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DOUBLE ROMAN AND DOUBLE ITALIAN DOMINATION

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Abstract

Let G be a graph with vertex set $V(G)$. A double Roman dominating function (DRDF) on a graph G is a function $f : V(G) \longrightarrow \{0, 1, 2, 3\}$ that satisfies the following conditions: (i) If $f(v) = 0$, then v must have a neighbor w with $f(w) = 3$ or two neighbors x and y with $f(x) = f(y) = 2$; (ii) If $f(v) = 1$, then v must have a neighbor w with $f(w) \geq 2$. The weight of a DRDF f is the sum $\sum_{v \in V(G)} f(v)$. The double Roman domination number equals the minimum weight of a double Roman dominating function on G. A double Italian dominating function (DIDF) is a function $f : V(G) \longrightarrow \{0, 1, 2, 3\}$ having the property that $f(N[u]) \geq 3$ for every vertex $u \in V(G)$ with $f(u) \in \{0, 1\}$, where $N[u]$ is the closed neighborhood of v. The weight of a DIDF f is the sum $\sum_{v \in V(G)} f(v)$, and the minimum weight of a DIDF in a graph G is the double Italian domination number. In this paper we first present Nordhaus-Gaddum type bounds on the double Roman domination number which improved corresponding results given in [N. Jafari Rad and H. Rahbani, *Some progress on the double Roman domination in graphs*, Discuss. Math. Graph Theory 39 (2019) 41–53]. Furthermore, we establish lower bounds on the double Roman and double Italian domination numbers of trees.

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1. Introduction

For definitions and notations not given here we refer to [12]. We consider simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* of G is $n = n(G) = |V(G)|$. The *open neighborhood* of a vertex v is the set $N(v) = N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N[v] =$

$N_G[v] = N(v) \cup \{v\}$. The *degree* of vertex $v \in V(G)$ is $d(v) = d_G(v) = |N(v)|$. The *maximum degree* and *minimum degree* of G are denoted by $\Delta = \Delta(G)$ and $\delta = \delta(G)$, respectively. The *complement* of a graph G is denoted by $\overline{G}$. A *leaf* is a vertex of degree one, and its neighbor is called a *support vertex.* A *strong support vertex* is a support vertex adjacent to more than one leaf. The *diameter* of a graph G, denoted by diam (G), is the greatest *distance* between two vertices of G. A subset D of $V(G)$ is a *dominating set* in G if $\bigcup_{v\in D} N[v] = V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set in G. A set S of vertices is *independent* if no two vertices in S are adjacent. We write P_n for the path of order n, C_n for the cycle of length n and K_n for the complete graph of order n. For $n \geq 2$, the *star* $K_{1,n-1}$ has one vertex of degree $n-1$ and $n-1$ leaves. By $S_{p,q}$ we denote the *double star*, where one center vertex is adjacent to p leaves and the other one to q leaves. Cockayne, Dreyer, S.M. Hedetniemi and S.T. Hedetniemi [10] introduced the concept of *Roman domination* in graphs, and since then a lot of related variations and generalizations have been studied (see [6–9]). In 2016, Beeler, Haynes and S.T. Hedetniemi [5] defined a stronger version of Roman domination which they called double Roman domination. A *double Roman dominating function* (DRDF) on a graph G is a function $f : V(G) \longrightarrow \{0, 1, 2, 3\}$ that satisfies the following conditions: (i) If $f(v) = 0$, then v must have a neighbor w with $f(w) = 3$ or two neighbors x and y with $f(x) = f(y) = 2$; (ii) If $f(v) = 1$, then v must have a neighbor w with $f(w) \geq 2$. The *weight* of a DRDF f is the sum $w(f) = \sum_{v\in V(G)} f(v)$. The *double Roman domination number* $\gamma_{dR}(G)$ equals the minimum weight of a double Roman dominating function on G. A DRDF of G with weight $\gamma_{dR}(G)$ is called a $\gamma_{dR}(G)$*-function.* Double Roman domination has been studied in [1, 2, 11, 13] and the survey paper [7].

