

WELL-COVERED TOKEN GRAPHS

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Abstract

The k -token graph $T_k(G)$ is the graph whose vertices are the k -subsets of vertices of a graph G , with two vertices of $T_k(G)$ adjacent if their symmetric difference is an edge of G . We explore when $T_k(G)$ is a well-covered graph, that is, when all of its maximal independent sets have the same cardinality. For bipartite graphs G , we classify when $T_k(G)$ is well-covered. For an arbitrary graph G , we show that if $T_2(G)$ is well-covered, then the girth of G is at most four. We include upper and lower bounds on the independence number of $T_k(G)$, and provide some families of well-covered token graphs.

Keywords: independence number, well-covered graph, token graph, double vertex graph, symmetric power of a graph.

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1. INTRODUCTION

Let G be a graph with vertex set $V = V(G)$ of order n and let $1 \leq k \leq n - 1$. The k -token graph of G , denoted $T_k(G)$, has as vertices the k -subsets of V with two vertices adjacent if their symmetric difference is an edge of G . Thus a vertex of

$T_k(G)$ can be thought of as a placement of indistinguishable tokens on k vertices of G with two vertices $u, v \in V(T_k(G))$ adjacent if u can be obtained from v by moving a single token along an edge of G . Hence, if G is a connected graph, then $T_k(G)$ is also connected. An example of a graph G and its 2-token graph $T_2(G)$ is given in Figure 1. We often abuse notation and write $i_1i_2 \cdots i_k$ instead of $\{i_1, \dots, i_k\}$ for a k -subset of V . Note the isomorphic graphs: $T_1(G) \cong G$ and $T_k(G) \cong T_{n-k}(G)$ for $1 \leq k \leq n-1$. Thus, when exploring properties of token graphs, it is sufficient to consider values of k satisfying $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$. If $\{u, v\}$ is an edge of $T_k(G)$, then $|u \cap v| = k-1$ and we refer to the set $u \cap v$ as the *anchor* of the edge $\{u, v\}$.

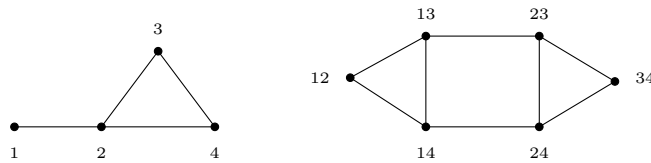


Figure 1. A graph G and its 2-token graph $T_2(G)$.

The k -token graphs appear in the literature under a number of different names. The k -token graphs are a generalization of the *Johnson graphs* (see e.g. [11]). In particular, if K_n is the complete graph on n vertices, then $T_k(K_n)$ is the Johnson graph $J(n, k)$. Thus $T_2(K_n)$ is also known to be the complement of the *Kneser graph* $Kn(n, 2)$. The k -token graph $T_k(G)$ is also known as the *symmetric k^{th} power* of G (see e.g. [4, 5]). Finally, the 2-token graph $T_2(G)$ is also called a *double vertex graph* (see e.g. [1, 2]).

Various properties of token graphs have recently been studied. For example, in [6], Carballosa, Fabila-Monroy, Leañós, and Rivera characterize when the token graphs are regular, as well as when a token graph is planar. In [16], Leañós and Trujillo-Negrete prove a conjecture about the connectivity of token graphs. In [18], Rivera and Trujillo-Negrete explore the Hamiltonicity of token graphs. The spectra of token graphs has been explored in various papers in the context of exploring cospectral graphs (see e.g. [3, 4]).

In this paper we explore properties of the independent sets of $T_k(G)$, and in particular, we focus on the problem of determining when $T_k(G)$ is well-covered. An *independent set* of a graph Γ is a subset S of vertices of Γ such that no two vertices in S are adjacent in Γ . The *independence number* of Γ , denoted $\alpha(\Gamma)$, is the maximum cardinality of any independent set of Γ . For example, for the graphs in Figure 1, $\alpha(G) = \alpha(T_2(G)) = 2$. A graph Γ is *well-covered* if all of its maximal independent sets have the same cardinality [17]. The graph G in Figure 1 is not well-covered but $T_2(G)$ is well-covered. The graph G in Figure 2 is not well-covered, nor is $T_2(G)$ well-covered, as illustrated in Example 3.10.

Some results are known about $\alpha(T_k(G))$. In [8], de Alba, Carballosa, Leaños, and Rivera bound the independence number of $T_k(G)$ when G is bipartite. When $k = 2$, they derive exact values for $\alpha(T_2(G))$ when G is either the complete bipartite graph, the cycle C_n , or the path P_n . Jiménez-Sepúlveda and Rivera [15] determine $\alpha(T_2(G))$ when G is the fan graph and the wheel graph. A sharp lower bound on $\alpha(T_2(G))$ appears in work of Deepalakshmi, Marimuthu, Somasundaram, and Arumugam [9] (also see Remark 3.5).

In the first part of this paper, we derive sharp upper and lower bounds for $\alpha(T_k(G))$ in terms of $\alpha(G)$ for all $k \geq 2$. In particular, in Theorem 2.1 and Corollary 3.4 we show that

$$\binom{\alpha(G)}{k} \leq \alpha(T_k(G)) \leq \frac{1}{k} \binom{n}{k-1} \alpha(G).$$

Interestingly, equality in the upper bound depends upon the existence of a specific combinatorial design. We also produce methods to construct maximal independent sets in $T_k(G)$, when $k = 2$, from independent sets of G (see e.g. Theorems 3.2, 3.9 and 3.11). We obtain some results about characteristics of graphs G for which $T_k(G)$ is well-covered. For example, we observe in Corollary 4.10 that if G is a bipartite graph, then $T_k(G)$ is well-covered if and only if $k = 1$ and G is well-covered. We determine in Corollary 5.2 that a graph G cannot have a large girth if $T_2(G)$ is well-covered, where *girth* is the length of the smallest cycle in G . We also provide some infinite families of graphs G for which $T_2(G)$ is well-covered.

We use the following outline in our paper. In Section 2 we prove our results about the upper bound, while Section 3 focuses on constructions of maximal independent sets. This section includes a general lower bound on the independence number. In Section 4, we characterize when $T_k(G)$ is well-covered if G is bipartite. In Section 5 we provide some restrictions on graphs G for which $T_2(G)$ is well-covered. Then in Section 6 we provide some families of graphs G for which $T_2(G)$ is well-covered. Section 7 contains some concluding remarks, and finally, in the Appendix we list all the graphs G on nine or fewer vertices such that $T_2(G)$ is well-covered.

We end this section with some common definitions that we will use throughout the paper. The subgraph of G induced by the set of vertices $A \subset V(G)$ is denoted $G[A]$, having vertex set A with two vertices adjacent in $G[A]$ if and only if they are adjacent in G . Given a $x \in V(G)$, the *neighbourhood* of x in G is the set $N_G(x) = \{y \mid \{x, y\} \in E(G)\}$. Given a set $X \subseteq V(G)$, the *neighbourhood* of X in G is $N_G(X) = \bigcup_{x \in X} N_G(x)$. The *closed neighbourhood* of X in G is $N_G[X] = X \cup N_G(X)$. For any $W \subseteq V(G)$, let $G \setminus W$ denote the graph obtained by removing all the vertices of W from G and all edges incident to a vertex in W .

2. INDEPENDENT SETS OF TOKEN GRAPHS AND COMBINATORIAL DESIGNS

In this section we describe a relationship between $\alpha(G)$ and $\alpha(T_k(G))$, and a connection with combinatorial designs. Recall that a t - (v, k, λ) design is a collection of k -subsets of a set of v elements, such that every t -subset appears in exactly λ of the k -subsets.

Theorem 2.1. *Let $k \leq \lfloor \frac{n}{2} \rfloor$. If G is a graph on n vertices with no isolated vertices, then*

$$\alpha(T_k(G)) \leq \frac{1}{k} \binom{n}{k-1} \alpha(G).$$

If equality occurs, then there exists a t - (n, k, λ) design with $t = k - 1$ and $\lambda = \alpha(G)$.

Proof. Consider an independent set $S \subset V(T_k(G))$ with $|S| = \alpha(T_k(G))$. Each $v \in S$ contains k potential anchors. Consider the multiset M consisting of all the subsets R of cardinality $k - 1$ such that $R \subset v$ for some $v \in S$. Then $|M| = k\alpha(T_k(G))$. Note that there are at most $\binom{n}{k-1}$ different potential anchors to be constructed from n vertices and each anchor can appear at most $\alpha(G)$ times in M . Thus $|M| \leq \binom{n}{k-1} \alpha(G)$. If we have equality, then every $k - 1$ subset must appear as an anchor exactly $\alpha(G)$ times, hence M is a t - $(n, k, \alpha(G))$ design with $t = k - 1$. ■

We note that equality is possible in Theorem 2.1. For example, let $G = C_5$, the cycle graph on five vertices. Then $\alpha(G) = 2$ and $\{12, 23, 34, 45, 15\}$ is a maximum independent set in $T_2(G)$ so that $\alpha(T_2(G)) = 5$.

Example 2.2. It was shown in [8, Corollary 3.10] that $\alpha(T_3(P_{2m+1})) = \frac{(2m+1)m(m+1)}{3}$, which meets the bound in Theorem 2.1. This corresponds to the existence of a 2 - $(2m + 1, 3, m + 1)$ design. It was also observed in [8, Corollary 3.10] that $\alpha(T_2(K_{m,m})) = m^2$ which again meets the bound in Theorem 2.1 and corresponds to the existence of a 1 - $(2m, 2, m)$ design.

Another example is the complete graph K_n with $k = 2$ and n even, as noted in the next remark. We provide a proof for completion but note that this is a known result since $T_2(K_n)$ is merely the line graph of the complete graph.

Remark 2.3. If $n \geq 2$, then $\alpha(T_2(K_n)) = \lfloor \frac{n}{2} \rfloor$.

Proof. If n is even, the set $\{12, 34, \dots, (n-1)n\}$ is an independent set of $T_2(K_n)$. If n is odd, then $\{12, 34, \dots, (n-2)(n-1)\}$ is an independent set. So $\alpha(T_2(K_n)) \geq \lfloor \frac{n}{2} \rfloor$. But by Theorem 2.1, $\lfloor \frac{n}{2} \rfloor$ is also an upper bound on $\alpha(T_2(K_n))$. ■

We can characterize when we get equality in Theorem 2.1 for the complete graphs. Note that $T_k(K_n)$ is isomorphic to the Johnson graph $J(n, k)$.

