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CHOOSABILITY WITH SEPARATION OF CYCLES AND OUTERPLANAR GRAPHS

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Abstract

We consider the following list coloring with separation problem of graphs. Given a graph G and integers a, b, find the largest integer c such that for any list assignment L of G with $|L(v)| \leq a$ for any vertex v and $|L(u) \cap L(v)| \leq c$ for any edge uv of G, there exists an assignment φ of sets of integers to the vertices of G such that $\varphi(u) \subset L(u)$ and $|\varphi(v)| = b$ for any vertex v and $\varphi(u) \cap \varphi(v) = \emptyset$ for any edge uv. Such a value of c is called the separation number of (G, a, b). We also study the variant called the free-separation number which is defined analogously but assuming that one arbitrary vertex is precolored. We determine the separation number and free-separation number of the cycle and derive from them the free-separation number of a cactus. We also present a lower bound for the separation and free-separation numbers of outerplanar graphs of girth $g \geq 5$.

Keywords: coloring, choosability, outerplanar graph.

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1. Introduction

Let a, b, c and k be integers and let G be a graph. A k-list assignment L of G is a function which associates to each vertex a set of at most k integers. The list assignment L is *c-separating* if for any $uv \in E(G)$, $|L(u) \cap L(v)| \leq c$. The graph G is *(a,b,c)-choosable* if for any c-separating a-list assignment L, there exists an (L,b)-coloring of G, i.e., a coloring function φ on the vertices of G that assigns to each vertex v a subset of b elements from $L(v)$ in such a way that $\varphi(u) \cap \varphi(v) = \emptyset$ for any $uv \in E(G)$.

This type of restricted list coloring problem, called choosability with separation, has been introduced by Kratochvíl, Tuza and Voigt [12]. Notice that Kratochvíl *et al.* [12, 13] defined (a,b,c)-choosability a bit differently, requiring for a c-separating a-list assignment L that the lists of two adjacent vertices u and v satisfy $|L(u) \cap L(v)| \leq a - c$. Among the first results on the topic, a complexity dichotomy was presented [12] and general properties given [13]. Since then, a number of papers has considered choosability with separation of planar graphs, mainly for the case $b = 1$ [3–7, 11, 15]. For this class of graphs, the two following questions are still open (see [13,15]). Does any planar graph is $(4,1,2)$-choosable? $(3,1,1)$-choosable? Other recent papers concern balanced complete multipartite graphs and k-uniform hypergraphs (for the case $b = 1$) [10]; bipartite graphs (for the case $b = c = 1$) [9] and a study with an extended separation condition [14].

In this paper, we concentrate on choosability and free-choosability with separation of cycles and outerplanar graphs in a little different point of view: as a (a,b,c)-choosable graph is also (a,b,c')-choosable for any $c' < c$, our aim is to determine, for given a, b, $a \geq b$, the largest c such that G is (a,b,c)-choosable. We find convenient to define the parameter $\mathsf{sep}(G,a,b)$ that we call the *(list) separation number* of G as

$$\mathsf{sep}(G,a,b) = \max\{c : G \text{ is } (a,b,c)\text{-choosable}\}.$$

The notion of free choosability [2], that consists in considering list assignments on graphs with a precolored vertex, easily extends to choosability with separation: a graph G is *(a,b,c)-free-choosable* if for any c-separating a-list assignment L, any $v \in V(G)$ and any $C \subset L(v)$ with $|C| = b$, there exists an (L,b)-coloring φ such that $\varphi(v) = C$. Alternatively, we can view free choosability as classical choosability but with a list of cardinality b on one arbitrary vertex. Analogously with the separation number, we define the *free-separation number* $\mathsf{fsep}(G,a,b)$ of a graph G as

$$\mathsf{fsep}(G,a,b) = \max\{c : G \text{ is } (a,b,c)\text{-free-choosable}\}.$$

Clearly, for any graph G and any integers a and b, we have $\mathsf{sep}(G,a,b) \geq \mathsf{fsep}(G,a,b)$. Moreover, since for any $a \geq b \geq 1$, every graph G is $(a,b,0)$-free-choosable, we have

$$0 \leq \mathsf{fsep}(G,a,b) \leq \mathsf{sep}(G,a,b) \leq a$$

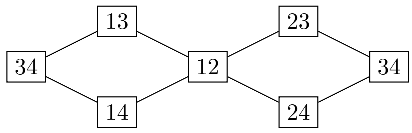


Figure 1. A cactus with a 1-separating 2-list assignment L for which no $(L, 1)$-coloring exists.

and thus both parameters are well defined for any graph. As a first example (and as we will prove in Proposition 15), the graph G depicted in Figure 1 is not $(2, 1, 1)$-choosable, thus implying that $\mathsf{sep}(G, 2, 1) = \mathsf{fsep}(G, 2, 1) = 0 < \mathsf{sep}(C_4, 2, 1) = 1$.

In this paper, we determine the separation number of the cycle in Section 2 (Theorem 5) and free-separation number (Theorem 10 and Proposition 11) of the cycle in Section 3. Contrary to the separation number, we show that the free-separation number of a cycle does not depend on the parity of its length and that C_3 is a special case. We then use these results to determine bounds and exact values for the same invariants on outerplanar graphs of girth at least 5 and tighter bounds for the subclass of cactuses in Section 4. Some possible directions for further works are given in Section 5.

Our proofs are all constructive and the proofs of upper bounds on sep and fsep rely on finding counter-example list assignments. These examples are constructed greedily, maximizing for each list, the intersections with lists of other adjacent vertices (while satisfying the c-separating condition). Our proofs for $\mathsf{fsep}(C_n, a, b)$ use special types of list assignments of the path (Lemmas 8 and 9) that may be of interest for obtaining other choosability results.

2. Separation Number of the Cycle

Using a similar argument than the one used by Kratochvíl *et al.* [13] (in the more general setting of graphs with bounded outdegree orientation) we have the following.

Proposition 1. *For any $n \geq 3$ and $a \geq b$, we have* $\mathsf{sep}(C_n, a, b) \geq a - b$.

Proof. Let $k = a - b$ and let L be a k-separating $(b + k)$-list assignment. Orient C_n clockwise, with x^- and x^+ being the predecessor and successor of vertex x, respectively. Since for any $x \in V(C_n)$, $|L(x) \cap L(x^+)| \leq k$, we have $|L(x) \setminus L(x^+)| \geq b + k - k = b$. Hence it is possible to assign a set $\varphi(x)$ of b colors from $L(x) \setminus L(x^+)$ to each vertex x.

Since any (a, b, c)-choosable graph is also (a', b, c)-choosable for any $a' \geq a$, we obtain that C_n is (a, b, c)-choosable when $c \leq a - b$. ∎

Since $a \geq b$, we can rewrite the above inequality as $\mathsf{sep}(C_n, b+k, b) \geq k$ for any $k \geq 0$.

