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TWO SUFFICIENT CONDITIONS FOR COMPONENT FACTORS IN GRAPHS

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Abstract

Let G be a graph. For a set $\mathcal{H}$ of connected graphs, a spanning subgraph H of a graph G is called an $\mathcal{H}$-factor of G if each component of H is isomorphic to a member of $\mathcal{H}$. An $\mathcal{H}$-factor is also referred as a component factor. If $G - e$ admits an $\mathcal{H}$-factor for any $e \in E(G)$, then we say that G is an $\mathcal{H}$-factor deleted graph. Let $k \geq 2$ be an integer. In this article, we verify that (i) a graph G admits a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor if and only if its binding number $bind(G) \geq \frac{2}{2k+1}$; (ii) a graph G with $\delta(G) \geq 2$ is a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor deleted graph if its binding number $bind(G) \geq \frac{2}{2k-1}$.

Keywords: graph, minimum degree, binding number, $\mathcal{H}$-factor, $\mathcal{H}$-factor deleted graph.

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1. Introduction

In this article, we discuss only finite simple graphs without loops or multiple edges. Given a graph G, we let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. For a vertex x of a graph G, we use $N_G(x)$ to denote the set of vertices adjacent to x in G, and use $d_G(x)$ to denote the degree of x in G. We write $\delta(G) = \min\{d_G(x) : x \in V(G)\}$. For a vertex subset X of a graph G, we denote by $G[X]$ the subgraph of G induced by X, and write $G-X = G[V(G)\setminus X]$ and $N_G(X) = \bigcup_{x\in X} N_G(x)$. We denote by $I(G)$ the set of isolated vertices of G, and write $i(G) = |I(G)|$. The binding number $bind(G)$ of a graph G is the minimum, taken over all $X \subseteq V(G)$ with $X \neq \emptyset$ and $N_G(X) \neq V(G)$, of the ratio $\frac{|N_G(X)|}{|X|}$.

We denote by C_n the cycle with n vertices, by K_n the complete graph with n vertices, and by $K_{n,m}$ the complete bipartite graph with partite sets X of size n and Y of size m, where $V(K_{n,m}) = X \cup Y$. For a tree T, we use $Leaf(T)$ to denote the set of leaves. An edge of T incident with a leaf is said to be a pendant edge.

We define a special class of trees $\mathcal{T}(2k+1)$, where $k \geq 2$ is an integer. Let R be a tree that satisfies the following conditions: for any $x \in V(R) - Leaf(R)$,

(a) $d_{R-Leaf(R)}(x) \in \{1, 3, \ldots, 2k+1\}$

and

(b) $2 \cdot$ (the number of leaves adjacent to x in R)$+d_{R-Leaf(R)}(x) \leq 2k+1$.

For such a tree R, we derive a new tree T_R as follows:

(c) insert a new vertex of degree 2 into each edge of $R - Leaf(R)$

and

(d) for every vertex x of $R - Leaf(R)$ with $d_{R-Leaf(R)}(x) = 2r+1 < 2k+1$, add $k-r-$(the number of leaves adjacent to x in R) pendant edges to x.

Then the set of such trees T_R for all trees R satisfying conditions (a) and (b) is denoted by $\mathcal{T}(2k+1)$.

For a set $\mathcal{H}$ of connected graphs, a spanning subgraph H of a graph G is called an $\mathcal{H}$-factor of G if every component of H is isomorphic to a member of $\mathcal{H}$. An $\mathcal{H}$-factor is also referred as a component factor. If $G - e$ admits an $\mathcal{H}$-factor for any $e \in E(G)$, then we say that G is an $\mathcal{H}$-factor deleted graph.

Tutte [8] derived a characterization for a graph admitting a $\{K_2, C_n : n \geq 3\}$-factor. Amahashi and Kano [2] showed a necessary and sufficient condition for the existence of a $\{K_{1,j} : 1 \leq j \leq k\}$-factor in a graph. Kano, Lu and Yu [4] presented a sufficient condition for a graph to have a $\{K_{1,2}, K_{1,3}, K_5\}$-factor. Kano and Saito [6] obtained a result on the existence of a $\{K_{1,j} : k \leq j \leq 2k\}$-factor in a graph. Zhang, Yan and Kano [11] posed a sufficient condition for the existence of $\{K_{1,j}, K_{2k} : k \leq j \leq 2k-1\}$-factors in graphs. Zhou [14] derived some results on the existence of component factors in graphs. For the relationships between binding number and graph factors, we refer the reader to [3,7,9,10,12,13,15–17].

Kano, Lu and Yu [5] gave a criterion for a graph having a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor, which is shown in the following.

Theorem 1 (Kano, Lu and Yu [5]). *Let k be an integer with $k \geq 2$. Then a graph G admits a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor if and only if*

$$i(G-X) \leq \left(k+\frac{1}{2}\right)|X|$$

for every $X \subseteq V(G)$.

In this article, we establishes some relationships between binding numbers and $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factors in graphs.

