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ON SUBGRAPHS WITH PRESCRIBED ECCENTRICITIES

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Abstract

A well-known result by Hedetniemi states that for every graph G there is a graph H whose center is G. We extend this result by showing under which conditions there exists, for a given graph G in which each vertex v has an integer label $\ell(v)$, a graph H containing G as an induced subgraph such that the eccentricity, in H, of every vertex v of G equals $\ell(v)$. Such a labelled graph G is said to be *eccentric*, and *strictly eccentric* if there exists such a graph H such that no vertex of $H - G$ has the same eccentricity in H as any vertex of G. We find necessary and sufficient conditions for a labelled graph to be eccentric and for a forest to be eccentric or strictly eccentric in a tree.

Keywords: distance, eccentricity, subgraph, tree.

2020 Mathematics Subject Classification: 05C12, 05C05.

1. Introduction

If G is a connected graph and v a vertex of G, then the *eccentricity* of v, denoted $e(v)$, is the maximum distance from v to any vertex of G. The minimum eccentricity of any vertex in G is its *radius*, $\operatorname{rad} G$, while the maximum such eccentricity is its *diameter*, $\operatorname{diam} G$. Knowledge of the eccentricities of the vertices of a graph provides information about the distance structure of that graph, and has been applied, for example, to facility location problems (a vertex of minimum eccentricity is an optimal location for a facility that minimizes maximum response time, see, for example, [18]) and as a predictor of the anti-HIV activity of dihydroseselins (via the eccentric distance sum [10, 14]).

Eccentricity is well-studied. The *eccentric sequence* (also called eccentricity sequence) of a graph or digraph, defined as the non-increasing sequence of the eccentricities of its vertices, has attracted much attention in the literature. The general problem of characterising eccentric sequences of connected graphs appears difficult. Lesniak [13] showed that a sequence s of nonnegative integers is eccentric if some subsequence that contains all values that appear in s is eccentric (since every sequence is a subsequence of itself, this does not lead to an algorithm to determine if a sequence is the eccentric sequence of a graph or digraph). She also characterised eccentric sequences of trees. Oellermann and Tian [17] extended these results to n-Steiner eccentric sequences. Apart from trees, maximal outerplanar graphs are the only other important class of graphs whose eccentric sequences have been characterised (see [5]). So-called minimal eccentric sequences were studied in [16] and [11]. Ferrero and Harary [8] gave a short survey on results on eccentric sequences of graphs.

The number of vertices of given eccentricity has also been studied. Lesniak [13] showed that the eccentric sequence has no gaps, i.e., every number between the smallest and largest number appears in the sequence, and — with the possible exception of the smallest value — every value appears at least twice in the sequence. In [4] it was shown that in k-connected graphs the entries that are not too far from twice the radius of the graph appear at least $2k$ times in the eccentric sequence. Upper and lower bounds on the number of vertices of given eccentricity in a graph in terms of order and diameter were given by Mubayi and West [15].

Eccentric sequences of digraphs have been investigated. Gimbert and López [9] proved that a sequence s of nonnegative integers is the eccentric sequence of a digraph if some subsequence that contains all values that appear in s is eccentric. The same authors further characterised eccentric sequences of strong digraphs of order n, diameter $n-1$ and out-radius $n-2$ or $n-3$. Eccentric sequences of tournaments have been characterised by Harminc and Ivančo [12].

Suppose that we wish to specify not only an allowed sequence of eccentricities but also the structure of the subgraph in which they appear. For example, we might ask whether there is a graph H that contains an induced subgraph G whose vertices have the eccentricities shown in Figure 1.

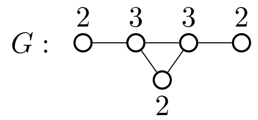

Figure 1. A labelled graph.

Hedetniemi's well-known proof [1] that every graph is the center of a connected graph shows that if every vertex of G has label 2, then we can always find a suitable ambient graph H. In [6], the analogous problem for various classes of digraphs is considered. In this paper, we consider the problem for undirected graphs.

2. Eccentric Labelled Graphs

Throughout this paper we assume that our graphs are finite and simple. For convenience, we use interval notation to denote integer rather than real intervals, e.g., $[3,6)=\{3,4,5\}$. If G and H are graphs, then by $G\cup H$ we mean the graph with $V(G\cup H)=V(G)\cup V(H)$ and $E(G\cup H)=E(G)\cup E(H)$. We denote by $G+H$ the graph obtained from $G\cup H$ by joining every vertex of G to every vertex of H. If k is a positive integer, then kG denotes the disjoint union of k copies of the graph G. The *Cartesian product* of G and H is the graph $G\times H$ with $V(G\times H)=V(G)\times V(H)$ and where two vertices (u_1,u_2) and (v_1,v_2) are adjacent if and only if ($u_1=u_2$ and $v_1v_2\in E(H)$) or ($v_1=v_2$ and $u_1u_2\in E(G)$).

If n is a positive integer, then the *complete graph* of order n is the graph K_n with n vertices that are pairwise adjacent, while the *path* of order n is the graph P_n with $V(P_n) = \{v_1, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : i \in [1, n-1]\}$. For $n \geq 3$, the *cycle* of order n is the graph obtained from P_n by joining v_n to v_1. If $S \subseteq V(G)$, then the *subgraph induced by* S is the graph $G[S]$ with vertex set S and edge set $E(G[S]) = \{uv : u, v \in S \text{ and } uv \in E(G)\}$. If G is a subgraph of H, we call H an *ambient graph* (of G). A vertex of degree 1 is an *endvertex*. For notation not defined here, we follow [2].

If G is a graph and $\ell : V(G) \to \mathbb{N}$ a labelling that assigns a positive integer $\ell(v)$ to every vertex v of G, then we denote by G_ℓ the associated *labelled graph*. If there is no possible ambiguity, we shall denote the labelled graph G_ℓ simply by G. We define $\ell(G) = \{\ell(v) : v \in V(G)\}$, $\ell_{\min}(G) = \min \ell(G)$ and $\ell_{\max}(G) = \max\ \ell(G)$. If there exists a graph H such that (i) G is an induced subgraph of H, and, (ii) for all $v \in V(G)$, $e_H(v) = \ell(v)$, then we say that G has an *eccentric embedding* in H and that G is *eccentric* in H. If there exists at least one graph H in which the labelled graph G is eccentric, then G is *eccentric*. As an example, consider the labelled graph G with $\ell_{\min}(G) = 3$ and $\ell_{\max}(G) = 4$ shown on the left side of Figure 2. On the right hand side is an eccentric embedding of G in a graph H. Hence the labelled graph G is eccentric.

