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AN UPPER BOUND ON THE CHROMATIC NUMBER OF 2-PLANAR GRAPHS

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Abstract

It is proved that any 2-planar graph (i.e., a graph which can be drawn on a plane such that any edge intersects at most two others) has a proper vertex coloring with 9 colors.

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1. Introduction

We consider graphs without loops and use the standard notation. For a graph G, we denote the set of its vertices by $V(G)$ and the set of its edges by $E(G)$. The number of vertices and edges of G we denote by $v(G)$ and $e(G)$, respectively.

For two graphs G and H, denote by $G \cup H$ the graph with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$.

We denote the *degree* of a vertex x in a graph G by $d_G(x)$.

For $R \subset V(G) \cup E(G)$, we denote by $G - R$ the graph obtained from G by deleting all vertices and edges of R and all edges incident to vertices of R.

Recall that a graph is *planar* if it can be drawn on the plane such that its edges do not intersect each other in inner points. The following definition generalizes this notion.

Definition. A graph is called *k-planar* if it can be drawn on the plane such that each edge intersects at most k others.

Recall that a vertex coloring of a graph is *proper* if any two adjacent vertices have distinct colors. The *chromatic number* of a graph G (denoted by $\chi(G)$) is the least number of colors in its proper vertex coloring. In what follows, we call a proper vertex coloring with k colors simply by *k-coloring*.

It is well known that every planar graph has a 4-coloring. Some bounds on the chromatic number of 1-planar graphs are known, their proofs are much simpler than the proof of the Four Color Theorem. In 1965, Ringel [1] proved that the chromatic number of a 1-planar graph does not exceed 7 and conjectured that the upper bound 6 also holds. Ringel's conjecture was proved in 1984 by Borodin [3]. This bound is, clearly, tight: the complete graph K_6 is 1-planar.

Concerning the chromatic number of 2-planar graphs, in 1997, Pach and Toth proved [2] that, for $k \le 4$ and any k-planar graph G, the bound $e(G) \le (k+3)(v(G)-2)$ holds. Hence, $e(G) < 5v(G)$ for a 2-planar graph G and, therefore, G has a 10-coloring.

The main result of our paper will strengthen this trivial bound.

Theorem 1. *Let G be a 2-planar graph. Then $\chi(G) \le 9$.*

We are not sure that this bound is tight. However, it is not trivial: one can construct an infinite series of 2-planar graphs with minimum degree 9.

2. Plane Drawings of 2-Planar Graphs

Speaking about plane drawings of graphs, we always mean that vertices are drawn as points and edges are drawn as polylines. The drawing of an edge contains only two vertices, namely, the ends of this edge. We mean that any two of these polylines have no common segment, i.e., have finite number of cross points. For each *cross point*, we mean that exactly two edges intersect each other in this point (if more than two edges pass through a cross point then one can easily change the drawing to avoid this problem). We put one more condition on plane drawings: if A is a cross point of edges e and f then each sufficiently small circle ω with the center A intersects both these edges twice and points of intersection with e and f alternate on ω. (If this condition does not hold then one can easily change the drawing such that the crossing of edges at A disappears, see Figure 1.)

2.1. Plane graphs

A *plane graph* is a crossing-free drawing of a planar graph on the plane. A plane graph G divides the plane into several parts called *faces*. We will denote by $F(G)$ the set of all faces of G and by $f(G)$ the number of faces of G.

Consider an edge e of a plane graph G. There are two possibilities: either e separates two distinct faces (then e is a *boundary* edge of these two faces), or the face on both sides of e is the same (then e is an *inner edge* of this face). *Boundary vertices* of a face a are all vertices incident to boundary or inner edges of a.

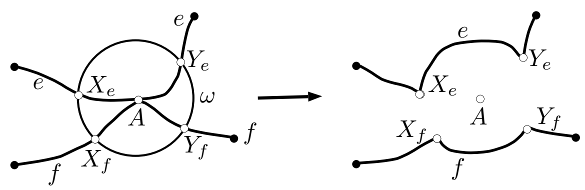

Figure 1. The neighborhood of a cross point.

Remark 2. Since each cycle of a plane graph G divides the plane into two regions and each face lies in one of these regions, an inner edge of a face is a *bridge* of G, i.e., does not belong to any cycle.

Definition. Let G be a plane graph and let $a \in F(G)$.

(1) The *boundary* of a face a is the subgraph $B(a)$ of G, induced on the set of all boundary and inner edges of a (the vertex set of $B(a)$ consists of boundary vertices of the face a).

(2) The *size* $b(a)$ of the boundary of a is the sum of the number of boundary edges of a and the double number of inner edges of a.

If $b(a) = k$ then we call a a *k-face.* The number of k-faces of the plane graph G will be denoted by $f^k(G)$.

(3) For a vertex $x \in V(G)$, denote by $d_a(x)$ the number of boundary edges of a incident to x plus the double number of inner edges of a incident to x.

Remark 3. Let G be a plane graph.

(1) The sum of sizes of boundaries of all faces of the graph G is equal to $2e(G)$.

(2) Let $x \in V(G)$ and $a \in F(G)$. Then $d_a(x)$ is even. Moreover, $d_a(x) \neq 0$ if and only if x is a boundary vertex of the face a.

(3) If the boundary of a face a is a simple cycle then $d_a(x) = 2$ for each boundary vertex x of a.

(4) Clearly, $\sum_{a \in F(G)} d_a(x) = 2d_G(x)$ for every vertex $x \in V(G)$.

Definition. Let G be a plane graph and $f \in F(G)$. A *diagonal* of f is a new edge drawn inside a and joining two vertices of f which are not adjacent in $B(f)$.

2.2. Drawings of 2-planar graphs and 2-diagonal graphs

All *drawings* of 2-planar graphs on the plane which are considered in this paper are such that each edge intersects at most two others. Mostly, we will consider drawings of 2-planar graphs with minimal number of cross-points. In [4], Theorem 1, it is proved that any two intersecting edges in a drawing with minimal number of cross-points have exactly one cross-point and have no common end. We will assume that each plane drawing of a 2-planar graph satisfy this condition. For convenience, we will allow multiple edges in 2-planar graphs.