As the next result shows, the above result is tight provided $k < b$.

Proposition 2. *For any $n \geq 3, b \geq 1$ and $k < b$, we have $\mathsf{sep}(C_n, b+k, b) \leq k$.*

Proof. We provide a $(k+1)$-separating $(b+k)$-list assignment L for which no (L, b)-coloring of C_n exists. Let X be a set of $n(b-1)+1$ colors and let C, D_i, F_i, $i \in \{0, \ldots, n-1\}$ be a partition of X with $|C| = 1, |D_i| = k, |F_i| = b-k-1$. Let $C_n = (x_0, \ldots, x_{n-1})$ and for any i, $0 \leq i \leq n-1$, let

$$L(x_i) = C \cup D_i \cup D_{i+1} \cup F_i,$$

with indices taken modulo n.

Now, observe that, by the construction of the list assignment L, the color of C is present in the color-list of every vertex and that every color of any set D_i is present in the lists of two consecutive vertices. Therefore, the color of C can be assigned to at most $\lfloor n/2 \rfloor$ vertices while for $0 \leq i \leq n-1$, every color of every set D_i can be given to at most one vertex. Hence, the total number of colors that can be given to vertices of C_n is $\lfloor n/2 \rfloor + nk + n(b-k-1) = \lfloor n/2 \rfloor + n(b-1) < nb$. Then, since the n vertices of C_n require nb colors in total, no (L, b)-coloring of C_n exists. ■

Reusing the method of the proof of Proposition 1 with a little more involved argument, we are able to prove.

Proposition 3. *For any a, b, c, $n \geq 3$ and $k \geq 1$, the following implication is true for the cycle C_n.*

If C_n is (a, b, c)-choosable, then C_n is $(a+2k, b+k, c+k)$-choosable.

Proof. Suppose $a \geq c$ and C_n is (a, b, c)-choosable and let L be a $(c+k)$-separating $(a+2k)$-list assignment of C_n. Orient C_n clockwise, with x^- and x^+ being the predecessor and successor of vertex x, respectively. Since for any $x \in V(C_n)$, $|L(x) \cap L(x^+)| \leq c+k$, we have $|L(x) \setminus L(x^+)| \geq a + 2k - (c+k) = a - c + k \geq k$. Hence it is possible to assign a set $\varphi(x)$ of k colors from $L(x) \setminus L(x^+)$ to each vertex x.

Now we still have to assign b more colors to each vertex to complete the coloring. For this, we are going to construct a new list assignment L' by removing from $L(x)$ the colors already assigned to x and also a maximum number of colors from $L(x) \cap L(x^+)$, including those assigned to x^+ that are in $L(x)$, if any, in order for L' to be a c-separating a-list assignment (observe that, by construction, $L(x) \cap \varphi(x^-) = \emptyset$). For any $x \in V(C_n)$, let $I^+(x) = L(x) \cap L(x^+)$ and let $S(x)$ be any subset of $I^+(x)$ of size $\min\{k, |I^+(x)|\}$ that contains $L(x) \cap \varphi(x^+)$. (Note that $I^+(x)$ and $S(x)$ may be empty.) Then, define a new list assignment L' on C_n by setting, for all $x \in V(C_n)$,

$$L'(x) = L(x) \setminus (\varphi(x) \cup S(x)).$$

We then have $|L'(x)| = a + 2k - k - \min\{k, |I^+(x)|\} \geq a$. Moreover, if $|I^+(x)| \geq k$ then $|S(x)| = k$ and thus $|L'(x) \cap L'(x^+)| \leq c + k - k = c$. Otherwise ($|I^+(x)| < k$), we have $S(x) = I^+(x)$ and thus $|L'(x) \cap L'(x^+)| = 0$ (and $|L'(x)| > a$ in this case). Therefore L' is a c-separating a-list assignment. Consequently, since by hypothesis G is (a, b, c)-choosable, there exists an (L', b)-coloring of G, which together with the coloring f, produces an $(L, b + k)$-coloring of G, proving that G is $(a + 2k, b + k, c + k)$-choosable. ■

For cycles of odd length, combining known choosability results on the cycle with Proposition 3 allows to determine the separation number in the remaining cases.

Proposition 4. *For any integers b, p and α with $0 \leq \alpha \leq \frac{b}{p}$ and $p \geq 1$, we have*

$$\mathsf{sep}(C_{2p+1}, 2b + \alpha, b) = b + (p + 1)\alpha.$$

Proof. It is known that cycles of length $2p+1$ are (a, b, a)-choosable if and only if $a/b \geq 2 + 1/p$ (see [2]). Hence, for $\alpha \geq 0$, C_{2p+1} is $((2p + 1)\alpha, p\alpha, (2p + 1)\alpha)$-choosable. Then by Proposition 3, C_{2p+1} is also $((2p + 1)\alpha + 2k, p\alpha + k, (2p + 1)\alpha + k)$-choosable for any $k \geq 0$. Setting $k = b - p\alpha$, we obtain that C_{2p+1} is $(2b + \alpha, b, b + (p + 1)\alpha)$-choosable.

Note that if p divides b and if $\alpha = b/p$, then $b + (p + 1)\alpha = 2b + \alpha$. Hence in this case, $\mathsf{sep}(C_{2p+1}, 2b + \alpha, b) = 2b + \alpha$. Otherwise, to prove that C_{2p+1} is not $(2b + \alpha, b, b + (p + 1)\alpha + 1)$-choosable for $\alpha < \frac{b}{p}$, we provide a $(b + (p + 1)\alpha + 1)$-separating $(2b + \alpha)$-list assignment L of C_{2p+1} for which no (L, b)-coloring exists. Let C be a set of $(2p + 1)\alpha + 2$ colors and for $0 \leq i \leq 2p$, let D_i be $2p + 1$ pairwise disjoint sets of $b - p\alpha - 1$ colors (also disjoint from C). Let $C_{2p+1} = (x_0, \ldots, x_{2p})$ and set

$$L(x_i) = C \cup D_i \cup D_{i+1},$$

with $0 \leq i \leq 2p$ and indices taken modulo $2p + 1$.

It can be checked that the lists of any two consecutive vertices have $(2p + 1)\alpha + 2 + b - p\alpha - 1 = b + (p + 1)\alpha + 1$ elements in common. Assume now that there exists an (L, b)-coloring of C_{2p+1}. Observe that, by the construction of the list assignment L, every color from C is present in the color-list of every vertex and that every color of any set D_i is present in the lists of two consecutive vertices. Therefore, every color of C can be assigned to at most p vertices while for $0 \leq i \leq 2p$, every color of every set D_i can be given to at most one vertex. Hence we have $p((2p+1)\alpha+2)+(2p+1)(b-p\alpha-1) = b(2p+1)-1$ available colors in total but we have to assign $b(2p + 1)$ colors to the vertices, a contradiction. ■

We are now ready to determine the separation number of the cycle.