Theorem 2.4. *Given $n \geq 2$ and $k \leq \lfloor \frac{n}{2} \rfloor$, then $\alpha(T_k(K_n)) = \frac{1}{k} \binom{n}{k-1}$ if and only if there exists a t - $(n, k, 1)$ design with $t = k - 1$.*

Proof. Let M be a collection of k subsets of an n set forming a t - $(n, k, 1)$ design with $t = k - 1$. Then the number of k -subsets in M is $\frac{1}{k} \binom{n}{k-1}$ (see e.g. [14, Corollary 1.4]) and each element of M is a vertex of $T_k(K_n)$. Since each $(k - 1)$ -subset of n appears in at most one block, no two vertices appearing in M are adjacent in $T_k(K_n)$. Thus M is an independent set in $T_k(K_n)$. From the proof of Theorem 2.1, we have $\alpha(T_k(K_n)) = |M|$. The converse follows directly from Theorem 2.1. ■

Example 2.5. It is known that for any $t \geq 1$ there exists a 2 - $(6t + 1, 3, 1)$ design and a 2 - $(6t + 3, 3, 1)$ design (see Steiner systems, e.g. [14, p. 174]). Thus, for $t \geq 1$,

$$\alpha(T_3(K_{6t+1})) = \frac{1}{3} \binom{6t+1}{2} \text{ and } \alpha(T_3(K_{6t+3})) = \frac{1}{3} \binom{6t+3}{2}.$$

3. CONSTRUCTING INDEPENDENT SETS IN TOKEN GRAPHS

In this section, we describe some methods of constructing independent sets of token graphs. We start with a remark that describes one way to visualize an independent set in a 2-token graph.

Remark 3.1. For a graph G on n vertices, one can picture an independent set in $T_2(G)$ as a set of edges E selected from K_n with the property that no two adjacent edges in E are part of a triangle whose third side is an edge of G (considering G as a subgraph of K_n).

To describe some constructions of independent sets in k -token graphs, we introduce the following notation. Given subsets $V_1, V_2, \dots, V_k \subseteq V(G)$, not necessarily distinct, we define

$$V_1V_2 \cdots V_k = \{x_1x_2 \cdots x_k \mid x_i \in V_i \text{ and } x_i \neq x_j \text{ for all } i \neq j\}.$$

Note that order does not matter; for instance, if $u, v \in V_i$ with $u \neq v$ then both uv and vu represent $\{u, v\} \in V_iV_i$. Observe that $V_1V_2 \cdots V_k$ is a subset of the vertices of $T_k(G)$. Indeed, $V(G)V(G) \cdots V(G)$ (k times) is the set of vertices of $T_k(G)$.

Theorem 3.2. *Let G be a graph with independent sets V_1, V_2, \dots, V_k such that $V_i \cap V_j = \emptyset$ or $V_i = V_j$ for all $i \neq j$. Then $V_1V_2 \cdots V_k$ is an independent set of $T_k(G)$.*

Proof. Let $W = V_1V_2 \cdots V_k$, and suppose that $x_1x_2 \cdots x_k, y_1y_2 \cdots y_k \in W$. Let $A \Delta B$ denote the symmetric difference of A and B . If

$$|x_1x_2 \cdots x_k \Delta y_1y_2 \cdots y_k| \neq 2,$$

then these vertices cannot be adjacent by the definition of $T_k(G)$. So, suppose

$$x_1x_2 \cdots x_k \Delta y_1y_2 \cdots y_k = \{x_i, y_j\}.$$

We have $x_i \in V_i$ and $y_j \in V_j$.

If $V_i = V_j$, then $\{x_i, y_j\}$ is not an edge in $E(G)$ since V_i is an independent set, and thus $x_1x_2 \cdots x_k$ and $y_1y_2 \cdots y_k$ are not adjacent in $T_k(G)$. So, suppose that $V_i \cap V_j = \emptyset$. Suppose that V_i appears a times among V_1, V_2, \dots, V_k , i.e. $V_{i_1} = \dots = V_{i_a} = V_i$. So, exactly a distinct elements of $\{x_1, x_2, \dots, x_k\}$ belong to V_i and the same is true for $\{y_1, \dots, y_k\}$. However, since $x_1x_2 \cdots x_k \Delta y_1y_2 \cdots y_k = \{x_i, y_j\}$ with $y_j \in V_j$ and $V_i \cap V_j = \emptyset$, all of the distinct elements in $\{y_1, \dots, y_k\}$ that belong to V_i must appear in $x_1x_2 \cdots x_k \setminus \{x_i\}$. But there are only $a - 1$ distinct elements of V_i in $x_1x_2 \cdots x_k \setminus \{x_i\}$. So, we cannot have a symmetric difference of the form $\{x_i, y_j\}$ with $x_i \in V_i, y_j \in V_j$ and $V_i \cap V_j = \emptyset$. ■

Corollary 3.3. *Let G be a graph with independent sets V_1, V_2, \dots, V_k such that $V_i \cap V_j = \emptyset$ or $V_i = V_j$ for all $i \neq j$. Suppose that $\{V_{i_1}, \dots, V_{i_l}\}$ are the distinct subsets that appear among V_1, \dots, V_k , and that V_{i_t} appears a_t times (so $a_1 + \dots + a_l = k$). Then*

$$\binom{|V_{i_1}|}{a_1} \binom{|V_{i_2}|}{a_2} \cdots \binom{|V_{i_l}|}{a_l} \leq \alpha(T_k(G)).$$

Proof. By Theorem 3.2, it is enough to show that

$$\binom{|V_{i_1}|}{a_1} \binom{|V_{i_2}|}{a_2} \cdots \binom{|V_{i_l}|}{a_l} = |V_1V_2 \cdots V_k|.$$

By definition of $V_1 \cdots V_k$, a_t of the elements of $x_1x_2 \cdots x_k \in V_1V_2 \cdots V_k$ belong to V_{i_t} , and these a_t elements are distinct. So, there are $\binom{|V_{i_t}|}{a_t}$ ways to pick these a_t elements. Since $V_{i_t} \cap V_{i_j} = \emptyset$ for all $i \neq j$, the result now follows. ■

We get a lower bound on the independence number for any token graph.

Corollary 3.4. *If G is a graph with independence number $\alpha(G)$, then*

$$\binom{\alpha(G)}{k} \leq \alpha(T_k(G)).$$

Proof. Let $W \subseteq V(G)$ be an independent set of G with $|W| = \alpha(G)$, and apply Corollary 3.3 with $V_1 = \dots = V_k = W$. ■

Remark 3.5. The lower bound in Corollary 3.4 is sharp. In particular, observe that $\alpha(K_{1,n-1}) = n - 1$ and, in [8], it was determined that $\alpha(T_k(K_{1,n-1})) = \binom{n-1}{k}$. When $k = 2$ and G is not isomorphic to $K_{1,n-1}$, then the lower bound in Corollary 3.4 can be improved. In particular, $\alpha(T_2(G)) \geq \binom{\alpha(G)}{2} + \lfloor \frac{n-\alpha(G)}{2} \rfloor$, as first shown in [9, Theorem 2.7].

When $k = 2$, Theorem 3.2 gives us the following consequences for the 2-token graphs of some family of bipartite graphs. In particular, we recover some of the formulas in [8].

Corollary 3.6. *If G is a bipartite graph on n vertices with bipartition $V(G) = V_1 \cup V_2$ such that $|V_1| = |V_2| = \alpha(G) = \frac{n}{2}$, then*

$$\alpha(T_2(G)) = \frac{n^2}{4}.$$

In particular, $\alpha(T_2(P_{2n})) = \alpha(T_2(C_{2n})) = \alpha(T_2(K_{n,n})) = n^2$.

Proof. By Theorem 3.2, V_1V_2 is an independent set of $T_2(G)$, and furthermore, $|V_1V_2| = |V_1||V_2|$ since $V_1 \cap V_2 = \emptyset$, and thus $\alpha(T_2(G)) \geq |V_1||V_2|$. By Theorem 2.1, $\alpha(T_2(G)) \leq \frac{n\alpha(G)}{2} = \frac{2|V_1||V_2|}{2}$. Therefore $\alpha(T_2(G)) = |V_1||V_2| = \frac{n^2}{4}$.

The last statement follows immediately since each of the listed bipartite graphs have $2n$ vertices with a bipartition $V_1 \cup V_2$ such that $|V_1| = |V_2| = n$. ■

The next result follows from Theorem 3.2 by noting that if V_1, V_2, V_3, V_4, V_5 are disjoint sets of G , then no vertex of V_1V_2 will be adjacent to any vertex in $V_3V_4 \cup V_5V_5$ in $T_2(G)$.

Corollary 3.7. *Let G be a graph having disjoint independent sets V_1, V_2, \dots, V_k . If k is even, then $V_1V_2 \cup V_3V_4 \cup \dots \cup V_{k-1}V_k$ is an independent set of $T_2(G)$. If k is odd, then $V_1V_2 \cup V_3V_4 \cup \dots \cup V_{k-2}V_{k-1} \cup V_kV_k$ is an independent set of $T_2(G)$.*

Corollary 3.8. *If G is the complete multipartite graph K_{n_1, n_2, \dots, n_k} of order n with k even, and if $n_i = \frac{n}{k}$ for $1 \leq i \leq k$, then $\alpha(T_2(G)) = \frac{n^2}{2k}$.*

Proof. By Corollary 3.7, $\alpha(T_2(G)) \geq \frac{n^2}{2k}$. Equality follows from Theorem 2.1. ■

We now give some constructions of maximal independent sets in $T_2(G)$; this enables us to derive lower bounds on $\alpha(T_2(G))$ for specific graphs. For the following, if A, B are disjoint independent sets of a graph G , we define

$$\phi(A, B) = \{x \in A \mid B \cup \{x\} \text{ is an independent set}\}.$$

In Theorem 3.9, we use a partition (in fact, a coloring) of the vertex set of a graph G to obtain a maximal independent set of $T_2(G)$. The condition that

$\phi(V_j, V_i) = \emptyset$ when $j > i$ implies that the partition $V_1 \cup \dots \cup V_k$ is constructed so that V_i is a maximal independent set on $G \setminus (V_1 \cup \dots \cup V_{i-1})$, for $1 \leq i \leq k-1$, with $V_0 = \emptyset$.