In general, our proof of Theorem 1 will follow the way of Borodin's proof for 1-planar graphs [3]. To estimate the chromatic number of 1-planar graphs, Borodin considered certain specific 4-proper vertex colorings of plane graphs (such colorings that all vertices of each 4-face must have distinct colors). It was proved that any plane graph has such a coloring with 6 colors. The question of constructing a proper 6-coloring of a 1-planar graph was reduced to constructing a 4-proper 6-coloring of a plane graph.

We need a special type of 2-planar graphs defined in [4]. Let us repeat definitions and some results of this paper.

Let $\mathcal{H}$ be a plane drawing of a 2-planar graph H.

Definition. An edge of $\mathcal{H}$ is *simple* if it does not intersect other edges and *non-simple* otherwise. Denote by $E'(\mathcal{H})$ the set of all simple edges of the drawing $\mathcal{H}$. The *plane graph* $P(\mathcal{H})$ of the drawing $\mathcal{H}$ is the graph with the vertex set $V(H)$ and the edge set $E'(\mathcal{H})$.

Clearly, $P(\mathcal{H})$ is a plane graph, we always will consider its plane drawing obtained from $\mathcal{H}$ by deleting all non-simple edges.

Definition. (1) In a plane drawing of a graph, for any vertex x, one can order drawings of edges incident to x clockwise. Two edges are called *neighboring at* x if they are incident to x and neighboring in this order.

If, among any three consecutive edges incident to x (in the clockwise order defined above), there is at least one simple edge then we say that the drawing is 2-*diagonal at* x.

(2) A graph G' is 2-*diagonal* if it has a plane drawing such that any edge intersects at most two others, any two crossing edges have at most one cross-point (in particular, two crossing edges have no common end) and the drawing is 2-diagonal at each vertex.

(3) We will always identify a 2-diagonal graph G' with its plane drawing $\mathcal{G}'$ which satisfies the conditions from the above definition. We will call $G = P(\mathcal{G}')$ the *plane graph* of the 2-diagonal graph G'. We will call edges of $E(G)$ *simple edges* of G'.

(4) In a 2-diagonal graph, multiple edges are allowed only in the case where all of them are simple in its plane drawing.

The following lemma is a simplified version of Theorem 3 from [4].

Lemma 4 [4, Theorem 3]**.** *Any 2-planar graph G' without multiple edges has a 2-diagonal supergraph H' (maybe, with multiple simple edges) such that $V(H') = V(G')$.*

Corollary 5. *Let $k, n \in \mathbb{N}$. Assume that all 2-diagonal graphs on n vertices have k-colorings. Then all 2-planar graphs on n vertices have k-colorings.*

Proof. Consider a 2-planar graph H^* with $v(H^*) = n$. Until it is possible, we will delete an edge from a pair of multiple edges. As a result, we will obtain a 2-planar spanning subgraph H' of H^* such that H' has no multiple edges and two vertices are adjacent in H' if and only if they are adjacent in H^*. By Lemma 4, H' has a 2-diagonal supergraph G' with $V(G') = V(H') = V(H^*) = n$. By the condition, G' has a proper k-coloring, which is also a proper k-coloring of H^*. ∎

Thus, it is enough to prove Theorem 1 only for 2-diagonal graphs.

Cross-points and ends divide edges into several parts which are also polylines. If A and B are two points on an edge e (possibly being its ends) then we denote by AeB the part of the edge e between A and B.

Definition. A part of a non-simple edge from its end to the nearest cross point is a *boundary part.*

Remark 6. Any non-simple edge has two boundary parts. If A is a cross point of edges e and f then at least one of two parts into which A divides e is a boundary part (since e contains at most two cross points with other edges).

Definition. Let $\mathcal{G}'$ be a plane drawing of a 2-planar graph G' and let L be a polyline with the ends $x, y \in V(G')$ which consists of parts of edges of G' and intersects no other edges in $\mathcal{G}'$. To *draw an edge f along L* means to draw a new edge f with the ends x, y in $\mathcal{G}'$ such that the region inside the closed polyline formed by f and L contains no vertices of G' and no parts of edges of the drawing $\mathcal{G}'$.

3. A Minimal Counterexample

Let $k \in \{8, 9\}$. In what follows, we consider all 2-diagonal graphs having no proper k-coloring with the minimal number of vertices. First, we choose among them all graphs having a drawing such that its plane graph has minimal number of edges. After that, among all graphs chosen on the first step, we choose a graph G' with the minimal number of edges.

Thus, our 2-diagonal graph G', its drawing $\mathcal{G}'$ and the plane graph $G = P(\mathcal{G}')$ are such that $e(G)$ is minimal and $e(G')$ is minimal among all 2-diagonal graphs having a drawing which plane graph has $e(G)$ edges. By Corollary 5, each 2-planar graph H' with $v(H') < v(G')$ has a proper k-coloring. Let us study properties of G' and G.

Claim 7. *The graph G' has no pair of multiple edges.*

Proof. Let G' have multiple edges e_1 and e_2 with the ends x, y. By the definition of a 2-diagonal graph, e_1 and e_2 are simple. These edges form a closed curve C which divides the plane into two regions O_1 and O_2. Let V_i be the set of vertices lying in O_i.

Assume that $V_1 = \emptyset$. Then, in O_1, no part of a non-simple edge incident to x or y is drawn. Hence, $G' - e_2$ is a 2-diagonal graph with drawing obtained from $\mathcal{G}'$ by deleting the edge e_2 and its plane graph is $G - e_2$. Clearly, $\chi(G' - e_2) = \chi(G')$. Since $e(G - e_2) < e(G)$, we have a contradiction with the choice of G'.