Theorem 5. *For any $p \geq 1$ and any a, b such that $a \geq b \geq 1$,*

$$\mathsf{sep}(C_{2p+2}, a, b) = \begin{cases} a-b, & b \leq a < 2b, \\ a, & a \geq 2b. \end{cases}$$

$$\mathsf{sep}(C_{2p+1}, a, b) = \begin{cases} a-b, & b \leq a < 2b, \\ b+(p+1)(a-2b), & 2b \leq a < 2b + \frac{b}{p}, \\ a, & a \geq 2b + \frac{b}{p}. \end{cases}$$

Proof. For even-length cycles, the result is obtained by combining Propositions 1 and 2 and noting that C_{2p} is $(2b, b, 2b)$-choosable.

For odd-length cycles, the combination of Propositions 1, 2 and 4 and the known fact that C_{2p+1} is (a, b, a)-choosable for any a, b such that $a/b \geq 2 + 1/p$ leads to the result. ■

Example 1. From the above theorem, we know that the cycle C_3 is $(5, 2, 4)$-choosable but not $(5, 2, 5)$-choosable (this was already known) and also $(7, 3, 5)$-choosable but not $(7, 3, 6)$-choosable (while it was only known before that C_3 is not $(7, 3, 7)$-choosable). In contrast, C_5 is $(7, 3, 6)$-choosable and $(9, 4, 7)$-choosable but not $(7, 3, 7)$-choosable and not $(9, 4, 8)$-choosable.

3. Free Separation Number of the Cycle

In order to determine the free-separation number of the cycle, we first set some notation and preliminary results.

The following Hall-type condition that we call the *amplitude condition* is necessary for a graph G to be (L, b)-colorable

$$\text{for all } H \subset G, \sum_{k \in C} \alpha(H, L, k) \geq b|V(H)|,$$

where $C = \bigcup_{v \in V(H)} L(v)$ and $\alpha(H, L, k)$ is the independence number of the subgraph of H induced by the vertices containing k in their color list. Notice that H can be restricted to be a connected induced subgraph of G. As shown by Cropper *et al.* [8] (in the more general context of weighted list coloring), this condition is also sufficient when the graph is a complete graph or a path (or some other specific graphs).

For a list assignment L on a graph G of order n with vertex set $V(G) = \{x_1, x_2, \ldots, x_n\}$, we let $L_i = L(x_i)$ and for $1 \leq i < j \leq n$, we write $\Sigma_{i,j}(L) = \sum_{k \in C} \alpha(H, L, k)$, where H is the subgraph of G induced by vertices $x_i, x_{i+1}, \ldots, x_j$. We also simplify $\Sigma_{1,n}(L)$ to $\Sigma(L)$.

From now on, a cycle of order n will have its vertices denoted by $x_1, x_2, \ldots, x_n$ following some order on the cycle and the vertices of a path of order n will be also denoted by $x_1, x_2, \ldots, x_n$ following the path from one end-vertex to the other. We will use the following relation to show that the cycle is (L, b)-colorable for some lists L.

Remark 6. A list assignment L on the cycle C_n with $|L_1| = b$ and $|L_i| = a$ for any $i, 2 \le i \le n$ can be transformed into a list assignment L' on the path P_{n+1} by setting $L'_{n+1} = L_1$ and for $1 \le i \le n$, $L'_i = L_i$ (i.e., P_{n+1} has been obtained by "cutting" the cycle on the vertex x_1). Clearly if P_{n+1} is (L', b)-colorable, then C_n is (L, b)-colorable.

As induced subgraphs of paths are sub-paths, the amplitude condition for a path P_n can be rewritten as

$$\text{for all } i, j,\ 1 \le i \le j \le n,\ \Sigma_{i,j}(L) \ge b(j - i + 1).$$

For integers n, a and b with $b \le a$, let

$$c(n,a,b) = \begin{cases} \frac{n-1}{n}(a-b), & \text{if } b \le a < \frac{2n-1}{n-1}b, \\ \frac{n-1}{n-2}(a-b) - \frac{2}{n-2}b, & \text{if } \frac{2n-1}{n-1}b \le a < 2\frac{n+1}{n}b, \\ a, & \text{if } 2\frac{n+1}{n}b \le a. \end{cases}$$

Remark 7. If $a < \frac{2n-1}{n-1}b$, then $c(n,a,b) \le \frac{n-1}{n}(a-b) < \frac{n-1}{n}(\frac{2n-1}{n-1}b - b) = b$ and if $\frac{2n-1}{n-1}b \le a < 2\frac{n+1}{n}b$, then $c(n,a,b) = \frac{n-1}{n-2}(a-b) - \frac{2}{n-2}b \ge \frac{n-1}{n-2}\frac{2n-1}{n-1}b - \frac{n+1}{n-2}b = b$.

We now prove two lemmas about conditions for a path with precolored end-vertices to be list colorable.

Lemma 8. *Let n, a, b, c be integers, $n \ge 3$, $b \le a < 2\frac{n+1}{n}b$ and $c = \lfloor c(n,a,b) \rfloor$. For any c-separating a-list assignment L of P_{n+1} such that $|L_1| = |L_{n+1}| = b$, there exists an (L, b)-coloring of P_{n+1}.*

Proof. It is sufficient to verify that the amplitude condition is satisfied. First, we show that for both the cases $a < \frac{2n-1}{n-1}b$ and $\frac{2n-1}{n-1}b \le a < 2\frac{n+1}{n}b$, we have $c \le a - b$. If $c \le \frac{n-1}{n}(a-b)$, then clearly $c \le a - b$. If $c \le \frac{n-1}{n-2}(a-b) - \frac{2}{n-2}b = \frac{n-2}{n-2}(a-b) + \frac{a-b}{n-2} - \frac{2}{n-2}b = a - b + \frac{a-3b}{n-2}$, then again, $c \le a - b$ since $a < 2\frac{n+1}{n}b \le 3b$ as soon as $n \ge 3$.

Now, since L is c-separating, if $j < n$ we have

$$|L_{j+1} \setminus L_j| \ge a - c$$

and thus, for any $1 \le i \le j$,

$$\Sigma_{i,j+1}(L) \ge \Sigma_{i,j}(L) + a - c. \tag{1}$$

Moreover, since $|L_{n+1}| = b$, we also have

$$\Sigma_{i,n+1}(L) \geq \Sigma_{i,n}(L) + \max\{b - c, 0\}. \tag{2}$$

Therefore, if $1 < i \leq j \leq n$, then $\Sigma_{i,j}(L) \geq (j - i + 1)(a - c)$. Hence the amplitude condition is satisfied in this case if $(j-i+1)(a-c) \geq (j-i+1)b$, i.e., if $a - c \geq b$, which is true since we have shown above that $c \leq a - b$.