Mojdeh and Volkmann [15] considered a variant of double Roman domination which they called double Italian domination. A *double Italian dominating function* (DIDF) on a graph G is a function $f : V(G) \longrightarrow \{0, 1, 2, 3\}$ having the property that for every vertex $u \in V(G)$, if $f(u) \in \{0, 1\}$, then $f(N[u]) \geq 3$. The *weight* of a DIDF f is the sum $w(f) = \sum_{v\in V(G)} f(v)$, and the minimum weight of a DIDF in a graph G is the *double Italian domination number*, denoted by $\gamma_{dI}(G)$. This concept was further studied in [3, 4, 17].

Clearly, $\gamma_{dI}(G) \leq \gamma_{dR}(G)$, since every double Roman dominating function is also a double Italian dominating function.

In this paper we first present Nordhaus-Gaddum type bounds on the double Roman domination number which improve corresponding results given in [13]. Furthermore, we establish lower bounds on the double Roman and double Italian domination numbers of trees.

We make use of the following known results.

Proposition 1 [5]**.** *If G is a graph, then $\gamma_{dR}(G) \leq 3\gamma(G)$.*

Using Proposition 1 and the classical bound $\gamma(G) \leq \frac{n}{2}$ of Ore for graphs G of order n with $\delta(G) \geq 1$, we obtain the next observation immediately.

Proposition 2. *If G is a graph of order n with $\delta(G) \geq 1$, then $\gamma_{dR}(G) \leq \frac{3n}{2}$.*

Let $\mathcal{H}$ be the family of connected graphs of order n that can be built from $n/4$ copies of P_4 by adding a connected subgraph on the set of centers of $\frac{n}{4}P_4$.

Theorem 3 [5]**.** *If G is a connected graph of order $n \geq 3$, then $\gamma_{dR}(G) \leq \frac{5n}{4}$, with equality if and only if $G \in \mathcal{H}$.*

Theorem 4 [14]**.** *If G is a graph of order n, minimum degree $\delta \geq 2$, and with no component isomorphic to C_5 or C_7, then $\gamma_{dR}(G) \leq \frac{11n}{10}$.*

Theorem 5 [1]**.** *If G is a graph of order n and minimum degree $\delta \geq 3$, then $\gamma_{dR}(G) \leq n$.*

Proposition 6 [15]**.** *If G is a graph of order $n \geq 2$, then $\gamma_{dI}(G) \geq 3$, with equality if and only if $\Delta(G) = n - 1$.*

Proposition 7 [1]**.** *If P_n is a path of order $n \geq 1$, then $\gamma_{dR}(P_n) = n$ if $n \equiv 0 \pmod 3$ and $\gamma_{dR}(P_n) = n + 1$ otherwise.*

2. Nordhaus-Gaddum Type Results

Results of Nordhaus-Gaddum type study the extreme values of the sum or product of a parameter on a graph and its complement. In their classical paper [16], Nordhaus and Gaddum discussed this problem for the chromatic number. Jafari Rad and Rahbani [13] presented Nordhaus-Gaddum type inequalities for the double Roman domination number. In the following let $K_n - e$ be the complete graph minus an edge and $K_n - \{e_1, e_2\}$ be the complete graph minus two independent edges.

Theorem 8 [13]**.** *If G is a graph of order $n \geq 2$, then $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 2n + 3$, with equality if and only if $G \in \{K_n, \overline{K_n}\}$.*

Theorem 9 [13]**.** *If $G \notin \{K_n, \overline{K_n}\}$ is a graph of order $n \geq 2$, then $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) = 2n + 2$ if and only if $G \in \{K_n - e, \overline{K_n - e}, P_4, C_5\}$.*

Note that Theorem 9 is incomplete, because we also have $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) = 2n + 2$ if $G \in \{C_4, 2K_2\}$. Next we improve these results.

Theorem 10. *Let $G \notin \{K_n, \overline{K_n}, K_n - e, \overline{K_n - e}, C_4, 2K_2, P_4, C_5\}$ be a graph of order $n \geq 4$. Then $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 2n + 1$, with equality if and only if $G \in \{K_n - \{e_1, e_2\}, \overline{K_n - \{e_1, e_2\}}\}$ and $n \geq 5$ or $G, \overline{G} \in \{P_5, 3K_2\}$.*

