac{q+1}{2}}{a}\binom{\frac{q+1}{2}}{b}$ ways. Next, choose a place $j_1\in[q]$ which can be done in q ways, and a type $t_1\in[s-2]$ which can be done in $s-2$ ways. Finally, choose a sequence of copies $1\le c_1,c_2,\dots,c_{a+b}\le s-1$ arbitrarily. This can be done in $(s-1)^{a+b}$ ways. Using the definition of G_q^s, one finds that the a vertices in the set

$$\{((i_z-1)q+j_1,t_1,c_z):1\le z\le a\}$$

are all adjacent to the b vertices in the set

$$\{((i_z-1)q+(j_1+1)_q,(t_1+1)_{s-2},c_z):a+1\le z\le a+b\}$$

(here $(j_1+1)_q$ is the unique integer ζ in $\{1,2,\dots,q\}$ for which $j_1+1\equiv\zeta(\mathrm{mod}\ q)$, and $(t_1+1)_{s-2}$ is the unique integer ζ' in $\{1,2,\dots,s-2\}$ for which $t_1+1\equiv\zeta'(\mathrm{mod}\ s-2)$). This gives a $K_{a,b}$ in G_q^s and so the number of $K_{a,b}$ in G_q^s is at least

$$\binom{(1/2)(q+1)}{a}\binom{(1/2)(q+1)}{b}q(s-2)(s-1)^{a+b}\ge C_{s,a,b}q^{a+b+1}\ge Cn^{(1/2)(a+b+1)}.$$

By Lemmas 11 and 10, G_q^s is a K_s-saturated n-vertex graph with

$$\Theta_{s,a,b}\left(n^{(1/2)(a+b+1)}\right)$$

copies of $K_{a,b}$.

3.2. Lower bounds on $\mathrm{sat}(n,K_{2,t},K_s)$

First we prove Theorem 6.

***Proof of Theorem* 6.** Let G be a K_s-saturated graph on n vertices. Note that we can assume

$$e(G)\le\frac{n^{\frac{7}{4}}}{10}. \tag{6}$$

Otherwise, a theorem of Erdős and Simonovits [10] implies that there is a positive constant γ such that

$$\mathcal{N}(K_{2,t},G)\ge\gamma\frac{e(G)^{2t}}{n^{3t-2}}=\Omega\left(n^{\frac{t}{2}+2}\right), \tag{7}$$

proving Theorem 6.

Let K_4^- be the graph consisting of 4 vertices and 5 edges obtained by removing an edge from K_4. For a copy of K_4^- with vertices x, y, u, v, where $uv \notin E(G)$, let xy be called the *base edge* of this K_4^-. We estimate the number of copies of K_4^- in a K_s-saturated graph G.

For every u, v with $uv \notin E(G)$ there is a set S such that $S \subseteq N(u, v)$ and S induces a K_{s-2} in G. Therefore, there are at least $\binom{s-2}{2}$ pairs $x, y \in S$ such that u, v, x, y form a copy of K_4^-. On the other hand, every $xy \in E(G)$ is the base edge of at most $\binom{d(x,y)}{2}$ copies of K_4^- in G.

Therefore,

$$\sum_{xy \in E(G)} \binom{d(x,y)}{2} \geq \mathcal{N}(K_4^-, G) \geq \sum_{uv \in E(\overline{G})} \binom{s-2}{2} \geq e(\overline{G}) \overset{(6)}{\geq} \frac{n^2}{4}.$$

Thus, there is a constant $c_t = c(t)$ such that the following holds.

$$\begin{aligned}
\mathcal{N}(K_{2,t}, G) &\geq \sum_{xy \in E(G)} \binom{d(x,y)}{t} \geq \frac{1}{t^t} \sum_{xy \in E(G)} \left(\binom{d(x,y)}{2}^{\frac{t}{2}} - t^t \right) \\
&\geq \frac{e(G)}{t^t} \left(\frac{\sum_{xy \in E(G)} \binom{d(x,y)}{2}}{e(G)} \right)^{\frac{t}{2}} - e(G) \geq \frac{(n^2/4)^{\frac{t}{2}}}{t^t e(G)^{\frac{t}{2}-1}} - e(G) \\
&= \frac{(n^2/4)^{\frac{t}{2}} - t^t e(G)^{t/2}}{t^t e(G)^{\frac{t}{2}-1}} \overset{(6)}{\geq} \frac{c_t n^t}{e(G)^{\frac{t}{2}-1}}.
\end{aligned}$$