If $i = 1$ or $j = n + 1$, we consider two cases depending on a, b and c (note that the ratio $\frac{2n-1}{n-1}$ has been chosen in such a way that $c < b$ in Case 1 and $c \geq b$ in Case 2, see Remark 7).

Case 1. $c = \left\lfloor \frac{n-1}{n}(a-b) \right\rfloor$ and $b \leq a < \frac{2n-1}{n-1}b$. By Remark 7, we have $c < b$. Hence, if $i = 1$ and $2 \leq j \leq n$, then by Equation (1), $\Sigma_{1,j}(L) \geq b + (a-c)j \geq (j+1)b$, since $a - c \geq b$. Otherwise, if $i > 1$ and $j = n+1$, then $\Sigma_{i,n+1}(L) \geq a + (a-c)(n-i) + b - c = b + (a-c)(n+1-i)$. Hence, since $a - c \geq b$, we have that $\Sigma_{i,n+1}(L) \geq (n+2-i)b$. Finally, since $c \leq \frac{n-1}{n}(a-b)$, we have $\Sigma(L) \geq b + (a-c)(n-1) + b - c \geq (n+1)b$ by Equations (1) and (2).

Case 2. $c = \left\lfloor \frac{n-1}{n-2}(a-b) - \frac{2}{n-2}b \right\rfloor$ and $a \geq \frac{2n-1}{n-1}b$. By Remark 7, we have $c \geq b$. Hence, if $i = 1$ and $2 \leq j \leq n$, then $\Sigma_{1,j}(L) \geq b + a - b + (a-c)(j-2) \geq b + (a-c)(j-1) \geq jb$, since $a - c \geq b$. Otherwise, if $i > 1$ and $j = n+1$, then $\Sigma_{i,n+1}(L) \geq b + a - b + (a-c)(n-i) \geq b + (a-c)(n+1-i) \geq (n+2-i)b$ since $a - c \geq b$. Finally, $\Sigma_{1,n+1}(L) \geq \Sigma_{2,n}(L) \geq a + (a-c)(n-2) \geq (n+1)b$ since $c \leq \frac{n-1}{n-2}(a-b) - \frac{2}{n-2}b$.

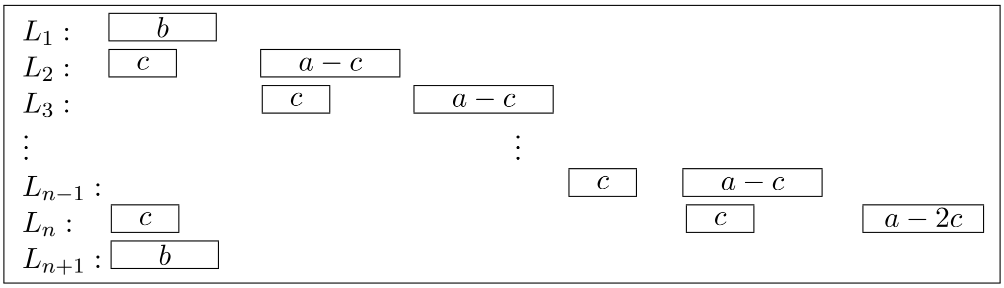
Therefore, in both cases, the amplitude condition is satisfied hence P_{n+1} is (L, b)-colorable. ■

As shown by the next lemma, the above condition in Lemma 8 is also necessary (provided that $n \geq 4$) and even with restricted list assignments in which the lists of the two endvertices are the same or are disjoint.

Lemma 9. *Let n, a, b, c be integers, $n \geq 4$, $b \leq a < 2\frac{n+1}{n}b$ and $c = \lfloor c(n,a,b) \rfloor + 1$. There exists a c-separating a-list assignment L of P_{n+1} with $|L_1| = |L_{n+1}| = b$ such that P_{n+1} is not (L, b)-colorable. Moreover, the same holds if in addition, the list assignment L is such that $L_1 = L_{n+1}$ or $L_1 \cap L_{n+1} = \emptyset$.*

Proof. We provide a counter-example in each of the two following cases.

Case 1. $b \leq a < \frac{2n-1}{n-1}b$. We show that for $c = \left\lfloor \frac{n-1}{n}(a-b) \right\rfloor + 1$ the following list assignment L is a c-separating a-list assignment of P_{n+1} is such that $L_1 = L_{n+1}$, $|L_1| = b$, but no (L, b)-coloring exists. In order to have a compact representation and to shorten the proof, L is described in a graphical way showing the intersections of sublists composing the lists L_i (each box represent a color subset and the number inside a box indicates its size).



Since $a < \frac{2n-1}{n-1}b$, we have $c = \left\lfloor \frac{n-1}{n}(a-b) \right\rfloor + 1 < \left\lfloor \frac{n-1}{n} \frac{2n-1-n+1}{n-1} b \right\rfloor + 1 = b+1$, i.e., $c \le b$. Moreover, since $\frac{n-1}{n} < 1$, we have $c < a-b+1$, hence $a > b+c-1 \ge 2c-1$. Consequently, the list assignment L is well defined.

Observe that for $2 \le i \le n-1$, each color of each list L_i can be used on only one vertex. Also, for any color k shared by the lists of L_1 and L_{n+1}, we have $\alpha(P_{n+1}, L, k) = 2$. Hence, we have

$$\Sigma(L) = 2b + (n-2)(a-c) + a - 2c = (n-1)(a-c) + 2b - c.$$

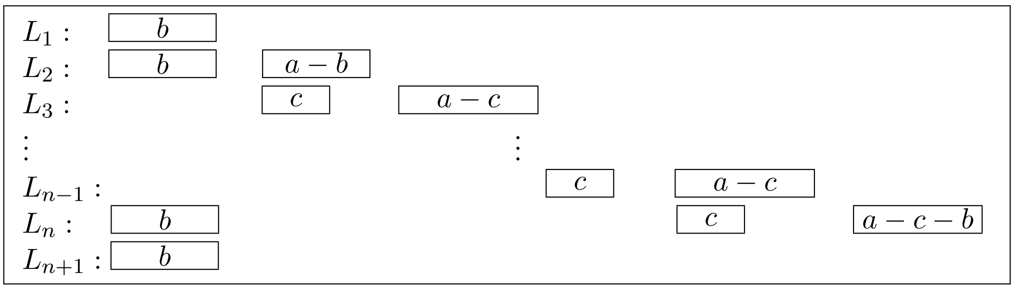
In order the amplitude condition to be satisfied, we must have $\Sigma(L) \ge (n+1)b$, i.e., $(n-1)(a-c) + 2b - c \ge (n+1)b$, which is equivalent to $c \le \frac{n-1}{n}(a-b)$. Consequently, P_{n+1} is not (L,b)-colorable when $c > \frac{n-1}{n}(a-b)$.