Then $V_1, V_2 \neq \emptyset$. Since C intersects no edge of G', the set $\{x, y\}$ separates V_1 from V_2 in G'. Let $G'_1 = G' - V_2$ and $G'_2 = G' - V_1$. It is easy to see that both G'_1 and G'_2 are 2-planar graphs, $G' = G'_1 \cup G'_2$ and $V(G'_1) \cap V(G'_2) = \{x, y\}$. Since 2-planar graphs G'_1 and G'_2 have less than $v(G')$ vertices, they have proper k-colorings, and the vertices x and y have distinct colors in these two colorings. Therefore, we may assume that the proper k-coloring of G'_1 agree with the proper k-coloring of G'_2 on vertices x and y. As a result, we obtain a proper k-coloring of G', a contradiction. ∎

3.1. The type and the contribution of a vertex

Set the following notation:

$$f_G^s(v) = \frac{1}{2} \sum_{f \in F(G),\, b(f)=s} d_f(v), \qquad f_G^{\geq s}(v) = \sum_{i=s}^{\infty} f^i(v),$$

$$s_G(v) = d_G(v) + f_G^4(v) + 2f_G^{\geq 5}(v), \quad V^k = \{v \in V(G) \ : \ s_G(v) = k\}.$$

Definition. (1) Vertices of V^k will be called *k-vertices.*

(2) The *type* of a vertex v is the ordered triple $(d_G(v), f_G^3(v), f_G^4(v))$.

Most commonly, we will use the above notation for the graph G. In this case, we will omit indexes and write simply $d(v)$, $s(v)$, $f^i(v)$ and so on. We will use the notation $d'(v)$ for $d_{G'}(v)$.

Remark 8. (1) Clearly, $\sum_{i=1}^{\infty} f^i(v) = d(v)$ for any vertex $v \in V(G)$.

(2) Since $\chi(G') \geq 9$, it is clear that $v(G) \geq 9$. Since G has no loops, $f^1(v) = 0$ for all $v \in V(G)$. By Claim 7, G has no multiple edges, and, therefore, $f^2(v) = 0$ for all $v \in V(G)$.

Claim 9. (1) *G has no isolated vertices.*

(2) *The boundary of any* 3*-face of* G *is a triangle.*

(3) *The boundary of any* 4*-face of* G *is a simple* 4*-cycle.*

Proof. (1) Let $d_G(v) = 0$. Then v is incident in G' to no simple edges and, by the definition of a 2-diagonal graph, v is incident in G' to at most two non-simple edges, i.e., $d_{G'}(v) \leq 2$. Since, as we know, $G' - v$ has a proper k-coloring, the graph G' also has. We obtain a contradiction.

(2) Let a be a 3-face which boundary is not a triangle. Assume that a has s boundary edges and t inner edges. Then $3 = s + 2t$, whence it follows $s = 1$, i.e., G has a loop, a contradiction.

(3) Let a be a 4-face which boundary is not a simple 4-cycle. Assume that a has s boundary edges and t inner edges. Then $4 = s + 2t$, whence it follows that $s = 2$ or $s = 0$. If $s = 2$ then G has a 2-cycle, i.e., two multiple edges. This contradicts Claim 7. Let $s = 0$. Then the face a is the whole plane. Since G has no isolated vertices, G consists of two inner edges of the face a and their ends. Hence, $v(G) \leq 4$, a contradiction. ∎

Remark 10. Let $v \in V(G)$ is a vertex of type (d, f_3, f_4). Then, by Claim 9, v belongs to exactly f_3 different 3-faces and to exactly f_4 different 4-faces. Therefore, $d - f_3 - f_4 = f^{\geq 5}(v)$.

Definition. A face of size at least 5 will be called *big*.

Claim 11. *For any vertex* $v \in V(G)$*, the following statements hold:*

(1) $s(v) \geq d'(v)$;

(2) $d(v) \geq \frac{s(v)}{3}$.

Proof. (1) In G', the vertex $v \in V(G')$ is incident to some edges of the graph G and some diagonals of faces of G. By Claim 7, G' has no multiple edges. Hence, all these diagonals are drawn in faces of size at least 4, and in each of $f^4(v)$ faces of size 4 containing v, at most one diagonal incident to v is drawn.

Let us prove that, in each big face a, at most $d_a(v)$ diagonals incident to v are drawn (this will give us at most $2f^{\geq 5}(v)$ diagonals incident to v in big faces). Indeed, let a be a big face and let v be its boundary vertex. Denote by $P(v)$ and $N(v)$ the sets of all simple and non-simple edges incident to v respectively. Consider all edges of G' incident to v in the clockwise order. By the definition of a 2-diagonal graph, for any edge of the set $N(v)$, at least one of the neighboring edges (on the left or on the right) belongs to $P(v)$. Thus, we can assign to every edge $f' \in N(v)$ a neighboring edge $f \in P(v)$. Clearly, if f' is a diagonal of the face a then f is a boundary or inner edge of a. Any boundary edge f of the face a can be assigned to at most one non-simple edge of $N(v)$, since the face a is disposed only on one side of the edge f. Any inner edge of the face a can be assigned to at most two non-simple edges of $N(v)$. Hence, at most $d_a(v)$ edges of $N(v)$ are diagonals of the face a.

(2) Let $d(v) = d$. It is enough to prove that $s(v) \le 3d$. By the definition, $s(v) \le d + \sum_{f\in F(G)} d_f(v) = d + 2d = 3d$. ∎

Claim 12. *Any vertex $v \in V(G)$ is adjacent in G' to at least k distinct vertices. In particular, $s(v) \ge d'(v) \ge k$ and $d(v) \ge 3$.*

Proof. Assume the converse; let v be adjacent in G' to at most $k-1$ vertices. Clearly, $G' - v$ is a 2-planar graph and $v(G'-v) < v(G)$. By the choice of G' and Corollary 5, the graph $G' - v$ has a k-coloring. Since v has at most $k-1$ neighbors, we can color this vertex and obtain a k-coloring of G', a contradiction. Therefore, v is adjacent in G' to at least k vertices and $d'(v) \ge k$. By Claim 11, $s(v) \ge d'(v)$ and $d(v) \ge \frac{s(v)}{3} \ge \frac{k}{3} > 2$. ∎

Definition. We define two variants of the *contribution* of a vertex $v \in V(G)$:

$$\mu(v) = d(v) - 10 + \sum_{i=3}^{\infty} \frac{4i-10}{i} f^i(v) \quad \text{and}$$
$$\nu(v) = d(v) - 10 + \frac{2}{3} f^3(v) + \frac{3}{2} f^4(v) + 2f^{\ge 5}(v).$$

Remark 13. Clearly, for any vertex $v \in V(G)$,

$$\mu(v) = \nu(v) + \sum_{i=6}^{\infty} \frac{2i-10}{i} f^i(v) \ge \nu(v) \quad \text{and}$$
$$\nu(v) = s(v) - 10 + \frac{2}{3} f^3(v) + \frac{1}{2} f^4(v).$$

Claim 14. $\sum_{v\in V(G)} \mu(v) < 0$.