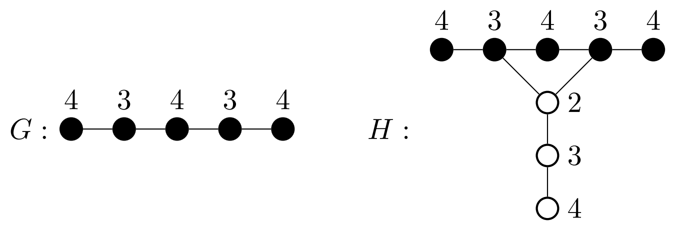

Figure 2. On the left is shown a labelled graph G, and on the right G as an eccentric subgraph of a graph H. Each vertex of H is labelled with its eccentricity.

There are three obvious necessary conditions for a labelled graph to be eccentric (we shall shortly see that these conditions are also sufficient). Suppose that G is a labelled graph and H a graph such that G is eccentric in H. If v is a vertex of G with label 1, then v must be adjacent to every other vertex of G (such a vertex is frequently called *universal*). Since the eccentricities of adjacent vertices in a graph cannot differ by more than 1, we must have that for every pair u, v of adjacent vertices of G, $|\ell(u) - \ell(v)| \leq 1$. Furthermore, since $\ell_{\max}(G) \leq \operatorname{diam} H \leq 2 \operatorname{rad} H \leq 2\ell_{\min}(G)$, we must have $\ell_{\max}(G) \leq 2\ell_{\min}(G)$. This leads directly to the following observation.

Observation 1. *If a labelled graph G with $\ell_{\max} = 2\ell_{\min}$ is eccentric in a graph H, then* $\operatorname{rad} H = \ell_{\min}$ *and* $\operatorname{diam} H = \ell_{\max}$.

If $\ell(G) = [\ell_{\min}, \ell_{\max}]$, then we say $\ell(G)$ is *connected*. If G is a connected labelled graph and u, v vertices of G with $\ell(u) = \ell_{\min}(G)$ and $\ell(v) = \ell_{\max}(G)$, then there is a $u - v$ path P. If G is eccentric, then consecutive vertices of P have labels that differ by at most 1. Hence we have the following.

Observation 2. *If G is a labelled graph that is connected and eccentric, then $\ell(G)$ is connected.*

If G and H are graphs, then the *strong product* of G and H, which we denote $G \boxtimes H$, is the graph with $V(G \boxtimes H) = V(G) \times V(H)$ and where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if (i) $u_1 = u_2$ and $v_1v_2 \in E(H)$, or, (ii) $v_1 = v_2$ and $u_1u_2 \in E(G)$, or, (iii) $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$. It is well-known that $G \boxtimes H$ is connected if and only if G and H are connected, and $d_{G\boxtimes H}((u_1, v_1), (u_2, v_2)) = \max\{d_G(u_1, u_2), d_H(v_1, v_2)\}$. The next result, which is also well-known, follows from this.

Lemma 3. *If G and H are connected graphs and $(u, v) \in V(G) \times V(H)$, then $e_{G\boxtimes H}(u, v) = \max\{e_G(u), e_H(v)\}$. Furthermore,* $\operatorname{rad} G \boxtimes H = \max\{\operatorname{rad} G, \operatorname{rad} H\}$ *and* $\operatorname{diam} G \boxtimes H = \max\{\operatorname{diam} G, \operatorname{diam} H\}$.

We shall in part of the proof below use a modified version of Hedetniemi's construction [1]. For some integer $r \geq 2$, let $\hat{G}_r$ be the graph obtained from $G \cup 2P_r$ by joining one endvertex of each P_r to every vertex of G. Then in $\hat{G}_r$, every vertex of G has eccentricity r and $\operatorname{rad} \hat{G}_r = r$.

Theorem 4. *A labelled graph G is eccentric if and only if* (i) $\ell_{\max}(G) \leq 2\ell_{\min}(G)$, (ii) $|\ell(u) - \ell(v)| \leq 1$ *for every pair u, v of adjacent vertices of G, and,* (iii) *every vertex w of G with $\ell(w) = 1$ is universal. If G is eccentric, then for every pair r, d of positive integers with $r \leq d \leq 2r$ and $\ell_{\min}(G), \ell_{\max}(G) \in [r, d]$, there is a graph H of radius r and diameter d such that G is eccentric in H.*

Proof. The necessity of the conditions has already been discussed, so we turn our attention to the other direction of the proof.

Suppose that G is a labelled graph satisfying conditions (i), (ii), and (iii), and let r and d be positive integers such that $r \leq d \leq 2r$ and $\ell_{\min}, \ell_{\max} \in [r, d]$ (where, for convenience, we have written $\ell_{\min}$ for $\ell_{\min}(G)$ and $\ell_{\max}$ for $\ell_{\max}(G)$). We show that there is a graph H of radius r and diameter d in which G is eccentric. Suppose that $\ell_{\min} = 1$. If $d = 1$, then $\ell_{\max} = 1$, i.e., $\ell(v) = 1$ for every vertex v of G and the graph G is complete, so we may take $H = G$. Suppose then that $d = 2$. If $\ell_{\max} = 1$, we let $H = G + \overline{K}_2$. If, on the other hand, $\ell_{\max} = 2$, then take H to be the graph formed from G by adding a new vertex x and joining x to every universal vertex of G.

Suppose that $\ell_{\min} \geq 2$. If $r = 1$, then $d = 2 = \ell_{\min} = \ell_{\max}$, and we may take H to be the graph formed from G by joining every vertex of G to one vertex of

P_2. If $r = 2$ and $d = 2$, then $\ell_{\min} = \ell_{\max} = 2$ and we may let H be the graph obtained by adding an edge between the two endvertices of $\hat{G}_2$. If $r = 2$ and $d = 3$, then there are three possibilities: (a) $\ell_{\min} = \ell_{\max} = 2$, in which case we let H be the graph obtained by deleting an endvertex of $\hat{G}_2$, or, (b) $\ell_{\min} = 2$ and $\ell_{\max} = 3$, in which case we let H be the graph obtained by joining one endvertex of P_3 to every vertex of G and then the middle vertex of P_3 to all the vertices v of G with $\ell(v) = 2$, or, (c) $\ell_{\min} = \ell_{\max} = 3$, in which case we let H be the graph obtained by joining one endvertex of P_3 to every vertex of G. If $r = d = 3$, then $\ell_{\min} = \ell_{\max} = 3$ and we may take H to be the graph obtained by joining every vertex of G to both endvertices of P_6.