Case 2. $\frac{2n-1}{n-1}b \le a < 2\frac{n+1}{n}b$. We show that for $c = \left\lfloor \frac{n-1}{n-2}(a-b) - \frac{2}{n-2}b \right\rfloor + 1$ there is a c-separating a-list assignment L of P_{n+1} for which $L_1 = L_{n+1}$, $|L_1| = b$, but no (L,b)-coloring exists. First, we show that $a \ge 2c-1$. Since $c \le \frac{n-1}{n-2}a - \frac{n+1}{n-2}b + 1$ and $a < 2\frac{n+1}{n}b$, i.e., $b > \frac{n}{2n+2}a$, then $2c - 2 \le \frac{2n-2}{n-2}a - \frac{2n+2}{n-2}b < \frac{2n-2}{n-2}a - \frac{2n+2}{n-2}\frac{n}{2n+2}a = a$.

Second, we show that $c \ge b+1$. As $c > \frac{n-1}{n-2}a - \frac{n+1}{n-2}b$ and since $a \ge \frac{2n-1}{n-1}b$, we obtain $c > \frac{n-1}{n-2}\frac{2n-1}{n-1}b - \frac{n+1}{n-2}b = b$.

Depending on the value of a and c, we consider two subcases and for each we provide a c-separating a-list assignment L for which $|L_1| = b$ and no (L,b)-coloring exists.

Subcase 2.a. $a \ge 2c$. Consider the list assignment L defined as follows.



Since we are in the case that $a \geq 2c$ and $c \geq b+1$, the list is well defined. We have

$$\Sigma(L) = 2b + a - b + (n-3)(a-c) + a - c - b = (n-1)a - (n-2)c.$$

Therefore, $\Sigma(L) \geq (n+1)b$ implies $c \leq \frac{n-1}{n-2}(a-b) - \frac{2}{n-2}b$. Consequently, P_{n+1} is not (L,b)-colorable when $c > \frac{n-1}{n-2}(a-b) - \frac{2}{n-2}b$.

Subcase 2.b. $a = 2c-1$. We are in the case that $a < 2\frac{n+1}{n}b$, i.e., $a < 2b + \frac{2b}{n}$. Hence, as $a = 2c-1$, c satisfies

$$c < b + \frac{b}{n} + \frac{1}{2}. \tag{3}$$

If n is odd, consider the list assignment L defined as follows.

L_1 :	b						
L_2 :	b	$2c-b-1$					
L_3 :		c	$c-1$				
L_4 :			$c-1$	c			
$\vdots$				$\vdots$			
L_{n-1} :					$c-1$	c	
L_n :	b					c	$c-b-1$
L_{n+1} :	b						

We have

$$\Sigma(L) = 2b + 2c - b - 1 + \frac{n-3}{2}(c-1) + \frac{n-3}{2}c + c - b - 1 = nc - \frac{n+1}{2}.$$

Therefore the amplitude condition is not satisfied if $nc - \frac{n+1}{2} < (n+1)b$, i.e., if $c < b + \frac{b}{n} + \frac{1}{2} + \frac{1}{2n}$ which is true by Equation (3).

If n is even, consider the list assignment L defined as follows.

L_1 :	b						
L_2 :	b	$2c-b-1$					
L_3 :		c	$c-1$				
L_4 :			$c-1$	c			
$\vdots$				$\vdots$			
L_{n-1} :					c	$c-1$	
L_n :	b					$c-1$	$c-b$
L_{n+1} :	b						

We have

$$\Sigma(L) = 2b + 2c - b - 1 + \frac{n-2}{2}(c-1) + \frac{n-4}{2}c + c - b = nc - \frac{n}{2}.$$

Therefore the amplitude condition is not satisfied if $nc - \frac{n}{2} < (n+1)b$, i.e., if $c < b + \frac{b}{n} + \frac{1}{2}$ which is true by Equation (3).

The counter-examples presented are such that $L_1 = L_{n+1}$, but they can be easily modified in order that $L_1 \cap L_{n+1} = \emptyset$ without changing the conclusion. For this, in each of the above four list assignments, instead of using the b colors of L_1 for L_n and L_{n+1}, just take b new colors (not used by any list L_i, $1 \leq i \leq n-1$). ■

Theorem 10. *For any $a \geq b \geq 1$ and any $n \geq 4$,*

$$\mathsf{fsep}(C_n, a, b) = \begin{cases} \left\lfloor \frac{n-1}{n}(a-b) \right\rfloor, & b \leq a < \frac{2n-1}{n-1}b, \\ \left\lfloor \frac{n-1}{n-2}(a-b) - \frac{2}{n-2}b \right\rfloor, & \frac{2n-1}{n-1}b \leq a < 2\frac{n+1}{n}b, \\ a, & 2\frac{n+1}{n}b \leq a. \end{cases}$$

Proof. First, if $2\frac{n+1}{n}b \leq a$, then we know from [2] that C_n is (a, b, a)-free-choosable.

For the two other cases, given a c-separating a-list assignment L of C_n with $|L_1| = b$ and $c \leq \lfloor c(n, a, b) \rfloor$, we consider the list assignment L' on P_{n+1} obtained from L on C_n by cutting the cycle at x_1 as in Remark 6. By Lemma 8, P_{n+1} is (L', b)-colorable. Hence C_n is (L, b)-colorable for $c \leq \lfloor c(n, a, b) \rfloor$. For the converse, if $c = \lfloor c(n, a, b) \rfloor + 1$, then we know, by Lemma 9 that there exists a c-separating a-list assignment L' of P_{n+1} with $L'_1 = L'_{n+1}$ and $|L'_1| = b$, such that P_{n+1} is not (L', b)-colorable. Therefore, identifying the vertices x_1 and x_{n+1} of P_{n+1}, we obtain a c-separating a-list assignment L on the cycle C_n such that $|L_1| = b$ and no (L, b)-coloring of C_n exists. ■

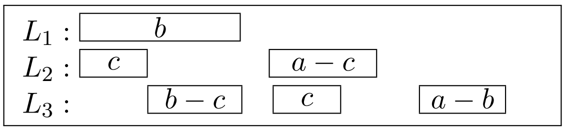
It only remains to determine the free separation number of the cycle of length 3 which has a special behavior.

Proposition 11.

$$\mathsf{fsep}(C_3, a, b) = \begin{cases} \left\lfloor \frac{2}{3}(a-b) \right\rfloor, & b \leq a < \frac{7}{4}b, \\ 2a - 3b, & \frac{7}{4}b \leq a < 3b, \\ a, & 3b \leq a. \end{cases}$$

Proof. We consider the three following cases depending on a.

Case 1. $b \leq a < \frac{7}{4}b$. Let $c = \left\lfloor \frac{2}{3}(a-b) \right\rfloor + 1$. We prove that C_3 is not (a, b, c)-free-choosable. For this, we give a c-separating a-list assignment L for which $|L_1| = b$ and no (L, b)-coloring exists.