Proof. First assume that $\delta(G) \geq 1$ and $\delta(\overline{G}) \geq 1$. Assume next that $\delta(G) = 1$ or $\delta(\overline{G}) = 1$, say $\delta(G) = 1$. Furthermore, assume that G has a component of order 2. Then we observe that $\gamma_{dR}(\overline{G}) \leq 4$, and hence Proposition 2 implies for $n \geq 7$ that $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq \frac{3n}{2} + 4 < 2n + 1$. If $n = 4$, then $G = 2K_2$, $\overline{G} = C_4$ (and so $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) = 10 = 2n + 2$), however, by the hypothesis $G \notin \{2K_2, C_4\}$. If $n = 5$, then G consists of K_2 and a component of order 3, and we deduce that $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 10 = 2n$. If $n = 6$ and $G = 3K_2$, then $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) = 9 + 4 = 13 = 2n + 1$. If $G \neq 3K_2$, then G consists of K_2 and a component of order 4, and it follows that $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 3 + 5 + 4 = 12 = 2n$.

Now assume that each component of G has order at least 3. Since $\delta(G) = 1$, the graph $\overline{G}$ has a vertex of degree $n - 2$, and hence we observe that $\gamma_{dR}(\overline{G}) \leq 5$. Therefore Theorem 3 implies for $n \geq 6$ that $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq \frac{5n}{4} + 5 < 2n + 1$. If $n = 4$, then G is connected. If G has a vertex of degree 3, then $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 3 + 5 = 8 = 2n$. If $\Delta(G) = 2$, then $G = P_4$, however by the hypothesis $G \neq P_4$. If $n = 5$, then G is connected. If $\Delta(G) \geq 3$, then $\gamma_{dR}(G) \leq 5$ and thus $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 10 = 2n$. If $\Delta(G) = 2$, then $G = P_5$. Now according to Proposition 7, we have $\gamma_{dR}(P_5) = 6$, and it easy to see that $\gamma_{dR}(\overline{P_5}) = 5$. Consequently $\gamma_{dR}(P_5) + \gamma_{dR}(\overline{P_5}) = 11 = 2n + 1$.

Second assume that $\delta(G) \geq 2$ and $\delta(\overline{G}) \geq 2$. Assume next that $\delta(G) = 2$ or $\delta(\overline{G}) = 2$, say $\delta(G) = 2$. Since $\delta(G) = 2$, the graph $\overline{G}$ has a vertex of degree $n - 3$, and hence we observe that $\gamma_{dR}(\overline{G}) \leq 7$. Now Theorem 3 yields for $n \geq 8$ that $\gamma_{dR}(G) < \frac{5n}{4} + 7 \leq 2n + 1$. The condition $\delta(G), \delta(\overline{G}) \geq 2$ leads to $n \geq 5$. If $n = 5$, then $\delta(G), \delta(\overline{G}) \geq 2$ shows that $G = C_5$, however, this is not allowed. If $n = 7$, then we deduce from Theorem 4 that $\gamma_{dR}(G) \leq \frac{11n}{10} = \frac{77}{10}$ or $G = C_7$. If $G = C_7$, then we observe that $\gamma_{dR}(\overline{C_7}) \leq 6$ and $\gamma_{dR}(C_7) = 8$ and therefore $\gamma_{dR}(C_7) + \gamma_{dR}(\overline{C_7}) \leq 8 + 6 = 14 = 2n$. In the remaining cases we obtain $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 7 + 7 = 14 = 2n$. Finally, let $n = 6$, and let u be a vertex of degree 2 in G, and let v and w be the neighbors of u in G. Theorem 4 implies $\gamma_{dR}(G) \leq \frac{11n}{10} = \frac{66}{10}$ and so $\gamma_{dR}(G) \leq 6$. If $vw \in E(\overline{G})$, then we see that $\gamma_{dR}(\overline{G}) \leq 6$ and thus $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 6 + 6 = 12 = 2n$ in this case. Now assume that $vw \in E(G)$, and let x, y, z be the neighbors of u in $\overline{G}$. Since $\delta(\overline{G}) \geq 2$, without loss of generality, the vertex x is a neighbor of v and w in $\overline{G}$. Now the function $f(x) = f(u) = 3$ and $f(x) = 0$ is a DRDF on $\overline{G}$ of weight 6. Consequently $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 6 + 6 = 12 = 2n$.

In the case $\delta(G) \geq 3$ and $\delta(\overline{G}) \geq 3$, it follows from Theorem 5 that $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq n + n = 2n$.