We hence assume that $r \geq 2$ and $d \geq 4$. Let $C_{r,d}$ be the graph formed from C_{2r} and $P_{d-r} : x_{r+1}, x_{r+2}, \ldots, x_d$ by joining a vertex x_r of C_{2r} to x_{r+1}. Note that $C_{r,d}$ has radius r and diameter d and for each $i \in [r, d]$, we have $e_{C_{r,d}}(x_i) = i$. Let $H = \hat{G}_2 \boxtimes C_{r,d}$. Since $\operatorname{rad} \hat{G}_2 = 2$ and $\operatorname{diam} \hat{G}_2 = 4$, by Lemma 3 we have $\operatorname{rad} H = r$ and $\operatorname{diam} H = d$. Consider the subgraph G^* of H induced by the set $S = \{(v, x_{\ell(v)}) : v \in V(G)\}$. By Lemma 3, for every vertex v of G, we have $e_H((v, x_{\ell(v)})) = \ell(v)$. We claim that $G^* \cong G$. If $uv \in E(G)$, then $|\ell(u) - \ell(v)| \leq 1$, which implies that $x_{\ell(u)} x_{\ell(v)} \in E(C_{r,d})$. Hence, $(u, x_{\ell(u)})(v, x_{\ell(v)}) \in E(G^*)$, and hence $G \leq G^*$. Suppose now that $(u, x_{\ell(u)})(v, x_{\ell(v)}) \in E(G^*)$. If $uv \notin E(G)$, then $u = v$ and $x_{\ell(u)} x_{\ell(v)} \in E(C_{r,d})$, which from our choice of the set S is impossible. It follows that $G^* \leq G$, and hence that G^* is an induced subgraph of H that is isomorphic to G and in which the vertices have the required eccentricities. ■

In the proof of Theorem 4, we made use of the fact that the graph $C_{r,d}$ contains a path $x_r, x_{r+1}, \ldots, x_d$ such that for each $i \in [r, d]$, we have $e_{C_{r,d}}(x_i) = i$. If G' is a connected graph with radius r and diameter d that contains such a path, then it is easy to see that the graph $C_{r,d}$ can be replaced by G'. It's worth noting, though, that not every graph has this property (see, for example, Figure 3).

3. Strictly Eccentric Labelled Graphs

Hedetniemi's construction implies that the subgraph induced by all the vertices of minimum eccentricity can have arbitrary structure. Suppose that we are interested in the subgraph induced by all vertices from a specified set of eccentricities. What structure can such a subgraph have? To address this question, we make the following definitions. For a labelled graph G, let $\ell(G) = \{\ell(v) : v \in V(G)\}$. If G is eccentric in H and there is no $v \in V(H) \setminus V(G)$ with $e_H(v) \in \ell(G)$, then G is *strictly eccentric* in H.

We note some obvious necessary conditions for a labelled graph G to be strictly eccentric. Clearly, every graph that is strictly eccentric is eccentric, and hence we must have that $\ell_{\max} \leq 2\ell_{\min}$, every edge uv satisfies $|\ell(u) - \ell(v)| \leq 1$, and

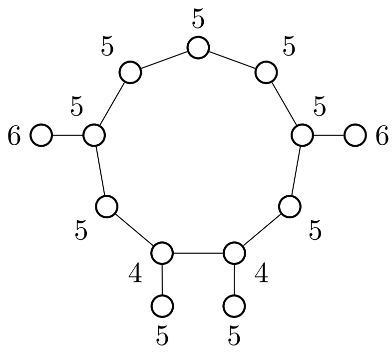

Figure 3. A graph of radius 4 and diameter 6 (the labels on the vertices are their eccentricities) that does not contain a path of length 2 between a vertex of eccentricity 4 and a vertex of eccentricity 6.

every vertex with label 1 is universal. If $\ell_{\max}(G) = 2\ell_{\min}(G)$, $\ell(G) = [\ell_{\min}, \ell_{\max}]$, and G is strictly eccentric, then by Observation 1, we must have that $\ell_{\min}(G)$ and $\ell_{\max}(G)$ are, respectively, the radius and diameter of the ambient graph H, and hence that $G = H$. Hence, if $\ell_{\max}(G) = 2\ell_{\min}(G)$ and $\ell(G) = [\ell_{\min}, \ell_{\max}]$, then G is strictly eccentric if and only if $e_G(v) = \ell(v)$ for all $v \in V(G)$. Lesniak [13] proved that for every connected graph G and for every integer $k \in (r(G), d(G)]$, there are at least two vertices of eccentricity k in G. We shall shortly strengthen this result.

Let G be a labelled graph that is connected and strictly eccentric in a graph H. We denote by V_i the set of vertices of H of eccentricity i in H. If u, v are vertices of G for which $d_G(u,v) = |\ell(u) - \ell(v)|$, then since $d_H(u,v) \leq d_G(u,v)$, we have $d_H(u,v) = |\ell(u) - \ell(v)|$. If, on the other hand, $d_H(u,v) = |\ell(u) - \ell(v)|$, then, since G is strictly eccentric in H, every vertex on a $u-v$ geodesic in H is a vertex of G, and thus $d_G(u,v) = |\ell(u) - \ell(v)|$. If u and v are vertices of G such that $d_G(u,v) = |\ell(u) - \ell(v)| = d_H(u,v)$, we say that u and v are an *ℓ-monotone pair*, or simply a *monotone pair* (and note that this relation is reflexive and symmetric). If u and v are a monotone pair and P a $u-v$ geodesic, then we say P is *ℓ-monotone* or simply *monotone*. For a vertex v, we define

$$\ell^+(v) = \max\{\ell(w) : v \text{ and } w \text{ are a monotone pair}\}.$$

Trivially, $\ell^+(v) \geq \ell(v)$.

Theorem 5. *Let G be a labelled graph that is connected and strictly eccentric, and let $v \in V(G)$. For every integer $k \in [\ell_{\min}, \ell^+(v)]$ there exists a vertex $w \in V(G)$ with $\ell(w) = k$ and $d_G(v,w) \geq \ell(v) + k - 2\,\ell_{\min}$.*

Proof. Suppose G is strictly eccentric in H and $v \in V(G)$. Suppose, first, that $k \in [\ell_{\min}, \ell(v)]$. Then there exists a vertex v' with $d_H(v,v') = \ell(v)$. Necessarily,

$e_H(v') \geq \ell(v)$. Let $u \in V_{\ell_{\min}}$ and P a shortest $u - v'$ path in H. Then P contains a vertex $w \in V_k$. Since $d_H(u, v') \leq e_H(u) = \ell_{\min}$, we have

$$d_H(u, w) + d_H(w, v') = d_H(u, v') \leq \ell_{\min}. \tag{1}$$

On the other hand, the triangle inequality yields

$$d_H(v, w) + d_H(w, v') \geq d_H(v, v') = \ell(v). \tag{2}$$

Subtracting (1) from (2), we get

$$d_H(v, w) - d_H(u, w) \geq \ell(v) - \ell_{\min},$$

and so

$$d_G(v, w) \geq d_H(v, w) \geq \ell(v) - \ell_{\min} + d_H(u, w) \geq \ell(v) - \ell_{\min} + k - \ell_{\min},$$

as desired. Suppose then that $k \in (\ell(v), \ell^+(v)]$ and let $z \in V_{\ell^+(v)}$ be in a monotone pair with v. By the first part of the proof, there exists $w \in V_k$ for which $d_G(z, w) \geq \ell(z) + k - 2\ell_{\min}$, so $d_G(v, w) \geq d_G(z, w) - d_G(v, z) \geq \ell(z) + k - 2\ell_{\min} - (\ell(z) - \ell(v)) = \ell(v) + k - 2\ell_{\min}$. ■

Note that Theorem 5 is no longer true if we allow $k > \ell^+(v)$ (see, for example, Figure 4).