Theorem 3.9. *Let G be a graph and $V_1 \cup V_2 \cup \dots \cup V_k$ be a partition of $V(G)$ into independent sets such that $\phi(V_j, V_i) = \emptyset$ when $j > i$. If k is even, let*

$$H = (V_1V_2 \cup V_3V_4 \cup \dots \cup V_{k-1}V_k) \\ \cup (\phi(V_1, V_2)\phi(V_1, V_2) \cup \dots \cup \phi(V_{k-1}, V_k)\phi(V_{k-1}, V_k)).$$

If k is odd, let

$$H = (V_1V_2 \cup V_3V_4 \cup \dots \cup V_{k-2}V_{k-1}) \cup V_kV_k \\ \cup (\phi(V_1, V_2)\phi(V_1, V_2) \cup \dots \cup \phi(V_{k-2}, V_{k-1})\phi(V_{k-2}, V_{k-1})).$$

Then H is a maximal independent set of $T_2(G)$.

Proof. Using Corollary 3.7 and the fact that $V_1V_1 \cup V_2V_2 \cup \phi(V_1, V_2)\phi(V_1, V_2)$ is an independent set, it follows that H is an independent set. We will show that H is maximal. Suppose $H \cup \{xy\}$ is an independent set for some $xy \in V(T_2(G))$. We will demonstrate that $xy \in H$.

Suppose $x, y \in V_i$ for some i . We consider three cases.

Case 1. Suppose $V_{i-1}V_i \subseteq H$. Since $\phi(V_i, V_{i-1}) = \emptyset$, x is adjacent to some vertex $w \in V_{i-1}$ in G . But then yw is adjacent to yx in $T_2(G)$, contradicting the fact that $H \cup \{xy\}$ is an independent set.

Case 2. Suppose $V_iV_{i+1} \subseteq H$. Then both x and y can not be adjacent to any vertex in V_{i+1} in G , hence $xy \in \phi(V_i, V_{i+1})\phi(V_i, V_{i+1}) \subseteq H$.

Case 3. Suppose $V_iV_i \subseteq H$. In this case, k is odd, and $i = k$, in which case $xy \in H$.

Suppose $x \in V_i$ and $y \in V_j$ for some $j > i$.

Case 1. Suppose $V_{i-1}V_i \subseteq H$. Since $H \cup \{xy\}$ is an independent set, xy is not adjacent to any vertex in xV_{i-1} in $T_2(G)$. But then y is not adjacent to any vertex in V_{i-1} in G , contradicting the fact that $\phi(V_j, V_{i-1}) = \emptyset$.

Case 2. Suppose $V_iV_{i+1} \subseteq H$. If $j = i+1$, then $xy \in H$. So suppose $j > i+1$. Then, since $\phi(V_j, V_{i+1}) = \emptyset$, y is adjacent to at least one vertex $w \in V_{i+1}$ in G . But then xy is adjacent to $xw \in V_iV_{i+1}$ in $T_2(G)$, contradicting the fact that $H \cup \{xy\}$ is an independent set. ■

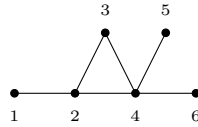


Figure 2. The graph G in Example 3.10.

Example 3.10. Consider the graph G in Figure 2. Let $V_1 = \{1, 3, 5, 6\}$, $V_2 = \{2\}$ and $V_3 = \{4\}$. Then $\phi(V_1, V_2) = \{5, 6\}$ and $H = V_1V_2 \cup \phi(V_1, V_2)\phi(V_1, V_2) = \{12, 23, 25, 26, 56\}$ is a maximal independent set in $T_2(G)$ by Theorem 3.9. If $V_1 = \{2, 5, 6\}$, $V_2 = \{1, 4\}$ and $V_3 = \{3\}$, then applying Theorem 3.9 we get $H = \{12, 15, 16, 24, 45, 46\}$ is an even larger maximal independent set. Further, if $V_1 = \{2, 5, 6\}$, $V_2 = \{1, 3\}$, and $V_3 = \{4\}$, then $\phi(V_1, V_2) = \{5, 6\}$ and $H = \{12, 15, 16, 23, 35, 36, 56\}$ is an even larger maximal independent set of $T_2(G)$.

The next theorem provides another construction of a maximal independent set in $T_2(G)$, starting with a vertex colouring of G .

Theorem 3.11. *Let G be a graph with a vertex partition into independent sets V_1, V_2, \dots, V_k such that $\phi(V_j, V_i) = \emptyset$ when $j > i$. Let E be a maximal set of edges from $E(G)$ such that the following conditions are satisfied.*

- (1) *If $e = \{u, r\} \in E$, $u \in V_i$ and $r \in V_j$ for $i \neq j$, then e is an isolated edge in $G[V_i \cup V_j]$.*
- (2) *If $e_1, e_2 \in E$ share a common endpoint in G , then there is no triangle in G containing e_1 and e_2 .*

Then $A = V_1V_1 \cup V_2V_2 \cup \dots \cup V_kV_k \cup E$ is a maximal independent set in $T_2(G)$.

Proof. We first show that A is an independent set in $T_2(G)$. The subset $A \setminus E$ is an independent set since each V_i is an independent set. Let $x = v_{ia}v_{ib} \in A \setminus E$ with $v_{ia}, v_{ib} \in V_i$. Let $y \in E$. If x, y do not share an anchor, then x and y are not adjacent. Thus, without loss of generality, suppose $y = v_{ia}u$. By definition of E , $u \in V_j$ for some $j \neq i$. Further $\{v_{ia}, u\}$ is an isolated edge in $G[V_i \cup V_j]$. Thus, y is not adjacent to x in $T_2(G)$.

Suppose now that $x, y \in E$. If x, y do not share an anchor, then x and y are not adjacent. So suppose $x = uv$ and $y = ut$. By condition (2), v and t are not adjacent in G , and hence x and y are not adjacent in $T_2(G)$. Therefore A is an independent set.

Now we show A is maximal. Suppose $x = x_ix_j \notin A$ for some $x_i \in V_i$ and $x_j \in V_j$. We know that $i \neq j$ (since otherwise, x would be in A). Without loss of generality $i < j$.

Suppose that no vertex of A is adjacent to x in $T_2(G)$. Then x_i is the only possible neighbour of x_j in V_i , and x_j is the only possible neighbour of x_i in V_j . Indeed, suppose x_i has a neighbour $z \neq x_j$ with $z \in V_j$. Then $x_i x_j$ is adjacent to $x_j z \in V_j V_j \subseteq A$. Similarly, suppose x_j has a neighbour $y \neq x_i$ and $y \in V_i$. Then, $x_i x_j$ is adjacent to $x_i y \in V_i V_i \subseteq A$. Since $\phi(V_j, V_i) = \emptyset$, it follows that $x \in E(G)$ and thus x is an isolated edge in $G[V_i \cup V_j]$. Suppose there is some $e \in E$ such that e shares a common endpoint with x . Then, without loss of generality, $e = x_i u$. If u is adjacent to x_j in G , then e is adjacent to x , but this contradicts the fact that x is not adjacent to any element of A . Therefore, u is not adjacent to x_j (and e is not adjacent to x). Then there is no triangle in G containing e and x . Since E does not contain x , it would follow from (1) and (2) that E is not maximal, a contradiction. ■

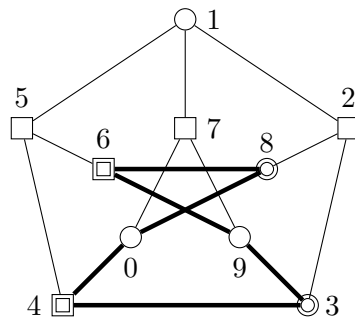


Figure 3. A colouring of the Petersen graph with some bolded edges.

Example 3.12. Let G be the Petersen graph depicted in Figure 3 with independent sets $V_1 = \{0, 1, 9\}$, $V_2 = \{2, 5, 7\}$, $V_3 = \{4, 6\}$, $V_4 = \{3, 8\}$ and bolded edges $E = \{04, 43, 39, 96, 68, 08\}$. Note that $\phi(V_j, V_i) = \emptyset$ for $j > i$. By Theorem 3.11, $V_1 V_1 \cup V_2 V_2 \cup V_3 V_3 \cup V_4 V_4 \cup E$ is a maximal independent set of $T_2(G)$ of cardinality fourteen.

Note that $\phi(V_1, V_2) = \emptyset$ and $\phi(V_3, V_4) = \emptyset$. Thus, by Theorem 3.9, $V_1 V_2 \cup V_3 V_4$ is a maximal independent set of $T_2(G)$ of cardinality thirteen. Thus, the 2-token graph of the Petersen graph is not well-covered.

Since G contains no triangles, the edges of G form an independent set in $T_2(G)$ of cardinality fifteen. This set is maximal since the addition of any further edge would form a triangle with the edges of G (see Remark 3.1).

Further, if $U_1 = \{5, 8, 9\}$, $U_2 = \{0, 2, 6\}$, $U_3 = \{3\}$, $U_4 = \{4\}$, $U_5 = \{1\}$, $U_6 = \{7\}$ and $F = \{39, 23, 04, 45, 15, 12, 07, 79\}$, then the hybrid construction $U_1 U_1 \cup U_2 U_2 \cup U_3 U_4 \cup U_5 U_6 \cup F$ is an independent set of $T_2(G)$ with cardinality sixteen. A computer check can verify that $\alpha(T_2(G)) = 16$.

4. CHARACTERIZATION OF BIPARTITE GRAPHS G WITH $T_k(G)$
WELL-COVERED

In this section we characterize when $T_k(G)$ is well-covered if G is a connected bipartite graph. In order to distinguish the vertices of G and those of $T_k(G)$ in this section, we use uppercase letters for vertices in $T_k(G)$. Before we address bipartite graphs, we present the following useful known result.

Theorem 4.1 [12]. *If G is a well-covered graph and I is an independent set in G , then $G \setminus N_G[I]$ is well-covered.*

A bipartite graph with bipartition $V(G) = L \cup R$ is *balanced* if $|L| = |R|$.