Finally assume that $\delta(G) = 0$ or $\delta(\overline{G}) = 0$, say $\delta(G) = 0$. Let I be the set of isolated vertices of G, and let $F = G - I$. We deduce from Proposition 2 that

$$\gamma_{dR}(G) \leq 2|I| + \frac{3n(F)}{2} = 2|I| + 2n(F) - \frac{n(F)}{2} = 2n - \frac{n(F)}{2}.$$

Since $\Delta(\overline{G}) = n-1$, we have $\gamma_{dR}(\overline{G}) = 3$, and this implies that

$$\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 2n - \frac{n(F)}{2} + 3 < 2n+1$$

if $n(F) \geq 5$. Let now $n(F) = 4$. Note that $\delta(G) = 0$ implies $n \geq 5$ in this case. If $F = 2K_2$, then $G = \overline{K_n - \{e_1, e_2\}}$ and so $\gamma_{dR}(G)+\gamma_{dR}(\overline{G}) = 2n-2+3 = 2n+1$. If $F \neq 2K_2$, then F is connected, and hence $\gamma_{dR}(G)+\gamma_{dR}(\overline{G}) \leq 2(n-4)+5+3 = 2n$. If $n(F) = 3$, then $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \leq 2(n-3)+3+3 = 2n$. If $n(F) = 2$, then $G = \overline{K_n - e}$ and if $n(F) = 0$, then $G = \overline{K_n}$, however, by the hypothesis, $G \notin \{\overline{K_n}, \overline{K_n - e}\}$. This completes the proof. ∎

Since $\gamma_{dI}(G) \leq \gamma_{dR}(G)$,

$$\gamma_{dI}(C_5) + \gamma_{dI}(\overline{C_5}) = 10 = 2n < 2n+2 = 12 = \gamma_{dR}(C_5) + \gamma_{dR}(\overline{C_5}),$$

$$\gamma_{dI}(C_4) + \gamma_{dI}(\overline{C_4}) = \gamma_{dR}(C_4) + \gamma_{dR}(\overline{C_4}) = 10 = 2n+2,$$

$$\gamma_{dI}(P_4) + \gamma_{dI}(\overline{P_4}) = \gamma_{dR}(P_4) + \gamma_{dR}(\overline{P_4}) = 10 = 2n+2,$$

$$\gamma_{dI}(P_5) + \gamma_{dI}(\overline{P_5}) = \gamma_{dR}(P_5) + \gamma_{dR}(\overline{P_5}) = 11 = 2n+1,$$

$$\gamma_{dI}(3K_2) + \gamma_{dI}(\overline{3K_2}) = \gamma_{dR}(3K_2) + \gamma_{dR}(\overline{3K_2}) = 13 = 2n+1,$$

$$\gamma_{dI}(K_n) + \gamma_{dI}(\overline{K_n}) = \gamma_{dR}(K_n) + \gamma_{dR}(\overline{K_n}) = 2n+3 \ (n \geq 2),$$

$$\gamma_{dI}(K_n - e) + \gamma_{dI}(\overline{K_n - e}) = \gamma_{dR}(K_n - e) + \gamma_{dR}(\overline{K_n - e}) = 2n+2 \ (n \geq 3),$$

$$\gamma_{dI}(K_n - \{e_1, e_2\}) + \gamma_{dR}(\overline{K_n - \{e_1, e_2\}})$$

$$= \gamma_{dR}(K_n - \{e_1, e_2\}) + \gamma_{dR}(\overline{K_n - \{e_1, e_2\}}) = 2n+1 \ (n \geq 5),$$

Theorem 10 yields the following Nordhaus-Gaddum type result for the double Italian domination number.

Corollary 11. *Let G be a graph of order $n \geq 4$ and suppose that $G \notin \{K_n, \overline{K_n}, K_n - e, \overline{K_n - e}, C_4, 2K_2, P_4\}$. Then $\gamma_{dI}(G) + \gamma_{dI}(\overline{G}) \leq 2n+1$, with equality if and only if $G \in \{K_n - \{e_1, e_2\}, \overline{K_n - \{e_1, e_2\}}\}$ and $n \geq 5$ or $G, \overline{G} \in \{P_5, 3K_2\}$.*

Observation 12. *If G is a graph of order $n \geq 3$, then $\gamma_{dR}(G) + \gamma_{dR}(\overline{G}) \geq \gamma_{dI}(G) + \gamma_{dI}(\overline{G}) \geq 8$, and this bound is sharp.*