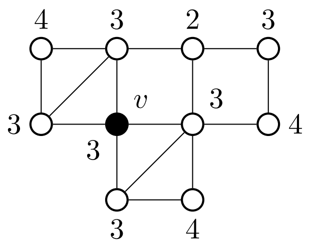

Figure 4. The vertices of the graph G shown above (taken from [3]) are labelled with their eccentricities. Every vertex u in this graph has $\ell^+(u) = 4$ except the black vertex v, which has $\ell^+(v) = 3$. Every vertex $u \in V(G) \setminus \{v\}$ has a vertex with label 4 at distance $\ell(u) + 4 - 4 = \ell(u)$ or more. Every vertex of eccentricity 4 is, however, distance 2 away from v.

Theorems 4 and 5 collectively give a set of necessary conditions for a labelled graph to be strictly eccentric. A natural question is whether these conditions are also sufficient (i.e., if G is a labelled graph for which (i) $\ell_{\max} \leq 2\ell_{\min}$, (ii) every vertex with label 1 is universal, (iii) $uv \in E(G) \Longrightarrow |\ell(u) - \ell(v)| \leq 1$, and, (iv) for every vertex $v \in V(G)$ and every integer $k \in [\ell_{\min}, \ell^+(v)]$ there exists a vertex $w \in V_k$ for which $d_G(v, w) \geq \ell(v) + k - 2\,\ell_{\min}$, is G strictly eccentric?) The answer is no. To see this, let G be the labelled graph shown in Figure 5 and v, x, y, z the indicated vertices.

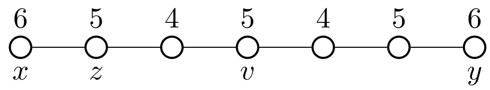

Figure 5. A labelled graph G that satisfies Theorems 4 and 5 but which is not strictly eccentric.

It is easy to verify that G satisfies Theorems 4 and 5. Suppose that G is strictly eccentric in some graph H, let v' be an eccentric vertex of v, and note that $e_H(v') \geq e_H(v) = 5$. Since v is distance 3 or less from every vertex of G, we have $v' \in V(H) \setminus V(G)$ and $\operatorname{diam} H \geq 7$, and thus $\operatorname{rad} H = 4$. Then $d_H(x, v') = 2 = d_H(y, v')$ (otherwise v' is too far from a vertex with label 4), there is a cycle containing x and y and $d_H(x, y) \leq 4$. Thus every eccentric vertex x' of x is in $V(H) \setminus V(G)$, but then $d_H(z, x') \geq 7$, a contradiction.

We now explore some consequences of Theorem 5.

Corollary 6. *Let G be a labelled graph that is connected and strictly eccentric. Then for every vertex $v \in V(G)$ there exists a vertex $w \in V(G)$ with $\ell(w) = \ell(v)$ and $d_G(v, w) \geq 2(\ell(v) - \ell_{\min})$.*

Corollary 7. *Let G be a labelled graph and $k \in \mathbb{N}$ with $\ell(v) = k$ for all $v \in V(G)$. Let $r \in \mathbb{N}$ with $r < k \leq 2r$. The following are equivalent.*

(i) *G is strictly eccentric in some graph of radius at most r.*

(ii) $\operatorname{rad} G \geq 2(k - r)$.

Proof. (i) $\Rightarrow$ (ii): Assume that G is strictly eccentric in a graph H with radius not more than r. Then H contains a vertex of eccentricity r. Let $G' = H[V_r \cup V_k]$ and ℓ' be the labelling of the vertices of G' that assigns r to the vertices in V_r and k to the vertices in V_k. Then G' has a strictly eccentric embedding in H. By Corollary 6, every vertex of V_k is at distance at least $2(k - r)$ in G from some other vertex in V_k. Hence $\operatorname{rad} G \geq 2(k - r)$.

(ii) $\Rightarrow$ (i): Assume that $\operatorname{rad} G \geq 2(k - r)$. We construct a graph H and show that G has a strictly eccentric embedding in H. Let the vertices of G be $u_1, u_2, \ldots, u_n$. For $i = 1, 2, \ldots, n$ let $P^{(i)}$ be a path on the vertices $v_r^i, v_{r+1}^i, \ldots, v_{2r}^i$. Let H be obtained from the disjoint union of the $P^{(i)}$ by identifying the vertices $v_r^1, v_r^2, \ldots, v_r^n$ to a single vertex v_r, and then adding an edge between $v_k^{(i)}$ and $v_k^{(j)}$ whenever $u_i u_j \in E(G)$. Let $V_k = \left\{v_k^i : i = 1, 2, \ldots, n\right\}$. Then clearly $H[V_k] = G$.

We now determine the eccentricity of every vertex of H. Clearly, $e_H(v_r) = r$. We show that for $i \in \{r+1, r+2, \ldots, 2r\}$ and $j \in \{1, 2, \ldots, n\}$ we have

$$e_H\left(v_i^j\right) = i. \tag{3}$$

Since $d_H\big(v_i^j, v_r\big) = i - r$, and since $e_H(v_r) = r$, there exists a path from v_i^j through v_r of length at most $(i-r)+r$ to every vertex of H, so $e_H\big(v_i^j\big) \leq i$. In order to prove

(3) it remains to show that $e_H\big(v_i^j\big) \geq i$. By our assumption $\operatorname{rad} G \geq 2(k-r)$ we have $e_G(u_j) \geq 2(k-r)$, so there exists a vertex $u_{j'}$ of G with $d_G(u_j, u_{j'}) \geq 2(k-r)$. We show that $d_H\left(v_i^j, v_{2r}^{j'}\right) \geq i$. Let P be a $\left(v_i^j, v_{2r}^{j'}\right)$-path. If P contains v_r, then P has length at least $d_H\big(v_i^j, v_r\big) + d_H\left(v_r, v_{2r}^{j'}\right) = (i-r) + r = i$. If P does not contain v_r, then P contains necessarily v_k^j and $v_k^{j'}$, and so P has length at least $d_H\left(v_i^j, v_k^j\right) + d_H\left(v_k^j, v_k^{j'}\right) + d_H\left(v_k^{j'}, v_{2r}^{j'}\right) = |i-k| + 2(k-r) + (2r-k) = |k-i| + k \geq i$. Hence (3) follows.

It follows from (3) that the vertices of the set $\left\{v_k^1, v_k^2, \ldots, v_k^n\right\}$ are exactly the vertices of eccentricity k in H. But $H\left[\left\{v_k^1, v_k^2, \ldots, v_k^n\right\}\right] = G$, so G is strictly eccentric in H. ■

Corollary 8. *Let G be a graph and $k \in \mathbb{N}$, $k \geq 2$. Then G is the subgraph induced by the vertices of eccentricity k of some graph if and only if* $\operatorname{rad} G \geq 2$.