Combining this with (7) we get

$$\mathcal{N}(K_{2,t}, G) \geq \min \left\{ \gamma \frac{e(G)^{2t}}{n^{3t-2}}, \frac{c_t n^t}{e(G)^{\frac{t}{2}-1}} \right\}.$$

Let $e(G) = n^{\alpha}$. Then

$$\mathcal{N}(K_{2,t}, G) \geq \min \left\{ \gamma n^{2\alpha t - 3t + 2}, c_t n^{t - \alpha t/2 + \alpha} \right\}.$$

Choosing $\alpha = \frac{8t-4}{5t-2}$ and $C = \min\{\gamma, c_t\}$, we get the desired lower bound

$$C n^{\frac{t}{5} - \frac{16}{125t-10} + \frac{41}{25}} > C n^{\frac{t}{5} + \frac{8}{5}}. \qquad \blacksquare$$

Next we turn to the proof of Theorem 7. We need the following lemma.

Lemma 12. *Let $s \geq 4$ and $2 \leq a \leq b$ be integers with $a \leq s-2$. Suppose that G is an n-vertex K_s-saturated graph with vertex set V. For any $v \in V$, there are at least*

$$c \left(\frac{n - d(v) - 1}{d(v)^{a-1}} \right)^{b/2}$$

copies of $K_{a,b}$ that contain v where $c = c(s,a,b)$ is a positive constant.

Proof. Let $v \in V$. For each $u \in V \backslash N[v]$, there is a set $S_u \subset N(v)$ such that S_u induces a K_{s-2} in G. Fix such an S_u and define an $(s-1)$-uniform hypergraph $\mathcal{H}$ to have vertex set $V \backslash \{v\}$, and edge set $E(\mathcal{H}) = \{\{u\} \cup S_u : u \in V \backslash N[v]\}$. By construction, $\mathcal{H}$ has $n - d(v) - 1$ edges, each of which contains exactly one vertex from $V \backslash N[v]$ and $s-2$ vertices from $N(v)$. Also, no two edges of $\mathcal{H}$ contain the same vertex from $V \backslash N[v]$. In what follows, we will add the subscript $\mathcal{H}$ if we are referring to degrees in $\mathcal{H}$, and no subscript will be included if we are referring to degrees or neighborhoods in G.

By averaging, there is a set $X \in \binom{N(v)}{a-1}$ such that

$$d_{\mathcal{H}}(X) \geq \frac{\binom{s-2}{a-1}(n - d(v) - 1)}{\binom{d(v)}{a-1}}.$$

We then have

$$\sum_{y \in N(v,X)} d_{\mathcal{H}}(y, X) \geq \frac{d_{\mathcal{H}}(X)}{(s-2) - |X|} \geq c_1 \frac{n - d(v) - 1}{d(v)^{a-1}} \tag{8}$$

for some constant $c_1 = c_1(s,a) > 0$. The number of $K_{a,b}$ with $X \cup \{y\}$ forming the part of size a (y is an arbitrary vertex from $N(v,X)$) and v in the part of size b is at least

$$\sum_{y \in N(v,X)} \binom{d_{\mathcal{H}}(y,X)}{b-1} \geq d(v,X) \binom{\frac{c_1(n-d(v)-1)}{d(v,X)d(v)^{a-1}}}{b-1} \geq \frac{c_2(n - d(v) - 1)^{b-1}}{d(v,X)^{b-2} d(v)^{(a-1)(b-1)}}.$$

Here we have used convexity, (8), and $c_2 = c_2(s,a,b)$ is some positive constant.