Proof. Since the left inequality is immediate, we only prove the right one. Assume, without loss of generality, that $\gamma_{dI}(G) \leq \gamma_{dI}(\overline{G})$. It follows from Proposition 6 that $\gamma_{dI}(G) \geq 3$. If $\gamma_{dI}(G) = 3$, then Proposition 6 implies $\Delta(G) = n-1$. If v is a vertex of maximum degree in G, then v is an isolated vertex in $\overline{G}$, and $\overline{G}-v$ is a graph of order at least 2. Therefore Proposition 6 leads to $\gamma_{dI}(\overline{G} - v) \geq 3$

and thus $\gamma_{dI}(\overline{G}) \geq 5$. We deduce that $\gamma_{dI}(G) + \gamma_{dI}(\overline{G}) \geq 8$. If $\gamma_{dI}(G) \geq 4$, then the assumption $\gamma_{dI}(G) \leq \gamma_{dI}(\overline{G})$ implies $\gamma_{dI}(G) + \gamma_{dI}(\overline{G}) \geq 8$.

Let H be a graph with $\Delta(H) = n-1$ such that $\Delta(\overline{H}) = n-2$ (for example a star). Then we note that $\gamma_{dR}(G)+\gamma_{dR}(\overline{G}) = \gamma_{dI}(H)+\gamma_{dI}(\overline{H}) = 8$. This example demonstrates that the given bounds in Observation 12 are sharp. ∎

3. Trees

If T is a tree, then Mojdeh and Volkmann [15] have shown that $\gamma_{dI}(T) = \gamma_{dR}(T)$. Thus all results in this section are also valid for $\gamma_{dI}(T)$ instead of $\gamma_{dR}(T)$.

Lemma 13. *Let T be a tree of order $n \geq 2$. If v is a leaf of T, then $\gamma_{dR}(T-v) \leq \gamma_{dR}(T)$.*

Proof. Let $u \in N(v)$, and let f be a $\gamma_{dR}(T)$-function. If $f(v) = 0$ then $f\,|_{V(T-v)}$ is a DRDF on $T - v$ and so $\gamma_{dR}(T - v) \leq w(f) = \gamma_{dR}(T)$. If $f(v) \in \{2, 3\}$, then we define a function g by $g(u) = \max\{f(u), f(v)\}$ and $g(x) = f(x)$ if $x \neq u, v$. Then g is a DRDF on $T - v$ and thus $\gamma_{dR}(T - v) \leq w(f) = \gamma_{dR}(T)$. Finally, assume that $f(v) = 1$. This leads to $f(u) = 2$. Now the function $f\,|_{V(T)-\{v\}}$ is a DRDF on $T - v$, and therefore $\gamma_{dR}(T - v) \leq \gamma_{dR}(T)$. ∎

Corollary 14. *If T is a tree of diameter d, then $\gamma_{dR}(T) \geq d+1$ if $d+1 \equiv 0 \pmod 3$ and $\gamma_{dR}(T) \geq d+2$ otherwise.*

Proof. If T is a tree of diameter $0 \leq d \leq 1$, then clearly $\gamma_{dR}(T) \geq d+2$. Let now $d \geq 2$. Let P be a diametrical path of T which is a copy of P_{d+1}. By Proposition 7, we note that $\gamma_{dR}(P_{d+1}) = d+1$ if $d+1 \equiv 0 \pmod 3$ and $\gamma_{dR}(P_{d+1}) = d+2$ otherwise. Now applying Lemma 13 for finite times leads to the desired result. ∎

The next examples will show that Corollary 14 is sharp.

Example 15. Let $P = v_1v_2 \cdots v_{3p}$ be a path of order $3p$ for an integer $p \geq 1$. If we add $t_{3i-1} \geq 0$ pendant edges to each vertex v_{3i-1} for $1 \leq i \leq p$, then let H be the resulting tree. If we define the function f by $f(v_{3i-1}) = 3$ for $1 \leq i \leq p$ and $f(x) = 0$ otherwise, then f is a DRDF on H of weight $3p$, and therefore $\gamma_{dR}(H) \leq 3p = \operatorname{diam}(H) + 1$. Corollary 14 implies $\gamma_{dR}(H) = \operatorname{diam}(H) + 1$.