4. Tree-Eccentric Labelled Graphs

Every labelled tree T that satisfies the conditions of Theorem 4 is eccentric. However, since the strong product of two nontrivial connected graphs contains a cycle, the ambient graph H containing T will almost always not be a tree. If a labelled forest or tree T is eccentric in a tree T', we shall say that T is *tree-eccentric*. Recall that every tree is *central* (if the center is K_1) or *bicentral* (if the center is K_2). If T is a central tree, then $\operatorname{diam} T = 2 \operatorname{rad} T$, while if T is a bicentral tree, then $\operatorname{diam} T = 2 \operatorname{rad} T - 1$. If u and v are vertices of a tree T, then the unique $u-v$ path in T is denoted $[u, v]$. A vertex x is said to *lie between* u and v if x lies on a shortest $u-v$ path, i.e., $d(u,v) = d(u,x) + d(x,v)$. We shall need the following results.

Lemma 9 [7]**.** *Let u and v be two vertices of a tree T having $e(u) = e(v)$.*

- *If $d(u,v)$ is odd, then T is bicentral and the center of T is the center of $[u,v]$.*
- *If $d(u,v)$ is even, then T may be central or bicentral, and the central vertex of $[u,v]$ lies between u and the center of T.*

The next result is well-known (see, for example, [7]).

Lemma 10. *If T is a tree, $v \in V(T)$ and C is the center of T, then $e(v) =$* $\operatorname{rad} T + d(v, C)$.

If T is a labelled tree and v a vertex of T with $\ell(v) = \ell_{\min}(T)$, we shall call v an *ℓ-central vertex.*

Lemma 11. *If a labelled tree T is eccentric in a tree T', then T has no more than two ℓ-central vertices. If T has one ℓ-central vertex, u, then u lies between the center of T' and every vertex of T. If T has two ℓ-central vertices u_1 and u_2, then T' is bicentral and u_1 and u_2 are the adjacent central vertices of T'.*

Proof. If T has one ℓ-central vertex, the result follows immediately from Lemma 10. Suppose then that u_1, u_2 are ℓ-central vertices of a tree T that is eccentric in a tree T'. If $d(u_1, u_2)$ is even, then by Lemma 9, the central vertex z of $[u_1, u_2]$ lies between u_1 and the center of T', but then $z \in [u_1, u_2] \subseteq T$ and by Lemma 10, $\ell(z) < \ell_{\min}$, a contradiction. If, on the other hand, $d(u_1, u_2)$ is odd, then by Lemma 9 the tree T' is bicentral and both central vertices of T' lie on $[u_1, u_2]$. Since $[u_1, u_2] \subseteq T$, it follows that the center of T' is in T and hence that u_1 and u_2 are central vertices of T. Hence, if there are two ℓ-central vertices, then they are adjacent and central in T' (and consequently, there cannot be three ℓ-central vertices). ■

It follows from Lemmas 10 and 11 that if v is a vertex in a tree-eccentric labelled tree T and c is the ℓ-central vertex closest to v, then $\ell(v) = \ell(c) + d(u, v)$. We hence define two classes of labelled trees.

- If T is a labelled tree with exactly one ℓ-central vertex u, and, for each vertex $v \in V(T)$, we have $\ell(v) = \ell_{\min}(T) + d(u, v)$, we say that $T \in \mathcal{T}_1$.
- If T is a labelled tree with exactly two ℓ-central vertices u_1, u_2, these two vertices are adjacent, and, for each vertex $v \in V(T)$, we have $\ell(v) = d(\{u_1, u_2\}, v) + \ell_{\min}(T)$, we say that $T \in \mathcal{T}_2$.

Theorem 12. *A labelled tree T is tree-eccentric if and only if $\ell_{\max} \leq 2\ell_{\min}$ and $T \in \mathcal{T}_1$, or $\ell_{\max} \leq 2\ell_{\min} - 1$ and $T \in \mathcal{T}_2$. If $T \in \mathcal{T}_1$ and $d \in [\ell_{\max}, 2\ell_{\min}]$, then there exists a tree T' of diameter d and radius $\lceil d/2 \rceil$ in which T is tree-eccentric. If $T \in \mathcal{T}_2$ and T is tree-eccentric in T', then T' is bicentral,* $\operatorname{rad} T' = \ell_{\min}$ *and* $\operatorname{diam} T' = 2\ell_{\min} - 1$.

Proof. The necessity is provided by Lemma 11 and the preceding discussion. We prove the sufficiency. Suppose first that $T \in \mathcal{T}_1$, let u be the ℓ-central vertex of T, and z a vertex of T with $\ell(z) = \ell_{\max}$. Let T' be the tree obtained from T and two paths $P : x_0, x_1, \ldots, x_{\ell_{\min}}$ and $Q : y_0, y_1, \ldots, y_{d-\ell_{\max}}$ by identifying u with x_0 and z with y_0. Every vertex v of T is distance at most $(\ell(v) - \ell_{\min}) + (\ell_{\max} - \ell_{\min}) \leq \ell(v)$ from every vertex of T, distance $\ell(v) - \ell_{\min} + \ell_{\min}$ from $x_{\ell_{\min}}$, and distance at most $\ell(v)$ from every vertex of Q, so $e_{T'}(v) = \ell(v)$ as required and T is eccentric in T', which has diameter d as required. Suppose, then, that $T \in \mathcal{T}_2$ and let the two ℓ-central vertices of T be u_1 and u_2. Let T_1 and T_2 be the components of $T - u_1u_2$ that contain the vertices u_1 and u_2, respectively. Let z_1, z_2 be vertices of T_1 and T_2, respectively, with $\ell(z_1) = \ell_{\max}(T_1)$ and $\ell(z_2) = \ell_{\max}(T_2)$, and for

$i \in \{1,2\}$ define $\delta_i = 2\ell_{\min} - \ell(z_i) - 1$. Let T' be the tree obtained from T by adding paths $P_1 : x_0, x_1, \ldots, x_{\delta_1}$ and $P_2 : y_0, y_1, \ldots, y_{\delta_2}$ and identifying z_1 with x_0 and z_2 with y_0. If, without loss of generality, $v \in V(T_1)$, then v is distance at most $(\ell(v) - \ell_{\min}) + (\ell(z_1) - \ell_{\min}) + \delta_1 = \ell(v) - 1$ from every vertex of $T_1 \cup P_1$, distance $(\ell(v) - \ell_{\min}) + 1 + (\ell(z_2) - \ell_{\min}) + \delta_2 = \ell(v)$ from y_{δ_2}, and distance at most $\ell(v)$ from every vertex of $T_2 \cup P_2$. Hence $e_{T'}(v) = \ell(v)$ and thus T is eccentric in T'. Furthermore, $\operatorname{rad} T' = \ell_{\min}$ and $\operatorname{diam} T' = 2\ell_{\min} - 1$. ■

If a labelled forest F is tree-eccentric, then every component of F is tree-eccentric. If T is a component of a tree-eccentric forest, we shall adopt the convention that $T \in \mathcal{T}_1 \cup \mathcal{T}_2$ if the underlying tree T together with the restriction of ℓ to $V(T)$ is in $\mathcal{T}_1 \cup \mathcal{T}_2$ (i.e., to judge which vertices of a component T are ℓ-central, we set $\ell_{\min} = \ell_{\min}(T)$). It follows from Theorem 12 that if F is a tree-eccentric labelled forest, then every component of F is in $\mathcal{T}_1 \cup \mathcal{T}_2$. A component of F that contains an ℓ-central vertex (i.e., a component T of F with $\ell_{\min}(T) = \ell_{\min}(F)$) is an *$\ell$-central* component.