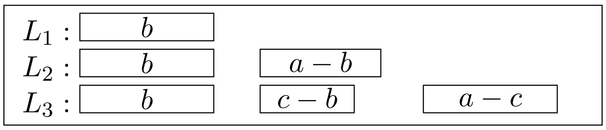
In order this list to be well defined, we must have $c \leq b$. If $b = 1$, then $a = 1$ and thus $c = 1 = b$. If $b \geq 2$, then $c \leq \frac{2}{3}(a-b) + 1 \leq b$ if $a \leq \frac{5b-3}{2}$ which is true since $\frac{7}{4}b \leq \frac{5b-3}{2}$ for $b \geq 2$. We must also have $a - c \geq c$ which is true since $a - 2c \geq a - 4(a-b)/3 = 4b/3 - a/3 \geq 0$ as $a < \frac{7}{4}b$. Moreover, we have $\Sigma(L) = b+a-c+a-b = 2a-c$. Since $c > \frac{2}{3}(a-b)$, $\Sigma(L) < 3b$ if $6a-2a+2b < 9b$, i.e., if $a < \frac{7}{4}b$ which is true by hypothesis. Hence the amplitude condition is not satisfied and thus C_3 is not (L, b)-colorable.

Now, we prove that C_3 is (a, b, c')-free-choosable with $c' = c-1 = \left\lfloor \frac{2}{3}(a-b) \right\rfloor$. First, observe that we have $b - \frac{2}{3}(a-b) = \frac{5b-2a}{3} \geq 0$, hence $b \geq c'$. Since C_3 is a complete graph, by Cropper *et al.'s* result [8], the amplitude condition is sufficient in order it is (L, b)-colorable. This is clearly true for a subgraph of C_3 reduced to one vertex and for a subgraph of two vertices since for any c'-separating a-list assignment L with $|L_1| = b$, we have $|L_2 \cup L_3| \geq 2a - c' \geq 2b$ as $a \geq b$ and, for $i = 2$ or 3, $|L_1 \cup L_i| \geq b + a - \min(b, c') = b + a - c' \geq 2b$ since $c' < a - b$. For the whole graph, since $c' \leq \frac{2}{3}(a-b)$, we have

$$\Sigma(L) \geq |L_1| + |L_2 \setminus L_1| + |L_3 \setminus (L_1 \cup L_2)| \geq b + a - c' + a - 2c' = 2a + b - 3c' \geq 3b.$$

Case 2. $\frac{7}{4}b \leq a < 3b$. Let $c = 2a - 3b + 1$. We prove that C_3 is not (a, b, c)-free-choosable. For this, we give a c-separating a-list assignment L for which $|L_1| = b$ and no (L, b)-coloring exists. We present a list for each of the two following subcases depending on whether $a \geq 2b$ or not (note that if $a \geq 2b$, then $b \leq c$ and if $a < 2b$, then $c < b$).

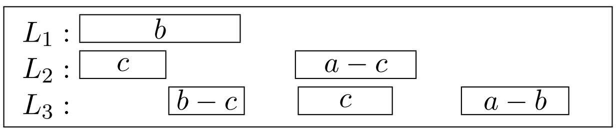
If $a \geq 2b$, L is made up as follows.



For this list to be well defined, we must have $a-c \geq 0$ which is true by hypothesis. In order for L to be c-separating, we must have $b \leq c = 2a-3b+1$, i.e., $a \geq 2b-\frac{1}{2}$ which is true.

Moreover, we have $\Sigma(L) = b + a - b + a - c = 2a - c = 3b - 1 < 3b$, hence the amplitude condition is not satisfied and thus C_3 is not (L, b)-colorable.

If $a < 2b$, L is made up as follows.



For this list to be well defined, we must have $c = 2a-3b+1 \geq b-c = 4b-2a-1$, i.e., $a \geq \frac{7}{4}b - 1/2$, which is true by hypothesis. We also have $c \leq a-c$ since $a < 2b$.

We have $\Sigma(L) = b+a-c+a-b = 2a-c = 3b-1 < 3b$, hence the amplitude condition is not satisfied and thus C_3 is not (L,b)-colorable.

Now, in both the cases $a \geq 2b$ and $a < 2b$, we prove that C_3 is (a,b,c')-free-choosable with $c' = c-1 = 2a-3b$. Again, it is sufficient to verify that the amplitude condition is satisfied by any c'-separating a-list assignment L with $|L_1| = b$. This is clearly true for a subgraph of C_3 reduced to one vertex and for a subgraph of two vertices since we have $|L_2 \cup L_3| \geq 2a - c' = 3b \geq 2b$ and $|L_1 \cup L_i| \geq b + a - \min(b,c') \geq 2b$ for $i = 2$ or 3. For the whole graph, we have

$$\Sigma(L) \geq |L_1| + |L_2 \setminus L_1| + |L_3 \setminus (L_1 \cup L_2)| \geq b + a - \alpha + a - \beta,$$

with $\alpha = |L_1 \cap L_2|$ and $\beta = |(L_1 \cup L_2) \cap L_3|$. Let $\beta = \beta_1 + \beta_2 - \gamma$, where $\beta_1 = |L_1 \cap L_3|$, $\beta_2 = |L_2 \cap L_3|$ and $\gamma = |L_3 \cap L_2 \cap L_1|$. Then we have $\Sigma(L) \geq b + 2a - \alpha - \beta_1 - \beta_2 + \gamma$. As, by definition, $\alpha + \beta_1 + \gamma \leq b$ and $\beta_2 \leq c'$, then we obtain $\Sigma(L) \geq b + 2a - b - c' = 2a - c' = 3b$.

Case 3. $a \geq 3b$. In this case, C_3 is trivially (a,b,a)-free-choosable and the result follows.

In conclusion, in each of the three cases, we have shown that the maximum value of c for which C_3 is (a,b,c)-free-choosable is the one given in the statement. ■

4. Outerplanar Graphs

An *outerplanar graph* is a graph that has a planar drawing in which all vertices belong to the outer face of the drawing. For an outerplanar graph G, we denote by $\mathcal{T}_G$ the *weak dual* of G, i.e., the graph whose vertex set is the set of all inner faces of G, and $E(\mathcal{T}_G) = \{\alpha\beta | \ \alpha \text{ and } \beta \text{ share a common edge}\}$. A *cactus* is a graph in which every edge is part of at most one cycle. Cactuses form a subclass of outerplanar graphs. The *girth* of a graph G is the length of a shortest cycle in G.

We will use the fact that if an outerplanar graph G is 2-connected, then $\mathcal{T}_G$ is a tree. Moreover, in order to restrict our argument to 2-connected graphs, we first show the following which has been proved in [1] in the case $c = a$.