If we add a vertex v_{3p+1} to H, adjacent to v_{3p}, then we denote the resulting tree by Q. If we define the function g by $g(v_{3i-1}) = 3$ for $1 \leq i \leq p$, $g(v_{3p+1}) = 2$ and $g(x) = 0$ otherwise, then g is a DRDF on Q of weight $3p + 2$, and therefore $\gamma_{dR}(Q) \leq 3p + 2 = \operatorname{diam}(Q) + 2$. Corollary 14 implies $\gamma_{dR}(Q) = \operatorname{diam}(Q) + 2$.

Theorem 16. *Let T be a tree of order $n \geq 4$ with $\ell(T)$ leaves. If T is not a star, then $\gamma_{dR}(T) \geq \frac{n+8-\ell(T)}{2}$.*

Proof. We use an induction proof on the order. If $n = 4$, then $T = P_4$, since T is not a star. Proposition 7 implies $\gamma_{dR}(T) = 5 = \frac{n+8-\ell(T)}{2}$. Thus assume that $n \geq 5$ and $\gamma_{dR}(T') \geq \frac{n'+8-\ell(T')}{2}$ for every tree T' of order n' which is not a star with $4 \leq n' < n$. If $\operatorname{diam}(T) = 3$, then T is a double star with $\gamma_{dR}(T) \in \{5, 6\}$ and $\ell(T) = n-2$, and thus $\gamma_{dR}(T) \geq 5 = \frac{n+8-(n-2)}{2} = \frac{n+8-\ell(T)}{2}$. Thus we assume that $\operatorname{diam}(T) \geq 4$.

Assume that T has a strong support vertex u, and let v be a leaf adjacent to u. Then $T - v$ is not a star, and it follows from Lemma 13 and the induction hypothesis that

$$\gamma_{dR}(T) \geq \gamma_{dR}(T - v) \geq \frac{n - 1 + 8 - (\ell(T) - 1)}{2} = \frac{n + 8 - \ell(T)}{2}.$$

Thus assume that T does not have a strong support vertex.

Let $v_1 v_2 \cdots v_k$ be a diametrical path in T, where v_1 and v_k are leaves and $k \geq 5$. Since T has no strong support vertex, $d(v_2) = d(v_{k-1}) = 2$. Let f be a $\gamma_{dR}(T)$-function.

If $f(v_2) = 2$, then $f(v_1) = 1$. Let $T' = T - v_1$ and $f' = f\mid_{V(T')}$. Then f' is a DRDF on T' such that $\gamma_{dR}(T') \leq w(f') = w(f) - 1 = \gamma_{dR}(T) - 1$, $\ell(T') = \ell(T)$, and T' is not a star. By the induction hypothesis, we have

$$\gamma_{dR}(T) \geq \gamma_{dR}(T') + 1 \geq \frac{n - 1 + 8 - \ell(T)}{2} + 1 > \frac{n + 8 - \ell(T)}{2}.$$

If $f(v_2) = 1$, then $f(v_1) = 2$. If we replace $f(v_2)$ by 2 and $f(v_1)$ by 1, then we obtain the desired bound as before.

Next assume that $f(v_2) = 0$. Then $f(v_1) \geq 2$. If $f(v_1) = 3$, then the function g with $g(v_1) = 0$, $g(v_2) = 3$ and $g(u) = f(u)$ otherwise is also a DRDF on T of weight $w(g) = w(f)$. However this we will discuss in the last case. Therefore assume now that $f(v_1) = 2$. Then $f(v_3) \geq 2$. If $T = P_5$, then Proposition 7 implies $\gamma_{dR}(T) = 6 \geq \frac{n+8-\ell(T)}{2}$. Next assume that $T \neq P_5$. Let $T'' = T - \{v_1, v_2\}$ and $f'' = f\mid_{V(T'')}$. Then f'' is a DRDF on T'', $\ell(T'') \leq \ell(T)$, and T'' is not a star. The induction hypothesis leads to

$$\begin{aligned}\gamma_{dR}(T) &= w(f) = w(f'') + 2 \geq \gamma_{dR}(T'') + 2 \geq \frac{n(T'') + 8 - \ell(T'')}{2} + 2 \\ &\geq \frac{n - 2 + 8 - \ell(T)}{2} + 2 = \frac{n + 8 - \ell(T)}{2} + 1 > \frac{n + 8 - \ell(T)}{2}.\end{aligned}$$

Finally assume that $f(v_2) = 3$. Then $f(v_1) = 0$. If $f(v_3) \geq 2$, then the function g with $g(v_1) = 2$, $g(v_2) = 0$ and $g(u) = f(u)$ otherwise is a DRDF on T with weight less than $w(f)$, a contradiction. If $f(v_3) = 1$, then we define the function g by $g(v_1) = 2$, $g(v_2) = 0$, $g(v_3) = 2$ and $g(u) = f(u)$ otherwise. Then

g is a DRDF on T of weight $w(g) = w(f)$, and as above, we obtain the desired result.