Proof. Let $e(G) = e$, $v(G) = v$, $f^i(G) = f^i$. Clearly,

$$\sum_{i=3}^{\infty} 4i \cdot f^i = 8e, \qquad \sum_{v\in V(G)} f^i(v) = if^i \quad \text{and} \quad \sum_{v\in V(G)} d(v) = 2e.$$

We obtain the following chain of calculations (the last equality follows from Euler's formula for the plane graph G):

$$\sum_{v\in V(G)} \mu(v) = \sum_{v\in V(G)} \left(d(v) - 10 + \sum_{i=3}^{\infty} \frac{4i-10}{i} f^i(v) \right)$$
$$= 2e - 10v + \sum_{i=3}^{\infty} (4i-10) f^i = 2e - 10v + 8e - 10f = -20.$$
∎

3.2. Short cycles in the minimal counterexample

Lemma 15. *Let a cycle S in G' and $ab \in E(S)$ be such that $v(S) \le 4$ and $E(S) \setminus \{ab\} \subset E(G)$. Let S divide the plane into two regions O_1 and O_2. Denote by V_i the set of vertices of G' lying strictly inside O_i. Assume that, for every vertex $c \in V(S) \setminus \{a, b\}$, the following two conditions hold:*

(1) *none of edges incident to c intersects ab;*

(2) *there exists $i \in \{1, 2\}$ such that c is incident in G' to at most two vertices of V_i.*

Then $V_1 = \emptyset$ or $V_2 = \emptyset$.

Proof. Assume that $V_1, V_2 \ne \emptyset$. Let E' be the set of all edges of G' intersecting ab. Then $|E'| \le 2$. Let $G_i = G - V_{3-i}$, $G'_i = G' - V_{3-i}$ and $\mathcal{G}'_i$ be the plane drawing of G'_i obtained from $\mathcal{G}'$ by deleting all surplus vertices and edges. Clearly, both G'_1 and G'_2 are 2-planar graphs.

If $E' = \emptyset$ then ab is a simple edge which separates in G two faces: q_1 lying in O_1 and q_2 lying in O_2. If $E' \ne \emptyset$ then non-simple edge ab and all edges of E' are diagonals of a certain face q of G. In this case, the diagonal ab splits q into a face q_1 of the graph G_1 and a face q_2 of the graph G_2. One more new face appears in G_1 — the face f_1 with the boundary S (recall that S is a cycle of length 2, 3 or 4). All other faces of G_1 are faces of G, we will call them *old* faces. Let us draw in G'_1 all possible diagonals of the face f_1 and denote the graph obtained by G^*_1. Clearly, G^*_1 is 2-planar. Similarly, one can define the new face f_2 of the graph G_2, its *old* faces and the 2-planar graph G^*_2.

Since $v(G^*_1) < v(G')$ and $v(G^*_2) < v(G')$, both graphs G^*_1 and G^*_2 have k-colorings. Vertices of the face f_1 have different colors in a k-coloring of G^*_1 and vertices of the face f_2 have different colors in a k-coloring of G^*_2 (these vertices are pairwise adjacent). Hence, any two k-colorings of the graphs G^*_1 and G^*_2 can be agreed on the set $V(S)$.

Only edges of E' join vertices of V_1 to vertices of V_2. If $E' = \emptyset$ then k-colorings of G^*_1 and G^*_2 can be glued into a k-coloring of the graph G', which does not exist, a contradiction. In we assume that $E' \ne \emptyset$. In this case, we will add some edges to G^*_1 and G^*_2. After that, for the graphs obtained, we will construct k-colorings agreed on the set $V(S)$ in which ends of each edge of E' will have distinct colors. This will prove that G' has a proper k-coloring and lead to a contradiction.

For every vertex $c \in V(S) \setminus \{a, b\}$, we choose $i \in \{1, 2\}$ such that c is adjacent to at most two vertices of V_i. Let S_1 consist of all c for which $i = 1$ and S_2 consist of all c for which $i = 2$. By condition (2), $V(S) = \{a, b\} \cup S_1 \cup S_2$.

We will consider several cases and, in each of them, construct auxiliary 2-planar supergraphs H'_1 and H'_2 from G^*_1 and G^*_2.

Case 1. $|E'| = 1$. Let $E' = \{e'_1\}$ and let T_1 be the cross point of $e = ab$ and $e'_1 = x_1y_1$ where $x_1 \in V_1$ and $y_1 \in V_2$ (this assumption is correct due to condition (1)). Then both aeT_1 and beT_1 are boundary parts and (by Remark 6) at least one of the parts into which T_1 divides e'_1 (say, $x_1e'_1T_1$) is a boundary part, see Figure 2a. Let us draw simple edges x_1a and x_1b along the polylines $x_1e'_1T_1ea$ and $x_1e'_1T_1eb$, respectively (see Figure 2b). Denote the graph obtained by H'_1. Clearly, H'_1 is 2-planar. Let $H'_2 = G^*_2$.

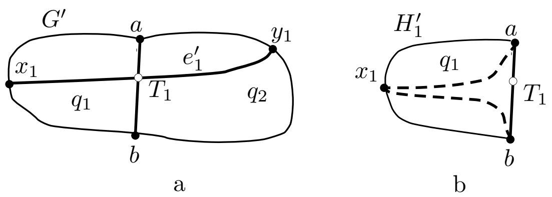

Figure 2. $|E'| = 1$.

Case 2. $|E'| = 2$. Let $E' = \{e'_1, e'_2\}$ and these edges intersect $e = ab$ at the points T_1 and T_2, respectively (say T_1 is closer to a, see Figure 3a). Let $e'_1 = x_1y_1$ and $e'_2 = x_2y_2$ where $x_1, x_2 \in V_1$ and $y_1, y_2 \in V_2$. The parts aeT_1 and beT_2 are boundary. At least one of the parts into which T_1 divides e'_1 (say, $x_1e'_1T_1$) is a boundary part, and at least one of the part into which T_2 divides e'_2 is a is boundary part. Consider two cases.