Lemma 12. *Let a, b, c be integers and let G_1, G_2 be two (a,b,c)-free-choosable graphs. Then the graph obtained from G_1 and G_2 by identifying any vertex of G_1 with any vertex of G_2 is (a,b,c)-free-choosable.*

Proof. Let G be the graph obtained by identifying vertex x_1 of G_1 with vertex x_2 of G_2, resulting in a vertex named x. Let $y \in V(G)$ and let L be a list assignment of G with $|L(v)| = a$ for $v \in V(G) \setminus \{y\}$ and $|L(y)| = b$ (i.e., y is the precolored vertex). Assume, without loss of generality, that $y \in V(G_1)$. For $i = 1, 2$, let L_i be the sublist assignment of L restricted to vertices of G_i. As G_1, G_2 are both (a, b, c)-free-choosable, there exists an (L_1, b)-coloring c_1 of G_1 and an (L_2, b)-coloring c_2 of G_2 such that $c_2(x) = c_1(x)$ (i.e., x is the precolored vertex of G_2). The union of colorings c_1 and c_2 is an (L, b)-coloring of G. ■

Lemma 13. *For any positive integers a, b, n and i, if $n \geq 4$, then*

$$\mathsf{fsep}(C_n, a, b) \leq \mathsf{fsep}(C_{n+i}, a, b).$$

Moreover, we have

$$\min(\mathsf{fsep}(C_3, a, b), \mathsf{fsep}(C_n, a, b)) = \begin{cases} \left\lfloor \frac{2}{3}(a-b) \right\rfloor, & \text{if } b \leq a < \frac{7}{4}b, \\ 2a - 3b, & \text{if } \frac{7}{4}b \leq a \leq \frac{2n+1}{n+1}b, \\ & \text{or } \frac{2n+2}{n}b \leq a < 3b, \\ \left\lfloor \frac{n-1}{n}(a-b) \right\rfloor, & \text{if } \frac{2n+1}{n+1}b < a < \frac{2n-1}{n-1}b, \\ \left\lfloor \frac{n-1}{n-2}(a-b) - \frac{2}{n-2}b \right\rfloor, & \text{if } \frac{2n-1}{n-1}b \leq a < \frac{2n+2}{n}b, \\ a, & \text{if } 3b \leq a. \end{cases}$$

Proof. The values of fsep for a cycle are given in Theorem 10. For the first assertion, we show that in each of the following cases we have $A = c(n, a, b) \leq B = c(n+1, a, b)$ (recall that $\mathsf{fsep}(C_n, a, b) = \lfloor c(n, a, b) \rfloor$).

- $b \leq a < \frac{2(n+1)-1}{n}b$.
  Since $\frac{n-1}{n} \leq \frac{n}{n+1}$, we have $A = \frac{n-1}{n}(a-b) \leq B = \frac{n}{n+1}(a-b)$.
- $\frac{2n+1}{n}b \leq a < \frac{2n-1}{n-1}b$.
  We want $A = \frac{n-1}{n}(a-b) \leq B = \frac{n(a-b)-2b}{n-1}$, i.e., $(n-1)^2(a-b) \leq n^2(a-b) - 2bn$. This can be rewritten as $a \geq \frac{4n-1}{2n-1}b$, which is true since $a \geq \frac{2n+1}{n}b$.
- $\frac{2(n+1)-1}{n}b \leq a < 2\frac{n+2}{n+1}b$.
  We want $A = \frac{(n-1)(a-b)-2b}{n-2} \leq B = \frac{n(a-b)-2b}{n-1}$, which simplifies to $a \leq 3b$ and is true by hypothesis.
- $2\frac{n+2}{n+1}b \leq a \leq 2\frac{n+1}{n}b$.
  We want $A = \frac{(n-1)(a-b)-2b}{n-2} \leq B = a$, i.e., $a \leq (n+1)b$ which is true.

For the second assertion, by Proposition 11 and Theorem 10, we infer the desired inequalities between $A = \mathsf{fsep}(C_3, a, b)$ and $B = \mathsf{fsep}(C_n, a, b)$, $n \geq 4$.

- For $b \leq a < \frac{7}{4}b$, we have $A = \left\lfloor \frac{2}{3}(a-b) \right\rfloor \leq B = \left\lfloor \frac{n-1}{n}(a-b) \right\rfloor$.

- For $\frac{7}{4}b \leq a \leq \frac{2n+1}{n+1}b$, we have $A = 2a - 3b \leq B = \left\lfloor \frac{n-1}{n}(a-b) \right\rfloor$.
- For $\frac{2n+1}{n+1}b < a < \frac{2n-1}{n-1}b$, we have $A = 2a - 3b \geq B = \left\lfloor \frac{n-1}{n}(a-b) \right\rfloor$.
- For $\frac{2n-1}{n-1}b \leq a < \frac{2n+2}{n}b$, we have $A = 2a - 3b \geq B = \left\lfloor \frac{n-1}{n-2}(a-b) - \frac{2}{n-2}b \right\rfloor$.
- For $\frac{2n+2}{n}b \leq a < 3b$, we have $A = 2a - 3b \leq B = a$.
- For $3b \leq a$, we have $A = B = a$. ■

From the free-separation number of the cycle we derive the free-separation number of cactuses.

Theorem 14. *Let G be a cactus with finite girth g and let $a \geq b \geq 1$ be integers. Then if $g \geq 4$ or G has only cycles of length three, then*

$$\mathsf{fsep}(G, a, b) = \mathsf{fsep}(C_g, a, b).$$

Otherwise, if G contains at least one triangle and ℓ is the length of a shortest cycle of G greater than three, then

$$\mathsf{fsep}(G, a, b) = \begin{cases} \mathsf{fsep}(C_\ell, a, b), & \text{if } \frac{2\ell+1}{\ell+1}b < a < \frac{2\ell+2}{\ell}b, \\ \mathsf{fsep}(C_3, a, b), & \text{otherwise.} \end{cases}$$

Proof. Let G be a cactus of finite girth g and let a, b, c be integers. Then, each of its blocks $B_1, B_2, \ldots, B_r$ is either a cycle (of length at least g) or a single edge, and they are connected in a treelike structure.

We first show that if $g \geq 4$ or all cycles of G are of length 3, then G is (a, b, c)-free-choosable if and only if $c \leq \mathsf{fsep}(C_g, a, b)$. Let $c \leq \mathsf{fsep}(C_g, a, b)$ and L be a c-separating a-list assignment of G such that $|L(x_1)| = b$ (i.e., x_1 is the precolored vertex) and suppose without loss of generality that $x_1 \in B_1$. By Lemma 12, it is sufficient to prove that each block is (a, b, c)-free-choosable. By Lemma 13, if $g \geq 4$, then each block consisting of a cycle C of length at least g can be colored if $c \leq \mathsf{fsep}(C_g, a, b)$. Also, trivially, any edge is (a, b, c)-choosable if $c \leq a - b$, hence $\mathsf{fsep}(K_2, a, b) \geq a - b \geq \mathsf{fsep}(C_g, a, b)$. Therefore, there exists an (L, b)-coloring of G and thus it is (a, b, c)-free-choosable. Moreover, since G contains a cycle of length g, then $\mathsf{fsep}(G, a, b) \leq \mathsf{fsep}(C_g, a, b)$.