Theorem 2. *Let k be an integer with $k \geq 2$. Then a graph G admits a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor if and only if its binding number $bind(G) \geq \frac{2}{2k+1}$.*

Theorem 3. *Let k be an integer with $k \geq 2$. Then a graph G with $\delta(G) \geq 2$ is a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor deleted graph if its binding number $bind(G) \geq \frac{2}{2k-1}$.*

Remark 4. For two graphs H_1 and H_2, $H_1 \cup H_2$ denotes the union of H_1 and H_2, and $H_1 \vee H_2$ denotes the join of H_1 and H_2. We show that the condition $bind(G) \geq \frac{2}{2k-1}$ in Theorem 3 cannot be replaced by $bind(G) \geq \frac{2}{2k}$. To explain this, we construct a graph $G = H_1 \vee ((2kK_1) \cup (2H_2))$, where $H_1 = K_2$, $H_2 = K_2$, and $k \geq 2$ is an integer. Choose $Y = V(2kK_1)$. Obviously, $Y \neq \emptyset$ and $N_G(Y) \neq V(G)$. Furthermore, we easily see that $bind(G) = \frac{|N_G(Y)|}{|Y|} = \frac{2}{2k}$. Set $e \in E(2H_2)$ and $G' = G - e$. We choose $X = V(H_1)$. Thus, we derive

$$i(G'-X) = 2k+2 > 2k+1 = 2\left(k+\frac{1}{2}\right) = \left(k+\frac{1}{2}\right)|X|.$$

By Theorem 1, G' has no $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor, and so, G is not a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor deleted graph.

2. THE PROOF OF THEOREM 2

We first verify the following lemma, which is very useful in the proof of Theorem 2.

Lemma 5. *Let G be a graph and $\lambda \geq 1$ be a real number. Then the following three statements are equivalent.*

(i) *$i(G-S) \leq \lambda|S|$ for all $S \subset V(G)$.*

(ii) $\lambda|N_G(X)| \geq |X|$ *for all independent set* X *of* G.

(iii) $\lambda|N_G(Y)| \geq |Y|$ *for all* $Y \subset V(G)$.

Proof. The equivalence of (i) and (ii) of Lemma 5 is known and easy (See Lemma 7.1 [1]). In what follows, we prove only that (ii) implies (iii).

We may assume that G is connected. Let $\emptyset \neq Y \subset V(G)$, and let $X = Y \cap N_G(Y)$. Then $Y - X$ is an independent set of G, $N_G(X) \supseteq X$ and $N_G(Y - X) \cap Y = \emptyset$. Then by (ii) and $\lambda \geq 1$, we have

$$\lambda|N_G(Y)| \geq \lambda(|N_G(Y - X)| + |X|) \geq |Y - X| + |X| = |Y|.$$

Hence, (iii) holds. Lemma 5 is verified. ■

Proof of Theorem 2. Set $\lambda = \frac{2k+1}{2}$ in Lemma 5. According to Theorem 1, Lemma 5 and the definition of *bind*(G), we see that

$$\begin{aligned}
&bind(G) \geq \frac{2}{2k+1} \\
&\Leftrightarrow \frac{2k+1}{2}|N_G(Y)| \geq |Y| \text{ for all } Y \subset V(G) \\
&\Leftrightarrow i(G - S) \leq \frac{2k+1}{2}|S| \text{ for all } S \subset V(G) \\
&\Leftrightarrow G \text{ admits a } \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}\text{-factor.}
\end{aligned}$$

We finish the proof of Theorem 2. ■

3. The Proof of Theorem 3

Proof of Theorem 3. Let $G' = G - e$ for any fixed $e = xy \in E(G)$ and $\mathcal{H} = \{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$. To prove Theorem 3, it suffices to verify that G' admits an $\mathcal{H}$-factor. On the contrary, we assume that G' has no $\mathcal{H}$-factor. Then it follows from Theorem 1 that

$$i(G' - X) > \left(k + \frac{1}{2}\right)|X| \tag{1}$$

for some vertex subset X of G.

We first demonstrate the following claim.

Claim 1. $|X| \geq 2$.

Proof. Since $\delta(G) \geq 2$, $\delta(G') \geq 1$ and so G' has no isolated vertex. Assume that $|X| = 1$. Then it follows from (1) and $k \geq 2$ that $i(G' - X) > (k + \frac{1}{2})|X| \geq \frac{5}{2}$, which implies

$$i(G' - X) \geq 3. \tag{2}$$

It is obvious that $i(G' - X) = i(G - e - X) \leq i(G - X) + 2$. Combining this with (2), we derive $i(G - X) \geq i(G' - X) - 2 \geq 3 - 2 = 1$, which implies that there exists at least one vertex u in $G - X$ with $d_{G-X}(u) = 0$. Thus, we have $d_G(u) \leq d_{G-X}(u) + |X| = 0 + 1 = 1$, which contradicts $\delta(G) \geq 2$. Therefore, we obtain $|X| \geq 2$. We finish the proof of Claim 1. □

It follows from $k \geq 2$, $bind(G) \geq \frac{2}{2k-1}$, Lemma 5 and Claim 1 that

$$i(G' - X) = i(G - e - X) \leq i(G - X) + 2 \leq \frac{2k-1}{2}|X| + 2 \leq \frac{2k+1}{2}|X|,$$

which contradicts (1). Hence, $G - e$ has an $\mathcal{H}$-factor by Theorem 1, which implies that G is an $\mathcal{H}$-factor deleted graph. This completes the proof of Theorem 3. ■

Finally, we present an open problem.

Problem. Find a criterion for a graph to be a $\{K_{1,1}, K_{1,2}, \ldots, K_{1,k}, \mathcal{T}(2k+1)\}$-factor deleted graph.

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