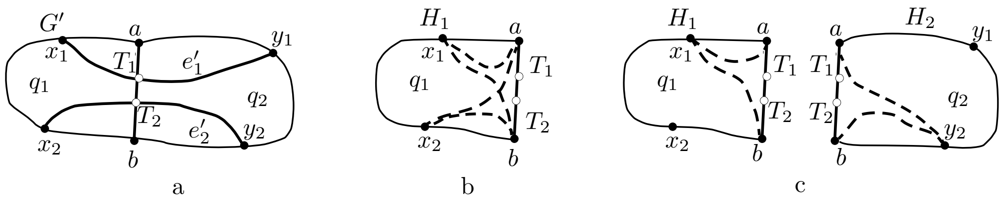

Figure 3. $|E'| = 2$.

Case 2a. *The part $x_2e'_2T_2$ is boundary.* Similarly to Case 1, we draw simple edges x_1a (along $x_1e'_1T_1ea$) and x_2b (along $x_2e'_2T_2eb$). If $x_1 = x_2$ then the construction of H'_1 is finished. If $x_1 \neq x_2$ then we draw in the face q_1 edges x_1b and x_2a along $x_1e'_1T_1eb$ and $x_2e'_2T_2ea$, respectively (see Figure 3b). These two new edges will intersect only each other. In both cases, the graph H'_1 constructed above is, clearly, 2-planar. Let $H'_2 = G^*_2$.

Case 2b. *The part $y_2e'_2T_2$ is boundary.* First, let us add edges to G^*_1: draw edges x_1a along $x_1e'_1T_1ea$ and x_1b along $x_1e'_1T_1eb$ (see Figure 3c). The edge x_1a is, clearly, simple. Since $x_1e'_1T_1eb$ intersects in G' only the edge e'_2 deleted in G^*_1, the edge x_1b is also simple. Clearly, the obtained graph H_1' is 2-planar. Similarly, we add to G_2^* simple edges y_2b along $y_2e_2'T_2eb$ and y_2a along $y_2e_2'T_2ea$ and obtain a 2-planar graph H_2'.

In all cases (1, 2a, 2b), we will similarly construct k-colorings ρ_1 and ρ_2 of H_1' and H_2', respectively. We need to match these colorings on S, but there is one more difficulty — ends of each edge of E' must have distinct colors. To provide this, we will construct colorings ρ_1 and ρ_2, and, at the same time, the correspondence of colors in ρ_1 and ρ_2.

Let us start with ρ_1. First, we will construct a k-coloring ρ_1 of the graph $H_1'-S_1$ (since $v(H_1'-S_1) < v(G')$, this coloring exists). After that, we successively consider vertices of the set S_1. Let $c \in S_1$. During the construction of H_1' no edges incident to c were added. Hence, $d_{H_1'}(c) = d_{G_1^*}(c) \le 5$ (the vertex c can be adjacent in G_1^* to at most two vertices of V_1 and to other $|S|-1 \le 3$ vertices of S). Since $k \ge 8$, we can choose the color $\rho_1(c)$ different from colors of all vertices of $\mathrm{N}_{G'}(c)$, $\rho_1(x_1)$ and (if $|E'| = 2$) from $\rho_1(x_2)$. Similarly, we construct a k-coloring ρ_2 of the graph H_2' (colors of vertices of the set S_2 will be different from colors of the vertices y_1 and y_2).

Now we will describe gluing the colorings ρ_1 and ρ_2 into a k-coloring of the graph G'. Both colorings color vertices of S in different colors. Thus, we can make this colorings agreed on $V(S)$. Denote by C_S the set of colors of $V(S)$, and by C' the set of all other colors. The colors of C_S will be fixed in both colorings, all other colors can be renumbered. On this step, choosing a color for a certain vertex in ρ_1, we always will choose the same color for all vertices of its color class. Similarly for ρ_2. Note, that $|C'| \ge 4$.

Consider the vertex x_1. We want the color $\rho_1(x_1)$ to be different from $\rho_2(y_1)$ and $\rho_2(y_2)$ (if the vertex y_2 exists). First, consider the case $\rho_1(x_1) \in C_S$, say, $\rho_1(x_1) = \rho_1(c)$ for a certain vertex $c \in V(S)$. Since, in all cases, $x_1a, x_1b \in E(H_1')$, we have $c \notin \{a, b\}$. By construction of ρ_1, we know that $c \notin S_1$. Hence, $c \in S_2$ and, therefore, $\rho_2(c) \notin \{\rho_2(y_1), \rho_2(y_2)\}$ by construction of ρ_2. Let $\rho_1(x_1) \notin C_S$. In this case, we can choose the color $\rho_1(x_1) \in C'$ such that it will be different from $\rho_2(y_1)$ and $\rho_2(y_2)$: at most two colors from C' are forbidden for $\rho_1(x_1)$ — namely, $\rho_2(y_1)$ and $\rho_2(y_2)$.

Now ends of the edge x_1y_1 have distinct colors. In the case where $|E'| = 2$ and $x_1 = x_2$ the ends of the edge x_2y_2 have distinct colors. The only case remaining is where $|E'| = 2$ and $x_1 \ne x_2$. First, consider the Case 2a. Then $x_2a, x_2b \in E(H_1')$. If $\rho_1(x_2) \in C_S$ then, similarly to the above case, we obtain that $\rho_1(x_2) \ne \rho_2(y_2)$. If $\rho_1(x_2) \notin C_S$ then we can choose the color $\rho_1(x_1) \in C'$ such that it will be different from $\rho_2(y_2)$: at most two colors from C' are forbidden for $\rho_1(x_1)$ — namely, $\rho_2(y_2)$ and the color $\rho_1(x_1)$ chosen before (in the case where $\rho_1(x_1) \in C'$ and $\rho_1(x_1) \ne \rho_1(x_2)$). In the Case 2b, we have $y_2a, y_2b \in E(H_2')$ and will do the same with changing H_1' and ρ_1 by H_2' and ρ_2: we will choose the color $\rho_2(y_2)$

different from $\rho_1(x_2)$. ■

Claim 16. (1) *In the plane graph G, two 3-faces cannot have a common edge.*

(2) *In the plane graph G, a 3-face cannot have a common edge with a 4-face.*

(3) *For any vertex $v \in V(G)$, $f^{\geq 5}(v) \geq f^3(v)$. If $f^{\geq 5}(v) = f^3(v)$ then $d(v) = 2f^3(v)$.*

Proof. (1) Assume that two 3-faces have a common edge. Since, by Claim 7, G has no multiple edges, these two faces can have only one common edge, say, e (Figure 4a). In the graph $G - e$, our two 3-faces are substituted by a 4-face f (Figure 4b), all other faces are the same as in G. Let us add to the drawing $\mathcal{G}' - e$ two diagonals of f — clearly, we obtain a drawing $\mathcal{H}'$ of a 2-diagonal supergraph H' of G'. The graph H' also has no k-coloring and $P(\mathcal{H}') = G - e$ is smaller than G. We obtain a contradiction with the choice of G'.