Theorem 4.2 ([8] and [11, Proposition 12]). *Let G be a connected bipartite graph with bipartition $V(G) = L \cup R$. Then $T_k(G)$ is bipartite with bipartition*

$$\{A \in V(T_k(G)) \mid |R \cap A| \text{ is even}\} \cup \{A \in V(T_k(G)) \mid |R \cap A| \text{ is odd}\}.$$

We require a sequence of technical lemmas.

Lemma 4.3. *Let G be a bipartite graph with bipartition $V(G) = L \cup R$ and suppose $A_1, A_2 \in V(T_k(G))$ for some $k \geq 2$. If $|A_1 \cap L| = \ell$ and A_2 is adjacent to A_1 in $T_k(G)$, then $|A_2 \cap L| \in \{\ell - 1, \ell + 1\}$.*

Proof. Since A_1, A_2 are adjacent, they must share an anchor. Hence $|A_2 \cap L| = \ell + i$ for some $i \in \{-1, 0, 1\}$. However, if $i = 0$, that would contradict Theorem 4.2. ■

Lemma 4.4. *Suppose G is a connected bipartite graph with bipartition $V(G) = L \cup R$ and $|L| \geq k \geq 2$. If $A \in V(T_k(G))$ with $|A \cap L| = k$, then $\deg(A) \geq 2$.*

Proof. Suppose $|A \cap L| = k$ and $A = A' \cup \{x, y\} \subseteq L$ for some A' with $|A'| = k - 2$. Since G is connected, let $u, v \in R$ be adjacent to x and y , respectively. Note that u and v need not be distinct. Then, A is adjacent to both $A' \cup \{x, v\}$ and $A' \cup \{u, y\}$. ■

Lemma 4.5. *Let G be a connected bipartite graph with bipartition $V(G) = L \cup R$ and let $k \geq 2$. Suppose that $|L| > k$ and $|R| \geq 1$. Let $T = T_k(G)$. If $A_1, A_2 \in V(T)$, $A_1 \neq A_2$ and $|A_1 \cap L| = |A_2 \cap L| = k$, then A_1 is not an isolated vertex in $T \setminus N_T[A_2]$.*

Proof. By Lemma 4.3, A_1 is not adjacent to A_2 . Since $A_1 \neq A_2$, without loss of generality, suppose $A_1 = l_1 l_2 \cdots l_k$ and $l_1 \notin A_2$. Since G is connected, there is a vertex $r \in R$ such that l_2 is adjacent to r in G . Thus A_1 is adjacent to $l_1 r l_3 \cdots l_k$. However, A_2 is not adjacent to $l_1 r l_3 \cdots l_k$ since $l_1, r \notin A_2$. Thus $l_1 r l_3 \cdots l_k \notin N_T[A_2]$. Therefore A_1 is not an isolated vertex in $T \setminus N_T[A_2]$. ■

Lemma 4.6. *Let G be a connected bipartite graph with bipartition $V(G) = L \cup R$ and let $k \geq 2$. Suppose that $|L| \geq k$ and $|R| \geq 2$. Let $T = T_k(G)$. Let $A = l_1 l_2 l_3 \cdots l_k \in V(T)$ with $|A \cap L| = k$ and $B \in V(T_k(G))$ with $|B \cap L| = k - 2$. If B is an isolated vertex in $T \setminus N_T[A]$, then $|B \cap A| = k - 2$ and, after relabelling, $B = r_1 r_2 l_3 \cdots l_k$ such that*

- (1) $N_G(\{l_3, l_4, \dots, l_k\}) \subseteq \{r_1, r_2\}$,
- (2) $N_G(\{r_1, r_2\}) \subseteq \{l_1, l_2, l_3, \dots, l_k\}$,
- (3) $\deg_T(B) \geq 2$.

Proof. Recall that since G is connected, T is connected. In particular, T contains no isolated vertices. Suppose B is an isolated vertex in $T \setminus N_T[A]$. Then $N_T(B) \subseteq N_T(A)$ and the vertices A and B must share at least one common neighbour. By Lemma 4.3, there is a vertex $C \in V(T)$ with $|C \cap L| = k - 1$ adjacent to B and A . Since C must share an anchor with A , without loss of generality, $C = r_1 l_2 l_3 \cdots l_k$ for some $r_1 \in R$. Since B shares an anchor with C , by Lemmas 4.3 and 4.4, $|B \cap A| = k - 2$. Without loss of generality, assume that $B = r_1 r_2 l_3 \cdots l_k$ for some $r_2 \in R$.

Suppose $N_G(\{l_3, l_4, \dots, l_k\}) \not\subseteq \{r_1, r_2\}$. Without loss of generality, suppose l_3 is adjacent to some $x \notin \{r_1, r_2\}$. Then B has a neighbour $r_1 r_2 x l_4 \cdots l_k$, but this vertex is not adjacent to A by Lemma 4.3. But then B is not isolated in $T \setminus N_T[A]$, a contradiction. Therefore $N_G(\{l_3, l_4, \dots, l_k\}) \subseteq \{r_1, r_2\}$.

Suppose $N_G(\{r_1, r_2\}) \not\subseteq \{l_1, l_2, \dots, l_k\}$. Without loss of generality, suppose r_2 is adjacent to some $x \notin \{l_1, l_2, \dots, l_k\}$. Then B is adjacent to $r_1 x l_3 \cdots l_k$, which is not adjacent to A , contradicting the fact that B is isolated in $T \setminus N_T[A]$. Therefore $N_G(\{r_1, r_2\}) \subseteq \{l_1, l_2, \dots, l_k\}$.

Finally, B is adjacent to both C and $l_1 r_2 l_3 \cdots l_k$. Therefore $\deg_T(B) \geq 2$. ■

Lemma 4.7. *Let G be a connected bipartite graph with bipartition $V(G) = L \cup R$ and let $k \geq 2$. Suppose that $|L| \geq k$, $|R| \geq 2$ and $|L| + |R| \geq 5$. Let $T = T_k(G)$. Suppose $A = l_1 l_2 l_3 \cdots l_k \in V(T)$ with $|A \cap L| = k$ and $B \in V(T)$ with $|B \cap L| = k - 2$. If B is an isolated vertex in $T \setminus N_T[A]$, then there is no isolated vertex in $T \setminus N_T[B]$.*

Proof. Suppose B is an isolated vertex in $T \setminus N_T[A]$. Note that, by Lemma 4.6, we can assume $B = r_1 r_2 l_3 \cdots l_k$ and $N_T(B) \subseteq N_T(A)$. Suppose that C is an isolated vertex in $T \setminus N_T[B]$, and so $N_T(C) \subseteq N_T(B)$.

By Lemma 4.3, $|C \cap L| \in \{k, k - 2\}$. We claim that $|C \cap L| = k - 2$. Suppose that $|C \cap L| = k$. We first show that $C \neq A$. Indeed, if $C = A$, then $N_T(C) = N_T(B) = N_T(A)$. If $k > 2$, let r be any vertex adjacent to l_k . Then, $A = l_1 l_2 \cdots l_k$ is adjacent to $W = l_1 l_2 \cdots l_{k-1} r$. Note that W cannot be adjacent to B , since l_1 and l_2 both appear in W , while neither appear in B . So W and B do not share an anchor, and thus $N_T(B) \neq N_T(A)$. If $k = 2$, then $A = l_1 l_2$

and $B = r_1r_2$. By Lemma 4.6 we have $N_G(\{r_1, r_2\}) \subseteq \{l_1, l_2\}$. Interchanging the roles of L and R in Lemma 4.6, since $B = r_1r_2$ and $|B \cap R| = 2$ and $|C \cap R| = 0$, we know that $C = l_1l_2 = A$ implies that $N_G(\{l_1, l_2\}) \subseteq \{l_1, l_2\}$. But this means $\{l_1, l_2, r_1, r_2\}$ is a maximal connected component of G , contradicting the hypothesis that G is connected and $|V(G)| \geq 5$. Therefore $C \neq A$.

Since $N_T(C) \subseteq N_T(B) \subseteq N_T(A)$, C is isolated in $T \setminus N_T[A]$, contradicting Lemma 4.5. Thus $|C \cap L| \neq k$, and by Lemma 4.3, $|C \cap L| = k - 2$.

Since C is isolated in $T \setminus N_T[A]$, C and B have a common neighbour D which is also adjacent to A . Thus, given that A and D have a common anchor, without loss of generality, $D = r_1xl_3 \cdots l_k$ for some $x \in \{l_1, l_2\}$.

Suppose $x = l_1$. Then since D and C have a common anchor, without loss of generality, either $C = r_1yl_3 \cdots l_k$ for some $y \in R$ or $C = r_1l_1zl_4 \cdots l_k$ for some $z \in R$. Note that if $k = 2$, the latter case does not occur. Suppose $C = r_1yl_3 \cdots l_k$. Then $y \neq r_2$ since $B \neq C$ and further, C is adjacent to $E = l_1yl_3 \cdots l_k$. However, E is not adjacent to B since $|E \cap B| = k - 2$, violating the fact that $N_T(C) \subseteq N_T(B)$. Therefore $C = r_1l_1zl_4 \cdots l_k$. Then $z = r_2$ by Lemma 4.6. In this case, C is adjacent to $F = r_1l_2l_1l_4 \cdots l_k$ and hence cannot be adjacent to B since $|F \cap B| = k - 2$. This again violates the fact that $N_T(C) \subseteq N_T(B)$. Therefore $x = l_2$. However, with $x = l_2$, a similar argument also leads to a contradiction. Therefore there is no vertex C that is isolated in $T \setminus N_T[B]$. ■

We pause to give an example that will be used to simplify our proof of Theorem 4.9.