Theorem 13. *A labelled forest F is tree-eccentric if and only if exactly one of the following holds.*

1. *F has exactly two ℓ-central vertices, they are adjacent, the ℓ-central component T_c is in $\mathcal{T}_2$, $\ell_{\max} \le 2\ell_{\min} - 1$, $\ell_{\min}(F - T_c) \ge \ell_{\min}(F) + 2$, and every component of $F - T_c$ is in $\mathcal{T}_1$.*
2. *F has no adjacent ℓ-central vertices, every component of F is in $\mathcal{T}_1$, and exactly one of the following holds.*
   - (a) *$\ell_{\max} \in \{2\ell_{\min} - 1, 2\ell_{\min}\}$, there is a unique ℓ-central component T_c, and $\ell_{\min}(F - T_c) \ge \ell_{\min}(F) + 2$.*
   - (b) *$\ell_{\max} \le 2\ell_{\min} - 2$.*

Proof. Suppose that F is a labelled forest that is eccentric in a tree T. If two ℓ-central vertices u_1, u_2 are adjacent, then by Lemma 11 the vertices u_1 and u_2 are central in T, every component of $F - T_c$ is in $\mathcal{T}_1$, the tree T' is bicentral, and consequently $\ell_{\max} \le 2\ell_{\min} - 1$. If some component of $F - T_c$ contains a vertex v with $\ell(v) = \ell_{\min}(F) + 1$, then v is adjacent in T to an ℓ-central vertex in T_c, a contradiction. We suppose, then, that no pair of ℓ-central vertices of F is adjacent. If $\ell_{\max} = 2\ell_{\min}$, then by Observation 1, $\operatorname{diam} T = \ell_{\max}$ and $\operatorname{rad} T = \ell_{\min}$, hence T is central, every ℓ-central vertex of F is central in T, and consequently there is exactly one ℓ-central vertex. Suppose, then, that $\ell_{\max} = 2\ell_{\min} - 1$. If there are two or more ℓ-central vertices, then $\operatorname{rad} T \le \ell_{\min} - 1$, implying that $\operatorname{diam} T \le 2\ell_{\min} - 2$, a contradiction. Thus there is exactly one ℓ-central vertex and, by the same argument, $\ell_{\min} = \operatorname{rad} T$. Whether $\ell_{\max} = 2\ell_{\min}$ or $\ell_{\max} = 2\ell_{\min} - 1$, in either case the ℓ-central vertices are central in T and hence, as previously, $\ell_{\min}(F - T_c) \geq \ell_{\min}(F) + 2$. This completes the first direction of the proof.

Suppose now that F is a labelled forest that satisfies exactly one of the stated conditions. If F satisfies condition 1, by Theorem 12 there exists a bicentral tree T' with $\operatorname{rad} T' = \ell_{\min}$ and $\operatorname{diam} T' = 2\ell_{\min} - 1$ in which T_c is eccentric. Form a new tree T'' from T' as follows. Add a path P of length $\operatorname{rad} T' - 1$, identify an end-vertex of P with a central vertex of T', then for each component T of $F - T_c$ join the ℓ-central vertex of T to that vertex v of P for which $e_{T''}(v) = \ell_{\min}(T) - 1$. Since no vertex of $F - T_c$ is adjacent to a central vertex of T', the forest F is an induced subgraph of T'', and it can be shown as previously that F is eccentric in T''. A similar argument holds for case 2(a). Suppose, then, that F satisfies condition 2(b). Let T'' be the tree obtained from F by adding a path P of length $2\ell_{\min} - 2$ and for each component T of F joining the ℓ-central vertex u of T to a vertex v of P having $e_P(v) = \ell(u) - 1$. Then F is an eccentric subgraph of the tree T'' with $\operatorname{rad} T'' = \ell_{\min} - 1$ and $\operatorname{diam} T'' = 2\ell_{\min} - 2$. ∎

A labelled tree or forest is *strictly tree-eccentric* if it is strictly eccentric in a tree. We consider first the general question of when a labelled forest is strictly tree-eccentric. A forest that is strictly tree-eccentric is tree-eccentric and hence must satisfy Theorem 13. Furthermore, every component of a strictly tree-eccentric forest is tree-eccentric, but not necessarily strictly (see, for example, Figure 6).

Figure 6. A labelled forest F that is strictly tree-eccentric in a tree T while, by Corollary 6, neither component of F is strictly tree-eccentric.

Suppose that F is a labelled forest. If for some integer $t \in (\ell_{\min}, \ell_{\max})$ there is no vertex labelled t, then t is called an *ℓ-gap* (or just a *gap* if there is no possible ambiguity); similarly, a set $S \subseteq \mathbb{Z}$ is a *gap* if every integer in S is a gap. If T is a component of a strictly tree-eccentric forest F with $\ell_{\min}(T) > \ell_{\min}(F)$, then, by essentially the same argument used in the proof of Theorem 13, $\ell_{\min}(T) - 1$ is a gap. Consequently, if T, T' are components of F with $\ell(T) \cap \ell(T') \neq \emptyset$, then $\ell_{\min}(T) = \ell_{\min}(T')$. If F is a labelled forest, recall that a $u - v$ path in F is ℓ-monotone (or simply monotone) if $d(u, v) = |\ell(u) - \ell(v)|$. A monotone path is *maximal* if it is not properly contained in another monotone path. If $a, b \in \ell(F)$ with $a \leq b$, then an *(a, b) virtual ℓ-path* (or simply *(a, b) virtual path*) is a sequence $P_1, \ldots, P_k$ of maximal monotone paths such that $a \in \ell(P_1)$, $b \in \ell(P_k)$, for each $i \in [1, k-1]$, we have $\ell_{\max}(P_i) < \ell_{\min}(P_{i+1})$, and if $c \in \ell(F) \cap [a, b]$, then $c \in \ell\big(\bigcup_{i=1}^{k} V(P_i)\big)$. If $P : P_1, \ldots, P_k$ is a virtual path, we let $V(P) = \bigcup_{i=1}^{k} V(P_i)$. The *intersection* of two virtual paths P and Q is $V(P) \cap V(Q)$; two virtual paths with an empty intersection are *disjoint*. The *ℓ-center* of a labelled tree T is the subgraph induced by its ℓ-central vertices.