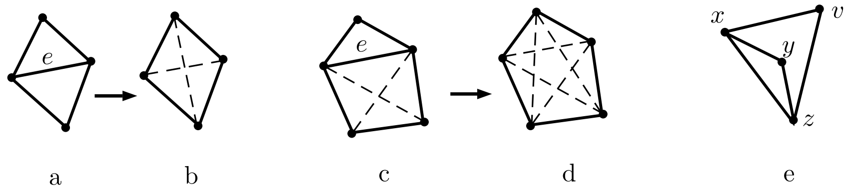

Figure 4. Deleting the common edge of 3- and 4-faces.

(2) Assume that a 3-face f and a 4-face f' have a common edge e. First, consider the case where e is the only common edge of these faces, see Figure 4c. In $G - e$, two faces f and f' are substituted by a 5-face f^* (Figure 4d), all other faces are the same as in G. Let us add to the drawing $\mathcal{G}' - e$ all diagonals of f^* — clearly, we obtain a drawing $\mathcal{H}'$ of a 2-diagonal supergraph H' of G'. The graph H' also has no k-coloring and $P(\mathcal{H}') = G - e$ is smaller than G. We obtain a contradiction with the choice of G'.

Clearly, the faces f and f' cannot have three common edges. Let they have two common edges. Then we may assume that the boundary of f is a triangle xyz and the boundary of f' is a 4-cycle $xyzv$, see Figure 4e. In this case, the triangle xzv divide the plane into two regions such that one of them contains exactly one vertex of G' — namely, y. Since edges of G' cannot intersect simple edges of the triangle xzv, the vertex y is adjacent in G' to at most three vertices, a contradiction with Claim 12.

(3) Consider all simple edges incident to v in the clockwise order: $e_1, \dots, e_n$ (the numeration is cyclic). Each pair of neighboring edges e_j, e_{j+1} belongs to a face containing v. If the pair e_i, e_{i+1} belongs to a 3-face then, by items (1) and (2), both pairs e_{i-1}, e_i and e_{i+1}, e_{i+2} belong to big faces. Conversely, each

pair e_i, e_{i+1} belonging to a big face is surrounded by two pairs, which can belong to 3-faces. Hence, $f^{\geq 5}(v) \geq f^3(v)$. Moreover, if $f^{\geq 5}(v) = f^3(v)$ then pairs of neighboring edges e_i, e_{i+1} forming a 3-face alternate with pairs belonging to big faces, therefore, $d(v) = 2f^3(v)$. ■

4. The Proof of Theorem 1

In what follows, we concentrate on the proof of our main Theorem, i.e., on the case $k = 9$. The graphs G' and G are the same as in previous section.

We will count the sum of contributions of vertices of G' in another way, and this sum will appear non-negative. This contradiction with Claim 14 will finish the proof of Theorem 1.

Definition. An edge is *big*, if on both sides of it big faces are disposed (maybe, it is the same big face).

Claim 17. (1) *Let* $v \in V(G)$ *be such that* $\mu(v) < 0$. *Then* $v \in V^9$, v *has the type* $(3, 0, 0)$ *and* $\mu(v) \geq -1$.

(2) *A vertex of type* $(3, 0, 0)$ *cannot be adjacent to a* 9*-vertex of another type.*

Proof. (1) By the definition and Remark 13, only 9-vertices can have negative contribution. Let us list all possible types of 9-vertices. In all cases, we will estimate $\nu(v)$ and take into account that $\mu(v) \geq \nu(v)$.

Let a 9-vertex v have type (d, f_3, f_4) and let $f_{\geq 5} = f^{\geq 5}(v)$. Then $d + f_4 + 2f_{\geq 5} = 9$. By Claim 12, we have $d \geq 3$. Consider several cases.

Case a. $d = 3$. Then $f_4 + 2f_{\geq 5} = 6$ and $f_3 + f_4 + f_{\geq 5} = 3$, whence it follows that $f_{\geq 5} = 3$. Thus, v has type $(3, 0, 0)$ and $\nu(v) = 9 - 10 = -1$ (see Figure 5a).

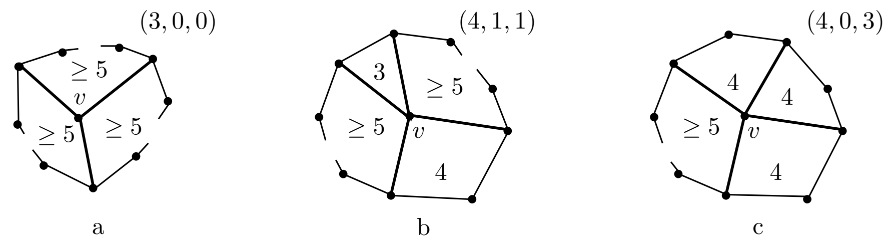

Figure 5. Types of 9-vertices.

Case b. $d = 4$. Then $f_4 + 2f_{\geq 5} = 5$ and $f_3 + f_4 + f_{\geq 5} = 4$. Therefore, $f_{\geq 5} \leq 2$.

First, consider the case $f_{\geq 5} = 2$. Then $f_3 = f_4 = 1$ and the vertex v has type $(4, 1, 1)$. Since the 3-face and the 4-face containing v cannot have a common edge, they are separated by big faces containing v (maybe, this is the same big face), and we obtain the configuration from Figure 5b. In this case, $\nu(v) = 9 - 10 + \frac{2}{3} + \frac{1}{2} = \frac{1}{6}$.

The case $f_{\geq 5} = 0$ is impossible: then $f_4 \leq d = 4$ and $s(v) = f_4 + d \leq 8$, a contradiction.