Example 4.8. Let $m \geq 4$ and $G = K_{1,m}$ be the complete bipartite graph (a *star* graph) with bipartition $V = \{x\} \cup \{y_1, \dots, y_m\}$. We show that $T_k(G)$ is not well-covered for any $1 \leq k \leq \lfloor \frac{m+1}{2} \rfloor$. Since $T_1(G) = G$ is not well-covered, we first consider $2 \leq k < \lfloor \frac{m+1}{2} \rfloor$. Note that the vertices of $V(T_k(G))$ come in two types: a k subset of $\{y_1, \dots, y_m\}$, and a k subset of V that contains x and a $(k - 1)$ subset of $\{y_1, \dots, y_m\}$. In fact, these two sets form the bipartition of $T_k(G)$. There are $\binom{m}{k}$ vertices of the first type, and $\binom{m}{k-1}$ vertices of the second type. When $k \neq \frac{m+1}{2}$, the parts of the bipartition have different cardinalities, and hence $T_k(G)$ is not well-covered. So, suppose $k = \frac{m+1}{2}$, and hence $\binom{m}{k} = \binom{m}{k-1}$. If $T = T_k(G)$ and we take the non-maximal independent set $I = \{y_1y_2 \cdots y_k\}$, then the bipartite graph $T \setminus N_T[I]$ is not well-covered since

$$N_T[I] = \{y_1y_2 \cdots y_k, xy_2 \cdots y_k, xy_1y_3 \cdots y_k, \dots, xy_1y_2 \cdots y_{k-1}\},$$

and so one part has $\binom{m}{k} - 1$ elements, and the other has $\binom{m}{k-1} - k$. By Theorem 4.1, $T_k(G)$ is not well-covered.

Theorem 4.9. Let $2 \leq k \leq \lceil \frac{n}{2} \rceil$. If G is a connected bipartite graph with $|V(G)| = n \geq 5$, then $T_k(G)$ and $T_{n-k}(G)$ are not well-covered.

Proof. Suppose G is a connected graph on n vertices with bipartition $V(G) = L \cup R$. Without loss of generality, assume that $|L| \geq |R|$ and hence $|L| \geq k$, since $2 \leq k \leq \lceil \frac{n}{2} \rceil$.

Suppose $|R| = 1$. Then G is the star graph which is not well-covered by Example 4.8.

Suppose $|R| \geq 2$. Let $T = T_k(G)$. If T is well-covered, it is necessary that the bipartition as described in Theorem 4.2 is balanced. Let $A \in V(T)$ with $|A \cap L| = k$. Let $Q = T \setminus N_T[A]$. Note that $\deg_T(A) \geq 2$ by Lemma 4.4. If Q contains no isolated vertices, then Q is a bipartite graph with nonempty bipartitions that is not balanced and hence Q is not well-covered. Therefore, by Theorem 4.1, T is not well-covered. Suppose Q contains an isolated vertex B . By Lemma 4.3, $|B \cap L| = k - 2$. Then, by Lemmas 4.5 and 4.7, it follows that $W = T \setminus N_T[B]$ contains no isolated vertices, and $\deg_T(B) \geq 2$. Thus W is an unbalanced bipartite graph with nonempty bipartitions and no isolated vertices. Therefore, by Theorem 4.1, $T_k(G)$ is not well-covered.

Since $T_{n-k}(G) \cong T_k(G)$, it follows that $T_{n-k}(G)$ is also not well-covered. ■

The following corollary gives the desired characterization.

Corollary 4.10. *Let G be a connected bipartite graph with $|V(G)| = n$. Given $1 \leq k \leq \lceil \frac{n}{2} \rceil$, then $T_k(G)$ is well-covered if and only if G is well-covered and $k = 1$ (in this case $T_k(G) \cong G$).*

Proof. If $n \geq 5$, then the result follows from Theorem 4.9. A direct computation on all bipartite graphs on four or fewer vertices finishes the proof. ■

5. RESTRICTIONS ON GRAPHS WITH WELL-COVERED 2-TOKEN GRAPHS.

In this section, we derive some restrictions on G , with regard to girth and independence number, when $T_2(G)$ is well-covered.

Theorem 5.1. *Suppose $|V(G)| \geq 3$, G is connected and $T_2(G)$ is well-covered. If $P = \{x_1, x_2, x_3\}$ is an induced path in G , then either P is part of a four cycle or at least one of the vertices of P is part of a 3-cycle in G .*

Proof. Let $T = T_2(G)$. Suppose that no vertex of P is part of a 3-cycle in G and that there is no induced four cycle in G that includes the vertices x_1, x_2 , and x_3 .

Let I_1, I_2, I_3 be the vertices of $H = G \setminus P$ that are adjacent to x_1, x_2, x_3 respectively. Since x_1, x_2, x_3 are not part of a triangle in G , we know that I_1, I_2, I_3 are independent sets in G . Likewise, $I_1 \cap I_2 = \emptyset = I_2 \cap I_3$ since x_1, x_2, x_3 are not part of a triangle. Further, $I_1 \cap I_3 = \emptyset$ since x_1, x_2 , and x_3 are not part of a 4-cycle.

Consider $T_2(P) = \{x_1x_2, x_1x_3, x_2x_3\}$. Then $N_T(T_2(P))$ is precisely the set of vertices $x_iI_j = \{x_iy \mid y \in I_j\}$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$.

Suppose $I_1 \neq \emptyset$ and $I_3 \neq \emptyset$. Consider the independent set $A = x_1I_1 \cup x_2I_2 \cup x_3I_3 \cup I_1I_3$. We have $N_T(T_2(P)) \subseteq N_T[A]$, and $T_2(P) \cap N_T[A] = \emptyset$. Thus $T_2(P)$ is a maximal connected component of $T \setminus N_T[A]$. Since $T_2(P)$ is not well-covered, it follows that $T_2(G)$ is not well-covered by Theorem 4.1.

Suppose $I_1 = \emptyset$ and $I_3 \neq \emptyset$. Consider the independent set $A = x_2I_2 \cup x_3I_3$. Let $T' = T \setminus N_T[A]$. Then $N_{T'}(T_2(P)) = x_1I_3$. Let B be a maximal independent set in $T' \setminus (N_T[A] \cup T_2(P) \cup x_1I_3)$. Then $A \cup B$ is an independent set and no vertex of B is adjacent to $T_2(P)$. Thus $T_2(P) \cup x_1(I_3 \setminus N_T[B])$ is a maximal connected component in $T \setminus N_T[A \cup B]$. Further, $T_2(P) \cup x_1(I_3 \setminus N_T[B])$ is not well-covered. To see this, consider that $\{x_1x_3\}$ is a maximal independent set, but $\{x_1x_2, x_2x_3\}$ is a larger independent set. Therefore, by Theorem 4.1, $T_2(G)$ is not well-covered.

Suppose $I_1 \neq \emptyset$ and $I_3 = \emptyset$. By symmetry, this case is similar the previous case.

Suppose $I_1 = I_3 = \emptyset$. Then $N_T(T_2(P)) = x_1I_2 \cup x_3I_2$. Consider the independent set $A = x_2I_2$. Note that $N_T(T_2(P)) \subseteq N_T[A]$ while $T_2(P) \cap N_T[A] = \emptyset$. Thus, $T \setminus N_T[A]$ contains an isolated path $T_2(P)$ and hence is not well-covered. Therefore $T_2(G)$ is not well-covered by Theorem 4.1. ■

Corollary 5.2. *If $|V(G)| \geq 3$, G is connected and $T_2(G)$ is well-covered, then $\text{girth}(G) \leq 4$.*

We finish this section with a bound on $\alpha(G)$ when $T_2(G)$ is well-covered.

Lemma 5.3. *Let G be a connected graph, and let V_1, V_2, \dots, V_r form a partition of $V(G)$ such that V_i is a maximum independent set in $G \setminus (\bigcup_{j < i} V_j)$. If $T_2(G)$ is well-covered, then $|\phi(V_{2i-1}, V_{2i})| \leq 1$ for $1 \leq i \leq \lfloor \frac{r}{2} \rfloor$.*

Proof. Suppose that $|\phi(V_1, V_2)| \geq 2$. Since V_1 is a maximum independent set of G , and because $V_2 \cup \phi(V_1, V_2)$ is an independent set, $|V_2 \cup \phi(V_1, V_2)| \leq |V_1|$. Thus $|V_2| \leq |V_1| - 2$.

Let $x \in \phi(V_1, V_2)$. Consider the sets $W_1 = V_1 \setminus \{x\}$ and $W_2 = V_2 \cup \{x\}$. Note that $|W_1W_2| = (|V_1| - 1)(|V_2| + 1) \geq |V_1V_2|$ since $V_2 \leq V_1 - 2$. Further $W_1W_2 \cup \phi(V_1, V_2) \cup \phi(V_1, V_2)$ is an independent set in $T_2(G)$. By Theorem 3.9, $A = V_1V_2 \cup \phi(V_1, V_2) \cup \phi(V_1, V_2) \cup V_2V_3 \cup \dots$ is a maximal independent set of $T_2(G)$. But $|A \setminus V_1V_2 \cup W_1W_2| > |A|$, and hence $T_2(G)$ is not well-covered. By a similar argument, $|\phi(V_{2i-1}, V_{2i})| \leq 1$ for $2 \leq i \leq \lfloor \frac{r}{2} \rfloor$. ■

Theorem 5.4. *Suppose G is a connected graph and $T_2(G)$ is well-covered. Let V_1, V_2, \dots, V_r form a partition of $V(G)$ such that V_i is a maximum independent*

set in $G \setminus (\bigcup_{j < i} V_j)$. Then for $1 \leq k \leq \lfloor \frac{r}{2} \rfloor$,

$$|V_{2k-1}| \leq \left\lfloor \frac{1 + 2|V_{2k}| + \sqrt{8|V_{2k}| + 1}}{2} \right\rfloor \leq \left\lfloor 1 + |V_{2k}| + \sqrt{2|V_{2k}|} \right\rfloor.$$

Proof. Note that Lemma 5.3 implies that $\phi(V_{2i-1}, V_{2i})\phi(V_{2i-1}, V_{2i}) = \emptyset$ for $i = 1, \dots, \lfloor \frac{r}{2} \rfloor$. Consequently, by Theorem 3.9, the set $V_1V_2 \cup V_3V_4 \cup \dots \cup V_{r-1}V_r$ if r even, or $V_1V_2 \cup \dots \cup V_{r-2}V_{r-1} \cup V_rV_r$ if r odd, is a maximal independent set of $T_2(G)$.

Let \mathcal{B}_k be the bipartite subgraph of G induced by $V_{2k-1} \cup V_{2k}$. For simplicity, consider the case $k = 1$. We first claim that $\alpha(T_2(\mathcal{B}_1)) = |V_1V_2|$. Suppose for contradiction that A is a maximal independent set of \mathcal{B}_1 with $|A| > |V_1V_2|$. Then, $T_2(G)$ contains an independent set with cardinality $|A \cup V_3V_4 \cup \dots| > |V_1V_2 \cup V_3V_4 \cup \dots|$, which contradicts the hypotheses that $T_2(G)$ is well-covered or that $V_1V_2 \cup V_3V_4 \cup \dots$ is maximal. Therefore $\alpha(T_2(\mathcal{B}_1)) = |V_1V_2|$.