Theorem 14. *A labelled forest F is strictly tree-eccentric if and only if exactly one of the following three conditions holds.*

1. *F has exactly two ℓ-central vertices, they are adjacent, and all of the following conditions are satisfied.*
   (a) $\ell_{\max}(F) \leq 2\ell_{\min}(F) - 1$,
   (b) *the ℓ-central component T_c is in $\mathcal{T}_2$,*
   (c) *if T' is a component of $F - T_c$, then $T' \in \mathcal{T}_1$ and $\ell_{\min}(T') - 1$ is a gap, and,*
   (d) *there exist two disjoint $(\ell_{\min}, \ell_{\max})$ virtual paths.*
2. *F has two or more ℓ-central vertices, they are pairwise non-adjacent, and all of the following conditions are satisfied.*
   (a) $\ell_{\max}(F) \leq 2\ell_{\min}(F) - 2$,
   (b) *if T' is a component of F, then $T' \in \mathcal{T}_1$ and $\ell_{\min}(T') - 1$ is a gap, and,*
   (c) *there exist two disjoint $(\ell_{\min}, \ell_{\max})$ virtual paths.*
3. *F has exactly one ℓ-central vertex and all of the following conditions are satisfied.*
   (a) $\ell_{\max}(F) \leq 2\ell_{\min}(F)$,
   (b) *if T' is a component of F, then $T' \in \mathcal{T}_1$ and $\ell_{\min}(T') - 1$ is a gap, and,*
   (c) *there exist two $(\ell_{\min}, \ell_{\max})$ virtual paths whose intersection is the ℓ-central vertex.*

Proof. Suppose first that F is a labelled forest that is strictly eccentric in a tree T. We consider two cases.

Case 1. F has two adjacent ℓ-central vertices, u_1 and u_2. By Lemma 11 and Theorem 12, T is bicentral, u_1 and u_2 are central vertices of T, conditions 1(a) and 1(b) in the current theorem are satisfied, and every component T' of F that is not ℓ-central is in $\mathcal{T}_1$. If $v \in V(T')$ with $\ell(v) = \ell_{\min}(T')$, then there is a vertex $x \in N_T(v) \setminus V(F)$ with $e_T(x) = \ell(v) - 1$, which implies that $\ell_{\min}(T') - 1$ is a gap. If e_1 is an eccentric vertex of u_1 in T, then the $u_1 - e_1$ path in T consists of u_1, the central edge u_1u_2, and the $u_2 - e_1$ path P_{21}. Since F is strictly eccentric in T, every vertex y of P_{21} with $e_T(y) \in \ell(F)$ is a vertex of F. Hence there is a vertex z of P_{21} with $e(z) = \ell_{\max}(F)$. Let $P_1, \ldots, P_k$ be the components of the subgraph induced by those vertices of P_{21} that are in F, numbered such that for each $i \in [1, k-1]$, we have $\ell_{\max}(P_i) < \ell_{\min}(P_{i+1})$. Clearly, $u_1 \in V(P_1)$ and the vertex with label $\ell_{\max}$ is in P_k. If $c \in \ell(F) - \ell\big(\bigcup_{i=1}^{k} V(P_i)\big)$, then there is a vertex of $V(P_{21}) \setminus V(F)$ that is labelled c, contradicting the fact that F is strictly eccentric in T. Thus $P_1, \dots, P_k$ is an $(\ell_{\min}, \ell_{\max})$ virtual path. A similar argument proves the existence of a disjoint second $(\ell_{\min}, \ell_{\max})$ virtual path containing u_1. Thus F satisfies all of the conditions 1(a)–(d).

Case 2. No ℓ-central vertices of F are adjacent. Then by Lemma 11, every component T' of T is in $\mathcal{T}_1$ and, as before, $\ell_{\min}(T') - 1$ is a gap.

Case 2.1. F has two or more ℓ-central vertices. Since these are not adjacent, $\operatorname{rad} T \leq \ell_{\min} - 1$, which implies that $\ell_{\max}(F) \leq 2\ell_{\min} - 2$. As previously, we can find two disjoint $(\ell_{\min}, \ell_{\max})$ virtual paths.

Case 2.2. F has exactly one ℓ-central vertex, u. Since F is strictly eccentric in T, by Lemma 11 the vertex u is the central vertex of T and we get the desired $(\ell_{\min}, \ell_{\max})$ virtual paths from two maximum length paths starting at u.

We now prove that a labelled forest F satisfying the stated conditions is strictly tree-eccentric. The construction is similar for all three cases, so we consider only the first. Suppose that F is a labelled forest satisfying the conditions 1(a)–(d). Let the ℓ-central vertices of F be p and q. Let $P : P_1, \dots, P_k$ and $Q : Q_1, \dots, Q_k$ be disjoint $(\ell_{\min}, \ell_{\max})$ virtual paths, where $p \in V(P_1)$ and $q \in V(Q_1)$. We now construct a tree T as follows. Let $X = [\ell_{\min}, 2\ell_{\min} - 1] - \ell(F)$ and for each $c \in X$, add two new vertices p_c and q_c. For each $i \in [2, k]$, the number $\ell_{\min}(P_i) - 1 = \ell_{\min}(Q_i) - 1$ is by assumption a gap. Hence there exist p_c and q_c with $c = \ell_{\min}(P_i) - 1$, so join p_c (respectively, q_c) to that vertex v of the component of F containing P_i (Q_i) that has $\ell(v) = \ell_{\min}(P_i)$. Whenever there are two consecutive integers c and $c + 1$ in X, join p_c to p_{c+1} and q_c to q_{c+1}. If $2\ell_{\min} - 1 > \ell_{\max}$, then join $p_{\ell_{\max}+1}$ to a vertex of P_k with label $\ell_{\max}$ and $q_{\ell_{\max}+1}$ to a vertex of Q_k with label $\ell_{\max}$. Finally, for each component T' of F that does not contain some P_i or Q_i, join the vertex v of T' with $\ell(v) = \ell_{\min}(T')$ to either $p_{\ell_{\min}(T')-1}$ or $q_{\ell_{\min}(T')-1}$ (the construction is illustrated in Figure 7). It is straightforward to prove that F is strictly eccentric in the tree T. ■

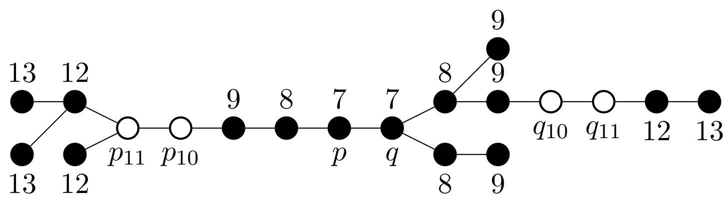

Figure 7. The construction of Theorem 14 applied to a labelled forest F to produce a tree T containing F as a strictly eccentric subgraph. Vertices of F are black and vertices of $T - F$ are white.

We now turn our attention to the special case when the forest F is a tree. If a, b are integers with $a \leq b$ and P an (a, b) virtual path that consists of a single monotone path, then we shall call P an (a, b) *path.* An immediate consequence of Theorem 14 is the following.