Consider the case $f_{\geq 5} = 1$. By Claim 16, $f_3 = 0$ (otherwise, $d(v) = 2$, a contradiction). Then $f_4 = 3$ and v has the type $(4, 0, 3)$, see Figure 5c. In this case, $\nu(v) = 9 - 10 + 3 \cdot \frac{1}{2} = \frac{1}{2}$.

In all possible cases, $\mu(v) \geq \nu(v) > 0$, a contradiction.

Case c. $d \geq 5$. By Claim 16, we have $f_3 \leq f_{\geq 5}$, whence it follows that $f_4 + 2f_{\geq 5} \geq f_3 + f_4 + f_{\geq 5} = d \geq 5$. This implies $s \geq 10$, a contradiction.

(2) It follows from the above-proved classification of 9-vertices that a 9-vertex of type $(3, 0, 0)$ is incident only to big edges and 9-vertices of other types are not incident to big edges. Hence, a vertex of type $(3, 0, 0)$ cannot be adjacent to other 9-vertices. ■

Definition. Let H' be a 2-planar graph with a drawing $\mathcal{H}'$ and let $a, b \in V(H')$ be non-adjacent vertices joined by a polyline L which does not intersect edges of H'. Then $H' \# ab$ is a graph obtained from H' by *merging* of vertices a and b, i.e., their joining into one vertex $a \# b$, which is incident to all vertices incident to a or b in H'. The graph $H' \# ab$ has a drawing $\mathcal{H}' \# ab$, in which the vertices a and b are merged into $a \# b$ along the polyline L. Multiple edges are admissible.

Remark 18. In the conditions of the above definition, the graph $H' \# ab$ is 2-planar (merging along the polyline L can be done such that no new intersection appears).

Clearly, for a 2-diagonal graph H', the graph $H' \# ab$ can be not 2-diagonal, but we do not need this.

Claim 19. *Let v be a vertex of type* $(3, 0, 0)$, $\mathrm{N}_G(v) = \{a_1, a_2, a_3\}$. *For each $i \in \{1, 2, 3\}$, let f_i be the face which boundary contains the part $a_{i-1}va_{i+1}$ (the numeration is cyclic modulo* 3, *see Figure* 6a, *some of these faces may coincide). Then, in each face f_i, the diagonal $a_{i-1}a_{i+1}$ exists.*

Proof. Let us prove that the diagonal a_2a_3 is drawn in the face f_1, the proof for two other diagonals is similar.

Assume that $e = a_2a_3 \in E(G')$. Consider the case where e is drawn outside $f_1 \cup f_2 \cup f_3$. Then e cannot intersect edges incident to v. No simple edges go from v in one of the regions into which the cycle $S = a_2a_3v$ divides the plane (namely, to the region D that contains f_4, see Figure 6a). Therefore, v is adjacent to at most two vertices inside this region (recall that, among any three successive edges incident to v, there must be at least one simple edge). Thus, we can apply Lemma

15 to the cycle S and obtain that one of the regions into which S divides the plane contains no vertices of G inside it. Clearly, this region is D and this is possible only if the boundary of f_1 is the triangle a_2a_3v (if a_2 and a_3 are non-neighboring in $B(f_1)$ then the vertex between them which is different from v is separated by S from a_1, but this contradicts the above-proved, see Figure 6a). Therefore, f_1 is not a big face and v cannot be a vertex of type $(3,0,0)$, a contradiction. Thus, e is drawn inside $f_1 \cup f_2 \cup f_3$. Since e cannot intersect simple edges, e is a diagonal of the face f_1 and the desired statement is proved.

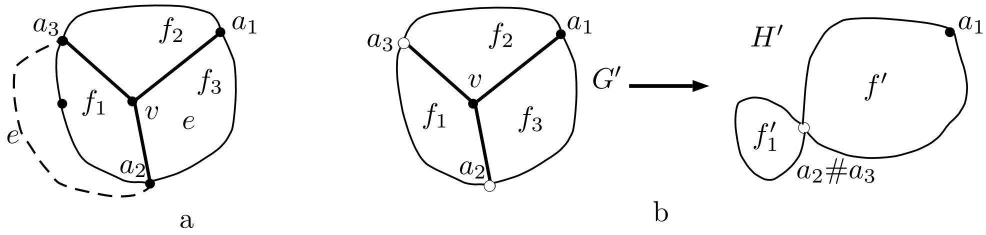

Figure 6. The diagonal a_2a_3.

Now assume that $a_2a_3 \notin E(G')$. In this case, we consider the graph $H' = (G'-v)\,\#\,a_2a_3$ (vertices a_2 and a_3 can be merged along the polyline a_2va_3, see Figure 6b). Since H' is a 2-planar graph and $v(H') < v(G')$, by Corollary 5, H' has a 9-coloring ρ. We will color both a_2 and a_3 with color $\rho(a_2\,\#\,a_3)$ and obtain a 9-coloring ρ' of the graph $G'-v$ in which at most 8 colors are forbidden for v (since $v \in V^9$ and two vertices a_2, a_3 adjacent to v have the same color). Then we can color v and obtain a 9-coloring of G', a contradiction. ∎

Claim 20. *Two 9-vertices of type* $(3,0,0)$ *cannot be adjacent in* G.

Proof. Assume that $w, v \in V(G)$ are two adjacent vertices of type $(3,0,0)$. Denote vertices of their neighborhoods and faces as it is shown on Figure 7a. By Claim 19, diagonals cv and bw are drawn in the face f_2, and diagonals dv and aw are drawn in the face f_4. Let $e = ac \in E(G')$. Clearly, e is drawn outside $f_1 \cup f_2 \cup f_3 \cup f_4$ (otherwise, e would intersect at least one simple edge, but this is impossible). Hence, edges incident to v or w cannot intersect e. Consider the cycle $S = avwc$. No simple edges go from v in one of the regions into which the cycle S divides the plane (namely, to the region that contains f_1, see Figure 7a). Therefore, v is adjacent to at most two vertices inside this region (recall that, among any three successive edges incident to v, there must be at least one simple edge). The similar argument holds for w. Thus, we can apply Lemma 15 to the cycle S and obtain that one of the regions into which S divides the plane contains no vertices of G inside it. Since S separates b from d (see Figure 7a), this is impossible. Hence, $ac \notin E(G')$.