Because $V_1V_1 \cup V_2V_2$ is an independent set of \mathcal{B}_1 , it follows that $\alpha(T_2(\mathcal{B}_1)) = |V_1V_2| = |V_1||V_2| \geq \binom{|V_1|}{2} + \binom{|V_2|}{2}$. Solving this inequality for $|V_1|$ (and using the fact that $|V_1|$ must be an integer) gives $|V_1| \leq \left\lfloor \frac{1+2|V_2|+\sqrt{8|V_2|+1}}{2} \right\rfloor \leq \left\lfloor 1 + |V_2| + \sqrt{2|V_2|} \right\rfloor$. The cases with $k \neq 1$ are similar. ■

Corollary 5.5. *Suppose G is connected and $|V(G)| = n \geq 3$. If $T_2(G)$ is well-covered, then $\alpha(G) \leq \left\lfloor \frac{n-1+\sqrt{n-1}}{2} \right\rfloor$.*

Proof. Partition the vertices of G into sets V_1, V_2, \dots, V_r such that $|V_1| = \alpha(G)$ and $|V_2| = \alpha(G \setminus V_1)$. Since $T_2(G)$ is well-covered and G is connected, Corollary 4.10 implies that G is not bipartite. Thus, there is at least one vertex which is not in $V_1 \cup V_2$ and so $|V_2| \leq n - \alpha(G) - 1$. Applying $|V_1| = \alpha(G)$ and $|V_2| \leq n - \alpha(G) - 1$ to the inequality from the Theorem 5.4 yields the required result. ■

6. CONSTRUCTIONS OF WELL-COVERED TOKEN GRAPHS

In this section we describe some graphs G for which $T_2(G)$ is well-covered. Many of the graphs fit within a certain family of graphs that we describe in Definition 6.3.

We first note that there is no direct inheritance with respect to being well-covered for token graphs. If G is well-covered, then there is no guarantee that $T_k(G)$ is well-covered. For example, the cycle C_4 is well-covered but $T_2(C_4)$ is isomorphic to the complete bipartite graph $K_{2,4}$ which is not well-covered. There

are also graphs for which G is not well-covered but $T_2(G)$ is well-covered, as observed in Figure 1 (and, for example, Theorem 6.9 with $s = m$ and $t = 0$).

Theorem 6.1. *For $n \geq 2$, $T_2(K_n)$ is well-covered.*

Proof. Let A be an independent set of vertices in $T_2(K_n)$. No vertex of K_n appears in more than one pair in A . If there exists $i, j \in V(K_n)$, with neither i nor j appearing in any pair in A , then $A \cup \{ij\}$ is also an independent set. It follows that if A is a maximal independent set, then $|A| = \lfloor \frac{n}{2} \rfloor$. Thus $T_2(K_n)$ is well-covered. ■

While $T_2(K_n)$ is well-covered, we expect that in general $T_k(K_n)$ is not well-covered for $k > 2$. For example, it is known that for each $n \geq 9$, there exists a partial Steiner triple system of order n that does not have an embedding of order v for any $v < 2n + 1$, demonstrating the existence of a maximal independent set in $T_3(K_n)$ that is not maximum when $n \equiv 1, 3 \pmod{6}$ (see [13] and [?]). For example, the maximal independent set $\{123, 367, 345, 147, 256\}$ in $T_3(K_7)$ cannot be completed to become a Fano plane.

We use the fact that if H is a subgraph of G , then $T_2(H)$ is a subgraph of $T_2(G)$. Taking a maximal independent set of a graph G and considering its restriction to a subset of vertices A of G will give an independent set in $G[A]$.

Remark 6.2. If $V(G) = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$, then $\alpha(G) \leq \alpha(G[V_1]) + \alpha(G[V_2])$.

Using a computer search, we found that there were few graphs G for which $T_2(G)$ is well-covered for small n (see the Appendix). Besides the complete graphs, many of the graphs G we found for which $T_2(G)$ is well-covered were in a class \mathcal{G} described below.

Definition 6.3. Define \mathcal{G} to be the set of graphs obtained by taking the disjoint union of K_m and K_n , $n \geq m$, and inserting some edges. An example of a graph in \mathcal{G} is given in Figure 4. Let $X = V(K_m) = \{x_1, x_2, \dots, x_m\}$ and $Y = V(K_n) = \{y_1, y_2, \dots, y_n\}$. Let $G \in \mathcal{G}$ and $H = T_2(G)$. Then the vertices of H can be partitioned as $V(H) = XX \cup XY \cup YY$ with $H[XX] = T_2(K_m)$, $H[YY] = T_2(K_n)$, and $H[XY] = K_m \square K_n$ (the Cartesian product of K_m and K_n). Further, if x_i is adjacent to y_k in G , then H contains the edges $\{\{x_j x_i, x_j y_k\} \mid 1 \leq j \leq m, j \neq i\}$ and $\{\{x_i y_\ell, y_k y_\ell\} \mid 1 \leq \ell \leq n, \ell \neq k\}$.

In the next theorems we determine classes of token graphs $T_2(G)$ with $G \in \mathcal{G}$ that are well-covered and classes that are not well-covered. We start by considering the independence number for some of the graphs in \mathcal{G} .

Lemma 6.4. *Let $G \in \mathcal{G}$ with at most $n - m$ vertices of Y having a neighbour in X . Then $\alpha(T_2(G)) = \alpha(T_2(K_m)) + \alpha(T_2(K_n)) + m$.*

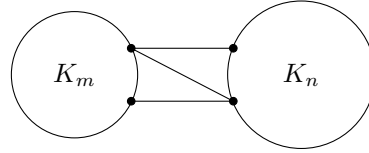


Figure 4. A graph $G \in \mathcal{G}$ with $T_2(G)$ not well-covered (Theorem 6.7).

Proof. By Remark 6.2, $\alpha(T_2(G)) \leq \alpha(T_2(K_m)) + \alpha(K_m \square K_n) + \alpha(T_2(K_n))$. If $n = m$, then $T_2(G)$ is just the disjoint union of $T_2(K_m)$, $K_m \square K_n$ and $T_2(K_n)$ and so equality holds in the previous inequality. Suppose that $n > m$. Note that $\alpha(K_m \square K_n) = m$. Without loss of generality, assume that none of the vertices y_1, y_2, \dots, y_m are adjacent to any of the vertices of K_m . Let $A = \{x_1x_2, x_3x_4, \dots\}$ be a maximal independent set of $T_2(K_m)$, $B = \{y_1y_2, y_3y_4, \dots\}$ be a maximal independent set of $T_2(K_n)$, and $C = \{x_1y_1, x_2y_2, \dots, x_{m-1}y_{m-1}\}$. Let $D = \{x_my_m\}$ if m is even and $D = \{x_my_{m+1}\}$ if m is odd. Then $A \cup B \cup C \cup D$ is an independent set of $T_2(G)$. Therefore $\alpha(T_2(G)) = \alpha(T_2(K_m)) + \alpha(T_2(K_n)) + m$. ■

Remark 6.5. The independent set constructed in the proof of Lemma 6.4 can be constructed via the construction of Corollary 3.7. In particular, taking $V_{2i-1} = \{x_{2i-1}, y_{2i}\}$ and $V_{2i} = \{x_{2i}, y_{2i-1}\}$ for $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$, with $W = \{x_m, y_m\}$ if m is odd and $W = \emptyset$ if m is even, and taking $U_i = \{y_i\}$ for $m + 1 \leq i \leq n$, then $A \cup B \cup C \cup D$ is the same as $\{V_1V_2 \cup V_3V_4 \cup \dots\} \cup \{U_{m+1}U_{m+2} \cup U_{m+3}U_{m+4} \cup \dots\} \cup WW$. While the tools of Section 3 are helpful for constructing independent sets in token graphs, in this section, such as in the previous proof, we give more direct descriptions of some independent sets.

For graphs $G \in \mathcal{G}$, the next three theorems provide forbidden configurations for $T_2(G)$ to be well-covered. The restriction of at most $n - m$ vertices of K_m having a neighbour in K_n allows us to provide the exact value of the independence number in Lemma 6.4. As such, we focus on graphs in \mathcal{G} having this restriction as we develop the next results. The next theorem provides a parity restriction for such graphs in \mathcal{G} that are well-covered.

Theorem 6.6. *Let $G \in \mathcal{G}$ be such that at most $n - m$ vertices of Y have a neighbour in X . If G is connected and either n or m is even, then $T_2(G)$ is not well-covered.*

Proof. Since G is connected, we know that $n > m$. By Lemma 6.4, $\alpha(T_2(G)) = \alpha(T_2(K_m)) + \alpha(T_2(K_n)) + m$. Suppose m is even. Suppose there is a vertex, say y_n , which is adjacent to every vertex in K_m . Let I be any maximal independent set of $T_2(G)$ with $x_1y_n \in I$. Then I can contain no edge $x_1x \in XX$. Thus $|I \cap XX| < \alpha(T_2(K_m))$ and hence G is not well-covered.