Corollary 15. *A labelled tree T is strictly tree-eccentric if and only if*

1. $T \in \mathcal{T}_2$, $\ell_{\max}(T) \leq 2\ell_{\min}(T) - 1$, *and there exist two disjoint* $(\ell_{\min}, \ell_{\max})$ *paths, or,*
2. $T \in \mathcal{T}_1$, $\ell_{\max}(T) \leq 2\ell_{\min}(T)$, *and there exist two* $(\ell_{\min}, \ell_{\max})$ *paths whose intersection is the ℓ-central vertex.*

With a little work, we can restate this result in simpler language. If $T \in \mathcal{T}_1$ and there exist two $(\ell_{\min}, \ell_{\max})$ paths whose intersection is the ℓ-central vertex, or if $T \in \mathcal{T}_2$ and there exist two disjoint $(\ell_{\min}, \ell_{\max})$ paths, then the labelled tree T is *ℓ-balanced.* A tree T is *ℓ-centered* if $T \in \mathcal{T}_1 \cup \mathcal{T}_2$ and its ℓ-center and center are equal. This is equivalent to the statement that there exists an integer k such that for all $v \in V(T)$, $\ell(v) = e(v) + k$.

Lemma 16. *A labelled tree is ℓ-balanced if and only if it is ℓ-centered.*

Proof. Let T be a labelled tree and suppose first that T is ℓ-balanced. If $T \in \mathcal{T}_1$ and u is the ℓ-central vertex, then $e(u) = \ell_{\max} - \ell_{\min}$, while for $v \in V(T) \setminus \{u\}$ we have $e(v) \geq 1 + \ell_{\max} - \ell_{\min}$. Thus u is a central vertex of T. If on other hand c is a central vertex of T, then $e(c) \geq d(c, u) + \ell_{\max} - \ell_{\min}$, which implies that $d(c, u) = 0$ and $c = u$. Thus T is ℓ-centered. A similar proof is easily found when $T \in \mathcal{T}_2$. Suppose then that T is ℓ-centered. If $T \in \mathcal{T}_1$, then T is a central tree with radius $\ell_{\max} - \ell_{\min}$ and diameter $2(\ell_{\max} - \ell_{\min})$, implying the existence of a pair of vertices with label $\ell_{\max}$ that are in different components of $T - u$. A similar proof holds when $T \in \mathcal{T}_2$. ■

The next results follow directly from Theorem 14 and Lemma 16.

Corollary 17. *A labelled tree T is strictly tree-eccentric if and only T is tree-eccentric and ℓ-centered.*

Corollary 18. *A labelled tree T is strictly tree-eccentric if and only if there exists an integer k such that $\ell(v) = e(v) + k$ for all $v \in V(T)$.*

5. Further Directions

Theorem 4 gives a complete characterization of eccentric labelled graphs. We have not, however, found an equivalent characterization of strictly eccentric labelled graphs. While such graphs must satisfy the conditions stated in Theorems 4 and 5, the discussion on page 692 shows that these conditions are not sufficient for a labelled graph to be strictly eccentric. It would clearly be interesting to find such a set of conditions.

In this paper, we have given necessary and sufficient conditions for a tree (or forest) to be eccentric or strictly eccentric in a tree. In a similar vein, given a class $\mathcal{C}$ of graphs, one might ask under what conditions a graph $G \in \mathcal{C}$ is eccentric or strictly eccentric in some graph $H \in \mathcal{C}$.

References

[1] F. Buckley, Z. Miller and P.J. Slater, *On graphs containing a given graph as center*, J. Graph Theory **5** (1981) 427–434.
https://doi.org/10.1002/jgt.3190050413

[2] G. Chartrand and L. Lesniak, Graphs and Digraphs (Chapman and Hall, New York, 1996).

[3] G. Chartrand, M. Schultz and S.J. Winters, *On eccentric vertices in graphs*, Networks **28** (1996) 181–186.
https://doi.org/10.1002/(SICI)1097-0037(199612)28:4¡181::AID-NET2¿3.0.CO;2-H

[4] P. Dankelmann, D.J. Erwin, W. Goddard, S. Mukwembi and H.C. Swart, *Eccentric counts, connectivity and chordality*, Inform. Process. Lett. **112** (2012) 944–947.
https://doi.org/10.1016/j.ipl.2012.08.022

[5] P. Dankelmann, D.J. Erwin, W. Goddard, S. Mukwembi and H.C. Swart, *A characterisation of eccentric sequences of maximal outerplanar graphs*, Australas. J. Combin. **58** (2014) 376–391.

[6] P. Dankelmann, D. Erwin, and H. Swart, *On digraphs with prescribed eccentricities*, J. Combin. Math. Combin. Comput. **104** (2018) 143–160.

[7] D.J. Erwin and F. Harary, *Destroying automorphisms by fixing nodes*, Discrete Math. **306** (2006) 3244–3252.
https://doi.org/10.1016/j.disc.2006.06.004

[8] D. Ferrero and F. Harary, *On eccentricity sequences of connected graphs*, AKCE Int. J. Graphs Comb. **6** (2009) 401–408.

[9] J. Gimbert and N. López, *Eccentricity sequences and eccentricity sets in digraphs*, Ars Combin. **86** (2008) 225–238.

[10] S. Gupta, M. Singh and A.K. Madan, *Eccentric distance sum: A novel graph invariant for predicting biological and physical properties*, J. Math. Anal. Appl. **275** (2002) 386–401.
https://doi.org/10.1016/S0022-247X(02)00373-6

[11] A. Haviar, P. Hrnčiar and G. Monoszová, *Minimal eccentric sequences with least eccentricity three*, Acta Univ. M. Belii Ser. Math. **5** (1997) 27–50.

[12] M. Harminc and J. Ivančo, *Note on eccentricities in tournaments*, Graphs Combin. **10** (1994) 231–234.
https://doi.org/10.1007/BF02986670

[13] L. Lesniak, *Eccentric sequences in graphs*, Period. Math. Hungar. **6** (1975) 287–293. https://doi.org/10.1007/BF02017925

[14] V. Mukungunugwa and S. Mukwembi, *On eccentric distance sum and minimum degree*, Discrete Appl. Math. **175** (2014) 55–61. https://doi.org/10.1016/j.dam.2014.05.019

[15] D. Mubayi and D.B. West, *On the number of vertices with specified eccentricity*, Graphs Combin. **16** (2000) 441–452. https://doi.org/10.1007/PL00007229

[16] R. Nandakumar, On Some Eccentric Properties of Graphs, Ph.D. Thesis (Indian Institute of Technology, New Delhi, 1986).

[17] O.R. Oellermann and S. Tian, *Steiner n-eccentricity sequences of graphs*, Recent Studies in Graph Theory (1989) 206–211.

[18] A. Tamir, *The k-centrum multi-facility location problem*, Discrete Appl. Math. **109** (2001) 293–307. https://doi.org/10.1016/S0166-218X(00)00253-5

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