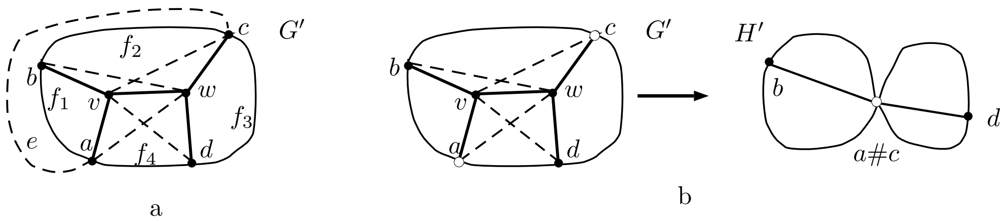

Figure 7. Two adjacent vertices of type $(3,0,0)$.

Consider the graph $H' = (G' - \{v,w\})\,\#\,ac$ (vertices a and c can be merged along the polyline $avwc$, see Figure 7b). Since H' is a 2-planar graph and $v(H') < v(G')$, by Corollary 5, H' has a 9-coloring ρ. We will color both a and c with the color $\rho(a\,\#\,c)$ and obtain a 9-coloring ρ' of the graph $G' - \{v,w\}$ in which at most 8 colors are forbidden for v (since w is not colored). Then we can color v and obtain a 9-coloring of $G' - w$ in which at most 8 colors are forbidden for w (since $a, c \in \mathrm{N}_{G'}(w)$ have the same color). Then we can color w and obtain a 9-coloring of G', a contradiction. ∎

***Proof of Theorem* 1.** We will call *big vertices* all k-vertices for $k \geq 10$.

Let us define the *corrected contribution* of a vertex. For each edge vw where v is a 9-vertex of type (3,0,0) and w is a big vertex, $\frac{1}{3}$ will be subtracted from $\mu(w)$ and added to $\mu(v)$. For each $x \in V(G)$, denote by $\mu'(x)$ the new contribution changed as said above.

We will prove that the corrected contribution of any vertex is nonnegative. Therefore, $\sum_{x\in V(G)} \mu(x) = \sum_{x\in V(G)} \mu'(x) \geq 0$. This contradiction with Claim 14 will finish the proof of Theorem 1.

The corrected contribution $\mu'(v)$ can be negative only in two cases: either $\mu(v) < 0$ (i.e., v is a vertex of type $(3,0,0)$) or v has given a part of its contribution to neighboring vertices of type $(3,0,0)$. Note that if w has given a part of its contribution to v then vw is a big edge.

Consider several cases.

Case 1. *v is a vertex of type* $(3,0,0)$. By Claim 20, v is not adjacent to 9-vertices of type $(3,0,0)$. By Claim 17, v is not adjacent to 9-vertices of other types. Therefore, v is adjacent to three big vertices and each of them gives $\frac{1}{3}$ to v. Hence, $\mu'(v) = \mu(v) + 3 \cdot \frac{1}{3} \geq 0$.

In what follows we assume that v has type (d, f_3, f_4) and $f_{\geq 5} = f^{\geq 5}(v)$. Recall that $\mu(v) \geq \nu(v) = s(v) - 10 + \frac{2f_3}{3} + \frac{f_4}{2}$.

Case 2. $s(v) \in \{10, 11\}$. Assume that v is incident to k big edges. Note that if v belongs only to big faces then $s(v) = 3f_{\geq 5}$. Since this does not hold in our case, $f_{\geq 5} \geq k+1$. By Claim 11, $d \geq \frac{s}{3}$, whence it follows $d \geq 4$ and $f_{\geq 5} \leq \frac{11-4}{2}$, i.e., $f_{\geq 5} \leq 3$ and $k \leq 2$. Consider two subcases.

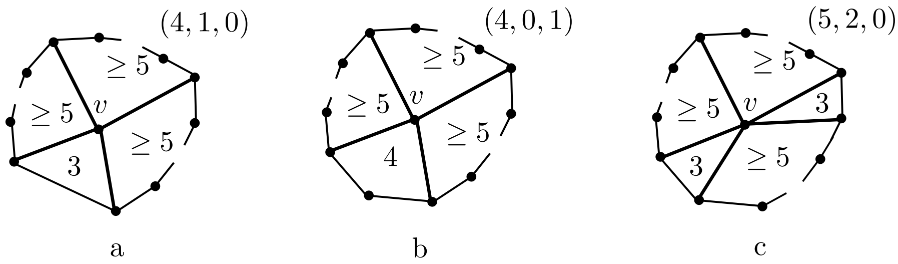

Figure 8. 10- and 11-vertices incident to big edges.

Case 2.1. $k=2$. Then $f_{\geq 5}=3$. If $s(v)=10$ then $d=4$, $f_3=1$ and $f_4=0$. In this case, v has type $(4,1,0)$ (see Figure 8a) and $\nu(v)=\frac{2}{3}$.

If $s(v)=11$ then two variants are possible: $d=4$, $f_3=0$, $f_4=1$ or $d=5$, $f_3=2$, $f_4=0$. In the first case, v has type $(4,0,1)$ (see Figure 8b) and $\nu(v)=\frac{3}{2}$. In the second case, v has type $(5,2,0)$ (see Figure 8c) and $\nu(v)=\frac{7}{3}$.

In all cases, $\mu'(v) \geq \nu(v) - 2\cdot\frac{1}{3} \geq 0$.

Case 2.2. $k=1$. As it is proved above, $f_3+f_4 \geq 1$, whence it follows that $\nu(v) \geq s(v)-10+\frac{1}{2} \geq \frac{1}{2}$ and $\mu'(v) \geq \nu(v)-\frac{1}{3} \geq 0$.

Case 3. $s(v)=s \geq 12$. Let v is incident to k big edges. Since $f_{\geq 5} \leq d$, we have $s \geq d+2f_{\geq 5} \geq 3f_{\geq 5}$. Therefore, $k \leq f_{\geq 5} \leq \frac{s}{3}$ and

$$\mu'(v) \geq \mu(v) - \frac{s}{3}\cdot\frac{1}{3} \geq s-10-\frac{s}{9} > 0$$

(the last inequality for $s \geq 12$ can be easily verified). ■

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