Suppose there is a vertex in Y adjacent to some vertex in X but not adjacent to every vertex in X . Assume that y_n is adjacent to x_1 but not x_m . Let I be a

maximal independent set with $\{y_n x_m\} \cup \{x_2 x_3, \dots, x_{m-2} x_{m-1}\} \subseteq I$. Note that $x_1 x_m \notin I$ since $y_n x_m \in I$ and x_1 is adjacent to y_n in G . Thus $|I \cap XX| < \alpha(T_2(K_m))$. Therefore $T_2(G)$ is not well-covered. The case with n even is similar to the previous case. ■

Theorem 6.7. *Let $G \in \mathcal{G}$ with $n \geq m + 2$, with $\{x_1, y_1\}, \{x_1, y_2\}, \{x_2, y_2\} \in E(G)$ and at most $n - m$ vertices of Y have a neighbour in X . Then $T_2(G)$ is not well-covered.*

Proof. By Lemma 6.4, $\alpha(T_2(G)) = \alpha(T_2(K_m)) + \alpha(T_2(K_n)) + m$. And if I is an independent set with $|I| = \alpha(T_2(G))$, then I must contain $\alpha(T_2(K_m))$ vertices from $T_2(K_m)$. If $m \geq 3$, consider a maximal independent set I of $T_2(G)$ containing the vertices $x_2 y_1$ and $x_3 y_2$, as well as the vertices $x_m x_{m-1}, x_{m-2} x_{m-3}, \dots, x_{t+1} x_t$ for $t \in \{3, 4\}$ (depending on the parity of m). Then I cannot include the vertices $x_2 x_3, x_1 x_3$, and $x_1 x_2$ since these vertices are all adjacent to either $x_2 y_1$ or $x_3 y_2$. If $m = 2$, construct a maximal independent set containing $x_2 y_1$, and hence $x_1 x_2 \notin I$. In either case, $|I \cap XX| < \alpha(T_2(K_m))$, and hence $|I| < \alpha(T_2(G))$. Thus $T_2(G)$ is not well-covered. ■

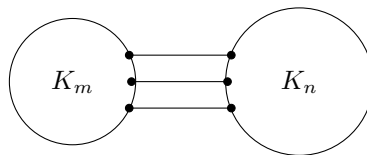


Figure 5. A graph $G \in \mathcal{G}$ with $T_2(G)$ not well-covered (Theorem 6.8).

Theorem 6.8. *Let $G \in \mathcal{G}$ with $n \geq m+3$, with $\{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_3\} \in E(G)$ and at most $n - m$ vertices of Y have a neighbour in X . Then $T_2(G)$ is not well-covered.*

Proof. By Lemma 6.4, $\alpha(T_2(G)) = \alpha(T_2(K_m)) + \alpha(T_2(K_n)) + m$. Consider a maximal independent set, say I , of $T_2(G)$ containing the vertices $x_1 y_3, x_2 y_1$, and $x_3 y_2$. Suppose also $x_{2i-2} x_{2i-1} \subseteq I$ for $3 \leq i \leq \lceil \frac{m}{2} \rceil$. Note that $x_1 y_3$ is adjacent to $x_1 x_3$, $x_2 y_1$ is adjacent to $x_1 x_2$, and $x_3 y_2$ is adjacent to $x_2 x_3$ in $T_2(G)$. Thus $x_1 x_3, x_1 x_2$, and $x_2 x_3$ are not in I . Therefore $|I \cap XX| < \alpha(T_2(K_m))$ and hence $|I| < \alpha(T_2(G))$. Thus $T_2(G)$ is not well-covered. ■

In the context of the previous two theorems, if $G \in \mathcal{G}$ is well-covered, then the edges between Y and X in G must consist of at most two distinct stars, and if there are two stars, they must be disjoint.

In the following theorem we consider graphs in $G \in \mathcal{G}$ having one vertex of X adjacent to s vertices of Y and another adjacent to t other vertices of Y , to get a well-covered graph $T_2(G)$ when $s + t \leq m$.

Theorem 6.9. *Let $G \in \mathcal{G}$ with $n > m$, both odd, $N_G[y_1] = \{x_1, x_2, \dots, x_s\} \cup Y$, $N_G[y_2] = \{x_{s+1}, x_{s+2}, \dots, x_{s+t}\} \cup Y$ and $N_G[y_i] = Y$ for all i , $3 \leq i \leq n$, with $0 \leq s + t \leq m$. Then $T_2(G)$ is well-covered.*

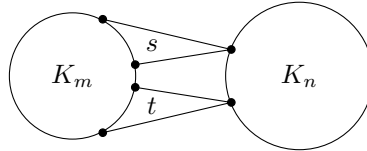


Figure 6. A graph $G \in \mathcal{G}$ with $T_2(G)$ well-covered if $s + t \leq m$ (Theorem 6.9).

Proof. By Lemma 6.4, $\alpha(T_2(G)) = \alpha(T_2(K_m)) + \alpha(T_2(K_n)) + m$. Let $I = A \cup B \cup C$ be a maximal independent set in $T_2(G)$ with $A \subseteq YY$, $B \subseteq XY$, $C \subseteq XX$. It is enough to show that $|A| = \alpha(T_2(K_n))$, $|B| = m$, and $|C| = \alpha(T_2(K_m))$.

Suppose $|A| < \alpha(T_2(K_n))$. Then there are at least three vertices $y_a, y_b, y_c \in Y$ which do not appear in any pair of A . Without loss of generality $y_c \notin \{y_1, y_2\}$. Suppose y_c appears in a pair xy_c of B . Note that x has at most one neighbour in Y . Thus, without loss of generality, x is not adjacent to y_b . In this case, and the case when y_c appears in no pair of B , $I = \{y_b y_c\} \cup I$ is an independent set. But then I is not maximal. Therefore, $|A| = \alpha(T_2(K_n))$.

Suppose $|B| < m$. Then there is some $x \in X$ that appears in no pair of B . Also, there are at least $n - m + 1 \geq 3$ vertices of Y that are not part of any pair in B ; say y_a, y_b, y_c . Let $Z = \{y_a, y_b, y_c\}$. We claim that $H = \{xy\} \cup I$ is an independent set of $T_2(G)$ for some $y \in Z$. If x does not appear in any pair in C , then there will be one less restriction on the possible $y \in Z$ (to ensure H is an independent set), so assume $xw \in C$ for some $w \in X$. Then w could be adjacent to y_1 or y_2 but not both. Thus there is at most one xy adjacent to xw in $T_2(G)$ for $y \in Z$. Without loss of generality, assume w is adjacent to y_a . Now, if either y_b or y_c does not appear in any pair of A , then H is an independent set for that y .

Suppose that $y_b y_c \in A$. Now x is adjacent to at most one of y_b and y_c . If x is adjacent to y_b , then let $y = y_b$, otherwise let $y = y_c$. In either case, H is an independent set.

Suppose that $y_b y_c \notin A$ but $y_b y_r, y_c y_q \in A$. Again x is adjacent to at most one of y_r and y_q . Without loss of generality, assume that y_q is not adjacent to x . Then take $y = y_c$, and H is an independent set. Since in each case, H is constructed to be an independent set, this would imply that I is not maximal. Therefore we conclude that $|B| = m$.

Suppose $|C| < \alpha(T_2(K_m))$. Then there are at least 3 vertices $x_a, x_b, x_c \in X$ that do not appear in any pair of C . If $x_a y_1 \notin B$, $x_a y_2 \notin B$, $x_b y_1 \notin B$ and $x_b y_2 \notin B$, then $\{x_a x_b\} \cup I$ is an independent set in $T_2(G)$. Without loss of

generality, assume $x_a y_1 \in B$. Note that then $x_a y_2 \notin B$ since y_1 is adjacent to y_2 . Likewise, $x_b y_1 \notin B$.

Suppose $x_b y_2 \notin B$. Then $\{x_b x_c\} \cup I$ is an independent set in $T_2(G)$. In either case, I is not maximal.

Suppose $x_b y_2 \in B$. Since x_c is not adjacent to both y_1 and y_2 , assume that x_c is not adjacent to y_1 . Then $\{x_a x_c\} \cup I$ is an independent set in $T_2(G)$, and so I is not maximal. Thus $|C| = \alpha(T_2(K_m))$.

Therefore $T_2(G)$ is well-covered. ■

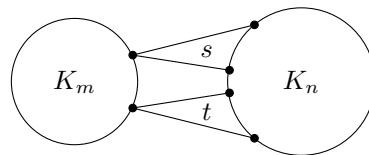


Figure 7. A graph $G \in \mathcal{G}$ with $T_2(G)$ well-covered if $s + t \leq n - m$ (Theorem 6.10).

We next consider the graph considered in Theorem 6.9 with the stars between K_m and K_n reversed.

Theorem 6.10. *Let $G \in \mathcal{G}$ with $n > m$, both odd, $N_G[x_1] = \{y_1, y_2, \dots, y_s\} \cup X$, $N_G[x_2] = \{y_{s+1}, y_{s+2}, \dots, y_{s+t}\} \cup X$ and $N_G[x_i] = X$ for all $i, 3 \leq i \leq m$, with $0 \leq s + t \leq n - m$. Then $T_2(G)$ is well-covered.*

Proof. As in the proof of Theorem 6.9, we let $I = A \cup B \cup C$ be a maximal independent set in $T_2(G)$ with $A \subseteq YY, B \subseteq XY, C \subseteq XX$. It is enough to show that $|A| = \alpha(T_2(K_n)), |B| = m$, and $|C| = \alpha(T_2(K_m))$. By similar arguments to those used for the proof of Theorem 6.9, we can show that $|A| = \alpha(T_2(K_n))$ and $|C| = \alpha(T_2(K_m))$.

Suppose $|B| < m$. Without loss of generality, there is some vertex $x \in X$ that belongs to no pair in B . Also, there are at least $n - m + 1 \geq s + t + 1$ vertices of Y that belong to no pair in B .

Suppose $x \in \{x_1, x_2\}$. Without loss of generality, $x = x_1$. If $xy \in XY$ is adjacent to a vertex of C , then $y \in \{y_{s+1}, \dots, y_{s+t}\}$. Thus at most t vertices of XY containing x are adjacent to a vertex in C . At most s vertices of A contain a member in $\{y_1, \dots, y_s\}$. Thus at most s vertices of XY containing x are adjacent to vertices in A . Since there are at least $s + t + 1$ vertices of Y that belong to no pair in B , there is some $v \in Y$ such that xv is not adjacent to any vertex of I . Hence $\{xv\} \cup I$ is an independent set in $T_2(G)$.

Suppose $x \notin \{x_1, x_2\}$. Since there are at least $s + t + 1$ vertices of Y that do not appear in any pair of B , there is some y_j with $j > s + t$ such that y_j does not appear in any pair of B . Hence $\{xy_j\} \cup I$ is an independent set in $T_2(G)$.

Since I is maximal, it follows that $|B| = m$. Thus $T_2(G)$ is well-covered. ■

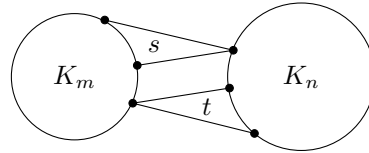


Figure 8. A graph $G \in \mathcal{G}$ with $T_2(G)$ well-covered if $t + 1 \leq n - m$ and $s + 1 \leq m$ (Theorem 6.11).