Recalling that $|X| = a-1$, there are $\binom{d(v,X)}{b}$ copies of $K_{a,b}$ where $\{v\} \cup X$ is the part of size a and the part of size b is contained in $N(v) \backslash X$. Thus, for some constant $c_3 = c_3(s,a,b) > 0$, the number of $K_{a,b}$ that contain v is at least

$$\frac{c_3(n - d(v) - 1)^{b-1}}{d(v,X)^{b-2} d(v)^{(a-1)(b-1)}} + c_3 d(v,X)^b.$$

By considering cases as to which is this the bigger term in this sum, we find that in both cases, there are at least

$$c_3 \left(\frac{n - d(v) - 1}{d(v)^{a-1}} \right)^{b/2}$$

copies of $K_{a,b}$ containing v. ■

Applying Lemma 12 to a vertex v with $d(v) = \delta(G)$ proves Theorem 7.

4. Concluding Remarks

An interesting open problem is determining the minimum number of copies of $K_{1,2}$ in a K_3-saturated graph. There is a connection between this problem and Moore graphs with girth 5. An n-vertex Moore graph with girth 5 and diameter 2 is K_3-saturated. If it's degree is d, then $d = \sqrt{n-1}$ and it will contain $n\binom{d}{2} = n\binom{\sqrt{n-1}}{2}$ copies of $K_{1,2}$. This value is always less than $\binom{n-1}{2}$ which is the number of copies of $K_{1,2}$ in $K_1 + \overline{K_{n-1}} = K_{1,n-1}$. Furthermore, one can duplicate vertices of a Moore graph and preserve the K_3-saturated property. Duplicating a vertex of the Petersen graph will lead to an 11-vertex K_3-saturated graph with 42 copies of $K_{1,2}$, but $K_{1,10}$ has 45 copies of $K_{1,2}$. Starting from the Hoffman-Singleton graph, one can duplicate a vertex up to 4 times and still have fewer copies of $K_{1,2}$ than the corresponding $K_{1,n-1}$. Duplicating a single vertex is not necessarily the optimal way to minimize the number of copies of $K_{1,2}$, but the point is that it is not just the known Moore graphs that have fewer copies of $K_{1,2}$ than the corresponding $K_{1,n-1}$.

It would also be interesting to determine the order of magnitude of

$$\mathrm{sat}(n, K_{2,t}, K_s).$$

There is a gap in the exponents (this is discussed in the introduction) and it would be nice to close this gap. It is not clear if our lower or upper bound is closer to the correct answer.

Another potential approach to studying $\mathrm{sat}(n, H, F)$ is via the random F-free process. This random process orders the pairs of vertices uniformly and then adds them one by one subject to the condition that adding an edge does not create a copy of F. The resulting graph is then F-saturated. This process was first considered in [4, 12, 21, 22] and has since been studied extensively. If $X_{H,F}$ is the random variable that counts the number of copies of H in the output of this process, then we have that $\mathrm{sat}(n, H, F) \leq \mathbb{E}(X_{H,F})$. Unfortunately, this random variable seems quite challenging to understand. If F satisfies a certain balancedness condition (that, e.g., complete graphs and cycles satisfy), and we let

$$p = n^{-\frac{v(F)-2}{e(F)-1}} (\log n)^{1/(e(F)-1)},$$

then many results on the F-free process show that the resulting graph behaves in certain ways like the Erdős-Rényi random graph with edge probability p. Bohman and Keevash [3] conjectured that the maximum degree of the final graph in the

F-free process will be $O(pn)$ and this was proven up to a logarithmic factor by Osthus and Taraz [19]. Bohman and Keevash also showed that in the early stages of the F-free process, the number of common neighbors of any fixed set of vertices is of the same order of magnitude as what it would be in a random graph with edge probability $n^{-\frac{v(F)-2}{e(F)-1}}$ ([3] Corollary 1.5). It seems very likely (though potentially technically difficult to prove) that

$$\mathrm{sat}(n, K_{a,b}, K_s) \leq \mathbb{E}\left(X_{K_{a,b},K_s}\right) \leq n^{a+b-\frac{2ab}{s+1}+o(1)}.$$

One might even guess that the $n^{o(1)}$ can be replaced by $(\log n)^{\frac{ab}{s-2}}$. This would be better than Theorem 5 for many choices of a, b and s.

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