Second, if G contains a triangle and cycles of length greater than three, let ℓ be the length of the shortest cycle of length at least 4. Then, by Lemma 13, $\mathsf{fsep}(C_3, a, b) > \mathsf{fsep}(C_\ell, a, b)$ if and only if $\frac{2\ell+1}{\ell+1}b < a < \frac{2\ell+2}{\ell}b$. We then proceed as for the first part of the proof, but with $c \leq \min(\mathsf{fsep}(C_g, a, b), \mathsf{fsep}(C_\ell, a, b))$. ■

As the free-separation number is a lower bound of the separation number, Theorem 14 provides a lower bound on the separation number of cactuses. Moreover, the following result shows that this lower bound is tight in some sense.

Proposition 15. *For any $a \geq b \geq 1$, and any $p \geq 3$, there exists a cactus G of girth p such that* $\mathsf{sep}(G, a, b) = \mathsf{fsep}(G, a, b)$.

Proof. Let $c = \mathsf{fsep}(C_p, a, b)$, and let $k = \binom{a}{b}$. Let G be the graph obtained by joining k copies $C^1, C^2, \ldots, C^k$ of the cycle C_p of length p at a shared universal vertex x_1 (see Figure 1 for an illustration in the case $a = 2$ and $b = 1$). Let $\mathcal{B} = \{B_1, \ldots, B_k\}$ be the family of b-element subsets in $\{1, \ldots, a\}$. In order to show that $\mathsf{sep}(G, a, b) = c$, we construct a $(c+1)$-separating a-list assignment L of G for which no (L, b)-coloring exists as follows. For each $i, 1 \leq i \leq k$, let L^i be a $(c+1)$-separating a-list assignment of C^i for which $L(x_1) = B_i$ and other lists of colors only use colors from B_i and from a set of colors A with $A \cap \{1, \ldots, a\} = \emptyset$ and such that C^i is not (L^i, b)-colorable. Since $\mathsf{fsep}(C_p, a, b) = c$, such an assignment exists. Let now L be the list assignment of G defined by

$$L(y) = \begin{cases} \{1, \ldots, a\}, & \text{if } y = x_1, \\ L^i(y), & \text{if } y \neq x_1 \text{ and } y \in C^i. \end{cases}$$

Then, by construction, L is a $(c+1)$-separating a-list assignment of G. Moreover, whatever the choice of the set of b colors for the vertex x_1, there will be a cycle C^i on which the (L, b)-coloring cannot be completed. Therefore, $\mathsf{sep}(G, a, b) \leq c$ and since $\mathsf{sep}(G, a, b) \geq \mathsf{fsep}(G, a, b)$, we have that $\mathsf{sep}(G, a, b) = \mathsf{fsep}(G, a, b)$. ■

Theorem 16. *Let G be an outerplanar graph with finite girth $g \geq 5$ and let a, b be integers, $a \geq b \geq 1$. Then we have*

$$\mathsf{fsep}(C_{g-1}, a, b) \leq \mathsf{fsep}(G, a, b) \leq \mathsf{fsep}(C_g, a, b).$$

Proof. First, we are going to prove that G is (a, b, c)-free-choosable for $c = \mathsf{fsep}(C_{g-1}, a, b)$. By Lemma 12, we may suppose that G is 2-connected. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the inner faces of G and let r be any vertex of any face, say α_1. Let L be a c-separating a-list assignment of G such that $|L(r)| = b$. We define an (L, b)-coloring of G by coloring the vertices of the faces, following a BFS order on the tree $\mathcal{T}_G$, starting with the face α_1. Since $g \geq 5$, by Lemma 13, we have $c = \mathsf{fsep}(C_{g-1}, a, b) \leq \mathsf{fsep}(\alpha_1, a, b)$, thus there exists an (L, b)-coloring of α_1. At each step, when coloring the vertices of a face $\alpha_i = (x_1, x_2, \ldots, x_\ell)$, this face shares an edge with at most one face α_j with already colored vertices. Assume, without loss of generalities, that $x_1 x_\ell \in \alpha_i \cap \alpha_j$. Then we have a path $P = (x_1, x_2, \ldots, x_\ell)$ of length ℓ with precolored endvertices and since $\ell \geq g-1$, we have $c \geq \mathsf{fsep}(C_{\ell,a,b}) = \lfloor c(\ell, a, b) \rfloor$. Therefore, by Lemma 8, there exists an (L, b)-coloring of P. By iterating the process on each face, we obtain an (L, b)-coloring of the whole graph G, hence proving that $\mathsf{fsep}(G, a, b) \geq c$.

Second, since G contains a cycle of length g, $\mathsf{fsep}(G, a, b) \leq \mathsf{fsep}(C_g, a, b)$. ■

Remark that lower bounds for the free-separation number of an outerplanar graph with girth four can be derived using a proof similar with the one for $g \geq 5$ but the formula will be more complex. For outerplanar graphs of girth 3, we cannot use Lemma 8 anymore since it needs the path P being of length at least 3.

5. CONCLUDING REMARKS

We have determined the separation and free-separation number of the cycle and the free separation number of cactuses, and only bounds for the free-separation-number of outerplanar graphs G of girth $g \geq 5$. For some values of g, a, b the lower and upper bounds of Theorem 16 are equal, but for some not. For instance, we have $\mathsf{fsep}(C_4, 9, 4) = 3$ and $\mathsf{fsep}(C_5, 9, 4) = 4$. We conjecture that for any $a \geq b \geq 1$ and any $g \geq 5$, there exists an outerplanar graph G of girth g such that $\mathsf{sep}(G, a, b) = \mathsf{fsep}(G, a, b) = \mathsf{fsep}(C_{g-1}, a, b)$.

The problem seems also hard for other simple graphs such as the complete graph K_n. In [13], the assymptotic on the minimum a such that K_n is $(a, 1, 1)$-choosable is given. For $n = 3$, $\mathsf{sep}(K_3, a, b)$ and $\mathsf{fsep}(K_3, a, b)$ are given in Theorem 5 and Proposition 11, respectively. For $n = 4$ we are able to determine both numbers for any values of a and b, but there are many cases in the formulae. For $n = 5$ even many more cases have to be considered. We conjecture that $f_n(a/b) = \mathsf{fsep}(K_n, a, b)$ is a piecewise linear function with the number of pieces growing exponentially with n.

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