In the next theorem we consider graphs $G \in \mathcal{G}$ with one vertex of X adjacent to t vertices of Y and one vertex of Y adjacent to s vertices of X . Due to Theorem 6.7, these stars will need to be disjoint if $T_2(G)$ is well-covered.

Theorem 6.11. *Let $G \in \mathcal{G}$ with $n > m$, both odd, $N_G[x_1] = \{y_2, y_3, \dots, y_{t+1}\} \cup X$, $N_G[y_1] = \{x_2, x_3, \dots, x_{s+1}\} \cup Y$, $N_G[x_i] = X$ for $s + 2 \leq i \leq m$ and $N_G[y_i] = Y$ for $t + 2 \leq i \leq n$, with $s + 1 \leq m$ and $t + 1 \leq n - m$. Then $T_2(G)$ is well-covered.*

Proof. By Lemma 6.4, $\alpha(T_2(G)) = \alpha(T_2(K_m)) + \alpha(T_2(K_n)) + m$. Let $I = A \cup B \cup C$ be a maximal independent set in $T_2(G)$ with $A \subseteq YY$, $B \subseteq XY$, $C \subseteq XX$. It is enough to show that $|A| = \alpha(T_2(K_n))$, $|B| = m$, and $|C| = \alpha(T_2(K_m))$.

Suppose $|A| < \alpha(T_2(K_n))$. Then there are at least three vertices $y_a, y_b, y_c \in Y$ that are not in any pair of A . At least two vertices in $\{y_a x_1, y_b x_1, y_c x_1\}$ cannot be in B ; without loss of generality, $y_a x_1, y_b x_1 \notin B$.

Suppose $y_1 x_1 \in B$. Then $y_a \neq y_1$ and $y_b \neq y_1$. Hence $\{y_a y_b\} \cup I$ is an independent set.

Suppose $y_1 x_1 \notin B$. If $y_a \neq y_1$, then $\{y_a y_b\} \cup I$ is an independent set. Suppose $y_a = y_1$. If $y_b w \notin B$ for all $w \in N_G(y_1)$, then $\{y_a y_b\} \cup I$ is an independent set. Likewise, if $y_c w \notin B$ for all $w \in N_G(y_1)$, then $\{y_a y_c\} \cup I$ is an independent set. Finally, if $y_b w \in B$ for some $w \in N_G(y_1)$ and $y_c t \in B$ for some $t \in N_G(y_1)$, then $\{y_b y_c\} \cup I$ is an independent set. Therefore, if $|A| < \alpha(T_2(K_n))$, then I is not a maximal independent set. Thus $|A| = \alpha(T_2(K_n))$.

Suppose $|B| < m$. Then there is at least one vertex $x' \in X$ that is in no pair of B . Let $M \subseteq Y$ be the set of vertices of Y that are not in any pair of B . Then M contains at least $n - m + 1 \geq t + 2$ vertices.

Suppose $x' = x_1$. Then at most t vertices $x' y \in XY$ are adjacent to vertices in A . Additionally, at most one vertex $x' y \in XY$ is adjacent to a vertex in C . Thus there is some $y' \in M$ such that $x' y'$ is not adjacent to any vertex of A or C . Thus $\{x' y'\} \cup I$ is an independent set.

Suppose $x' \neq x_1$. If $x' x_1 \in C$, then at most t vertices $x_1 y \in XY$ are adjacent to $x' x_1$. Additionally, there is at most one vertex $yy_1 \in A$. Thus there is at least one vertex, say $x' y'$, which is not adjacent to any vertex in A or C . Then $\{x' y'\} \cup I$ is an independent set.

Suppose $x' x_1 \notin C$. Then there is at most one vertex $x'' \in X$ such that $x' x'' \in C$. Additionally, there is at most one vertex $yy_1 \in A$. Thus there exists

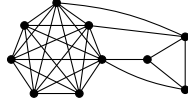


Figure 9. A graph G with $T_2(G)$ well-covered (Theorem 6.11).

at least one vertex of the form $x'y'$ such that $\{x'y'\} \cup I$ is an independent set.

In each case, we have seen that if $|B| < m$, then I is not maximal. Thus $|B| = m$.

Suppose $|C| < \alpha(T_2(K_m))$. A similar argument to that used for A shows that $|C| = \alpha(T_2(K_m))$. Thus $T_2(G)$ is well-covered. ■

7. CONCLUDING COMMENTS

By computer calculation, one can check that if G is a graph on eight or fewer vertices and $T_2(G)$ well-covered, then $G \in \mathcal{G}$. In fact, all these graphs are accounted for by Theorems 6.1, 6.9 and 6.10 (see the Appendix). The graphs covered in Theorem 6.11 must have at least ten vertices, such as in Figure 9.

The theorems in Section 6 considered graphs in \mathcal{G} when $n > m$. We do not expect that $T_2(G)$ is well-covered for any non-complete graph $G \in \mathcal{G}$ with $n = m$. The following theorem is an illustration.

Theorem 7.1. *Suppose $G \in \mathcal{G}$ and $m = n > 1$ and x_1y_1 is the only edge with one endpoint in Y and one in X . Then $T_2(G)$ is not well-covered.*

Proof. Let $A = \{x_1x_2, x_3x_4, \dots\}$, $B = \{y_1y_2, y_3y_4, \dots\}$ and suppose that $C = \{x_1y_1, x_2y_2, \dots, x_ny_n\}$. Then $A \cup B \cup C$ is an independent set and so by Remark 6.2, $\alpha(T_2(G)) = 2\alpha(T_2(K_n)) + \alpha(K_n \square K_n)$. Let $I = \{x_1x_2, x_3x_4, \dots\} \cup \{y_1y_2, y_3y_4, \dots\} \cup \{x_1y_n\} \cup \{x_3y_2, \dots, x_ny_{n-1}\}$. Then $|I| = \alpha(T_2(G)) - 1$ and yet I is maximal. Therefore $T_2(G)$ is not well-covered. ■

Remark 7.2. If $G \in \mathcal{G}$ and $m = n$ and there is at least one edge xy with one endpoint in Y and one in X and $\alpha(T_2(G)) = \alpha(T_2(G)[XX]) + \alpha(T_2(G)[XY]) + \alpha(T_2(G)[YY]) = 2\alpha(T_2(K_n)) + n$, then $T_2(G)$ is not well-covered. In particular, suppose x_1y_1 is an edge of G . Let I be a maximal independent set of $T_2(G)$ with $\{x_1y_1\} \cup \{x_1y_n\} \cup \{x_3y_2, \dots, x_ny_{n-1}\} \subseteq I$. Then $|I \cap XY| = n - 1 < \alpha(T_2(G)[XY])$ and so $|I| \neq \alpha(T_2(G))$.

One of the reasons that we are interested in well-covered graphs is that they are candidate Cohen-Macaulay graphs (for details and definitions, see e.g. [10]). As an example, we can show that if G is a non-complete graph G of order 4 with $T_2(G)$ well-covered, then $T_2(G)$ is vertex-decomposable and hence $T_2(G)$ is

Cohen-Macaulay. Future work could be done to determine when a well-covered token graph is vertex-decomposable and/or Cohen-Macaulay.

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8. APPENDIX: GRAPHS G WITH $T_2(G)$ WELL-COVERED

The number of graphs G of order at most 9 with $T_2(G)$ well-covered are listed in Table 1 as determined by a computer search. In Figures 10, 11 and 12, we display all the non-complete graphs G of order at most 9 with $T_2(G)$ well-covered.

n	2	3	4	5	6	7	8	9
number of graphs	1	1	3	1	5	1	13	9

Table 1. Number of graphs G of order n with $T_2(G)$ well-covered for $n \leq 9$.

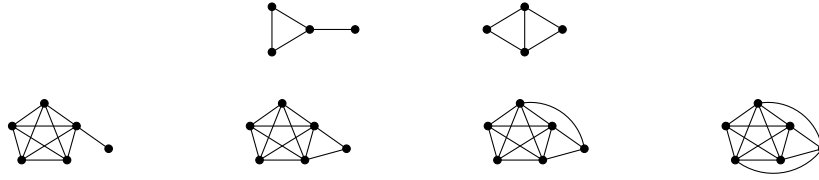


Figure 10. Non-complete graphs G of order 4 and 6 with $T_2(G)$ well-covered.

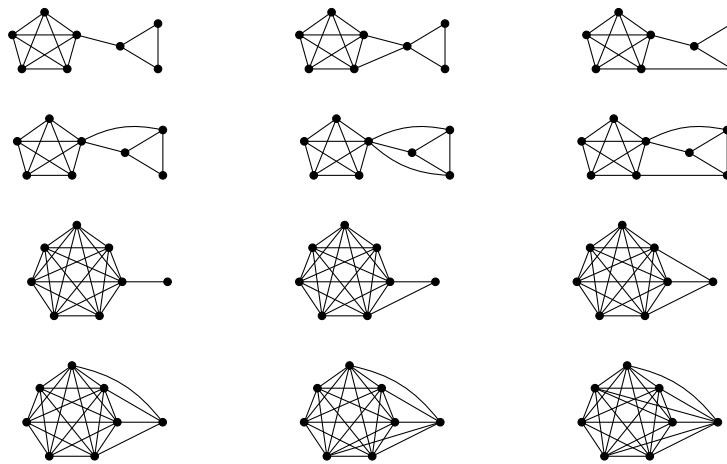


Figure 11. Non-complete graphs G of order 8 with $T_2(G)$ well-covered.

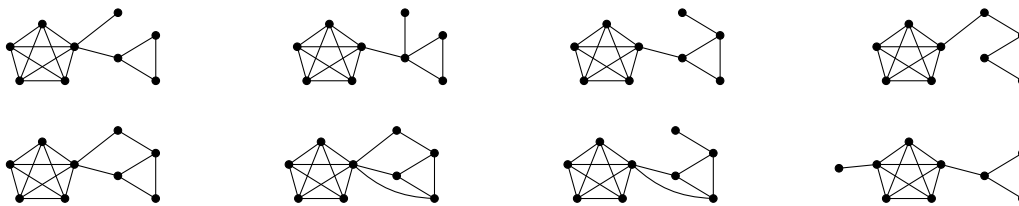


Figure 12. Non-complete graphs G of order 9 with $T_2(G)$ well-covered.