Thus assume that $f(v_3) = 0$. Assume first that $d(v_3) = 2$. If $k = 5$, then $T = P_5$, and we have seen above that the desired result is valid. So assume that $k \geq 6$. If $T = P_6$, then Proposition 7 implies $\gamma_{dR}(T) = 6 = \frac{n+8-\ell(T)}{2}$. Next assume that $T \neq P_6$. Let $T' = T - \{v_1, v_2, v_3\}$ and $f' = f\mid_{V(T')}$. Then f' is a DRDF on T', $\ell(T') \leq \ell(T)$, and T' is not a star. It follows from the induction hypothesis that

$$\begin{aligned}\gamma_{dR}(T) &= w(f) = w(f') + 3 \geq \gamma_{dR}(T') + 3 \geq \frac{n(T') + 8 - \ell(T')}{2} + 3\\ &\geq \frac{n - 3 + 8 - \ell(T)}{2} + 3 = \frac{n + 8 - \ell(T)}{2} + \frac{3}{2} > \frac{n + 8 - \ell(T)}{2}.\end{aligned}$$

Next assume that $d(v_3) \geq 3$. Let $u_2 \neq v_2, v_4$ be a further neighbor of v_3. Assume that $d(u_2) = 1$. This implies that $f(u_2) = 2$. Then the function g defined by $g(v_1) = 2$, $g(v_2) = 0$, $g(v_3) = 2$, $g(u_2) = 1$ and $g(u) = f(u)$ otherwise is a DRDF on T of weight $w(g) = w(f)$. If we consider $T - \{v_1, v_2\}$, then we obtain the desired result as above. Assume next, without loss of generality, that $d(u_2) = 2$, and let $u_1 \neq v_3$ be a neighbor of u_2. Clearly, u_1 is a leaf and $f(u_1) + f(u_2) = 3$. Then the function g defined by $g(v_1) = g(v_3) = g(u_1) = 2$, $g(v_2) = g(u_2) = 0$ and $g(u) = f(u)$ otherwise is a DRDF on T of weight $w(g) = w(f)$. Now the result follows as before, and the proof is complete. ∎

The next examples will show that Theorem 16 is sharp.

Example 17. If $S_{p,1}$ is a double star, then $\gamma_{dR}(S_{p,1}) = 5 = \frac{n(S_{p,1})+8-\ell(S_{p,1})}{2}$.

Let $S_{p,q}$ be the double star with the center vertices u and v. Now let $H_{p,q}$ be the tree constructed by subdividing the edge uv in $S_{p,q}$ twice. Then we observe that $\gamma_{dR}(H_{p,q}) = 6 = \frac{n(H_{p,q})+8-\ell(H_{p,q})}{2}$.

A DIDF f on a graph G is called in [4] an *outer-independent double Italian dominating function* (OIDIDF) if $V_0 = \{v \in V(G) : f(v) = 0\}$ is an independent set. The minimum weight of an OIDIDF on a graph G is called the *outer-independent double Italian domination number* of G and is denoted by $\gamma_{oidI}(G)$. The definitions lead to $\gamma_{oidI}(G) \geq \gamma_{dI}(G)$. Since $\gamma_{dI}(T) = \gamma_{dR}(T)$ for each tree, Theorem 16 leads to the next bound immediately.

Corollary 18. *Let T be a tree of order $n \geq 4$ with $\ell(T)$ leaves. If T is not a star, then $\gamma_{oidI}(T) \geq \frac{n+8-\ell(T)}{2}$.*

If T is a star, then $\gamma_{oidI}(T) = 3 = \frac{n(T)+5-\ell(T)}{2}$. Therefore Corollary 18 implies the next known result.

Corollary 19 [4]. *If T is a tree of order $n \geq 3$ with ℓ leaves, then $\gamma_{oidI}(T) \geq \frac{n+5-\ell(T)}{2}$, with equality if and only if T is a star.*

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