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EDGE-MAXIMAL GRAPHS WITH CUTWIDTH AT MOST THREE [1]

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Abstract

The cutwidth minimization problem consists of finding an arrangement of the vertices of a graph G on a line P_n with $n = |V(G)|$ vertices, in such a way that the maximum number of edges between each pair of consecutive vertices is minimized. A graph G with cutwidth k ($k \geq 1$) is edge-maximal if $c(G + uv) > k$ for any $uv \in \{uv : u, v \in V(G)$ and $uv \notin E(G)\}$. In this paper, we provide a complete insight to structural properties of edge-maximal graphs with cutwidth at most 3.

Keywords: combinatorics, graph labeling, cutwidth, edge-maximal graph.

2020 Mathematics Subject Classification: 05C75, 05C78, 90C27.

1. Introduction

Graphs in this paper are finite and simple with undefined notations following [1]. The cutwidth of a graph G is the smallest integer k such that the vertices of G are arranged in a linear layout $[v_1, v_2, \ldots, v_n]$ in such a way that, for each $i = 1, 2, \ldots, n-1$, there are at most k edges with one endpoint in $\{v_1, v_2, \ldots, v_i\}$ and the other in $\{v_{i+1}, \ldots, v_n\}$. The cutwidth problem for graphs, together with a class of optimal labeling (or embedding) problems, have significant applications in VLSI designs, network communications and others. In particular, the cutwidth is closely related to a basic parameter, called the congestion, in designing microchip circuits [2, 5, 12]. Here, a graph G may be thought of as a model of the wiring diagram of an electronic circuit, with the vertices representing components and

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the edges representing wires connecting them. When a circuit is laid out on a certain architecture (say a path P_n), the maximum number of overlap wires is the congestion, which is one of major parameters determining the electronic performance. This motivates the cutwidth problem in graph theory practically. Theoretically, the cutwidth is closely related to other graph-theoretic parameters such as bandwidth, modified bandwidth, pathwidth and treewidth among other domains [3, 5, 7, 11].

Deciding whether the cutwidth of G is at most k for a given graph G and an integer k is an NP-complete problem [6], even for graphs with maximum vertex degree 3 [11], but it admits a polynomial algorithm within the family of trees [14]. The cutwidth problem has been extensively examined [5]. First, much work has been done for determining the exact value of the cutwidth of special classes of graphs (see e.g., [5, 8, 10, 13]) and algorithms computing the cutwidth of trees [4, 14]. As to the structure of graphs with cutwidth k ($k \geq 1$), relatively little work has been done. A graph G is called k-cutwidth critical if (1) $c(G) = k$, and (2) $c(G - uv) < k$ for any edge $uv \in E(G)$. In 2004, all five 3-cutwidth critical graphs H'_1, H'_2, H'_3, H'_4 and H'_5 were presented in [9], where H'_1 is star $K_{1,5}$, H'_2 is a tree with diameter 4 obtained by identifying a pendant vertex in three copies of star $K_{1,3}$, H'_3 is obtained from H'_2 by replacing a $K_{1,3}$ by a triangle K_3, H'_4 is a 'crown' made of a cycle C_3 with a pendant edge in each vertex of it, and H'_5 is a cycle C_4 with a chord. It was proved that any 2-cutwidth graph contains no one of H'_1, H'_2, H'_3, H'_4 and H'_5 being an induced subgraph. Similarly, the 4-cutwidth acyclic critical graph class has 18 graphs each of which can be decomposed into three 3-cutwidth minimal subtrees [15, 16]. For $k > 4$, although the structure of the acyclic critical graphs with cutwidth k is obtained in [17], the structural characterization of general graphs with cutwidth k is also a task to study further.

A graph G is k-cutwidth edge-maximal for an integer $k \geq 1$ if (1) $c(G) = k$, and (2) $c(G + uv) > k$ for any edge $uv \in \{v_iv_j : v_i, v_j \in V(G) \text{ and } v_iv_j \notin E(G)\}$. For any integer $k \geq 1$, the k-cutwidth edge-maximal graphs have not been previously studied. In this paper, we present a graph structure which precisely characterizes the class of k-cutwidth edge-maximal graphs for $k \leq 3$.

The rest of this paper is as follows. In Section 2, some preliminaries are presented. Section 3 gives 2-connected forbidden subgraphs of 3-cutwidth graphs. The 2-cutwidth edge-maximal graphs are characterized in Section 4. Section 5 is devoted to presenting the structure of 3-cutwidth edge-maximal graphs. A short remark is given in Section 6.

2. Preliminaries

Suppose that $G = (V(G), E(G))$ is a graph with $|V(G)| = n$. A *labeling* of a graph G is a bijection $\phi : V(G) \to \{1, 2, \ldots, n\}$, viewed as an embedding of G

into the path P_n with vertices in $\{1, 2, \ldots, n\}$, where consecutive integers are the adjacent vertices. The *cutwidth of G with respect to ϕ* is

$$c(G, \phi) = \max_{1 \le j < n} \left|\{uv \in E(G) : \phi(u) \le j < \phi(v)\}\right|, \tag{1}$$

which is also the *congestion* of the embedding. If $k = c(G, \phi)$, then ϕ, as well as the embedding induced by ϕ, is called a *k-cutwidth embedding* or *labeling* of G. The *cutwidth* of G is defined by

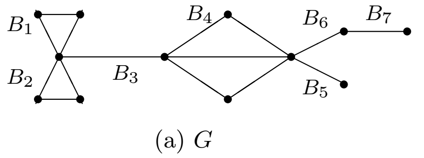
$$c(G) = \min_{\phi} c(G, \phi), \tag{2}$$

where the minimum is taken over all labelings ϕ. A labeling ϕ attaining the minimum in (2) is an optimal labeling. For a graph G, let $S \subset V(G), \bar{S} = V(G) \setminus S$. The edge cut $E[S, \bar{S}]$, i.e., the set of edges of G with one end in S and the other end in $\bar{S}$, is called the *coboundary* of S and denoted by $\partial(S)$, i.e., $\partial(S) = E[S, \bar{S}]$. For a labeling ϕ of G and each $1 \le j < n$, let $S_j^{\phi} = \{v \in V(G) : \phi(v) \le j\}$. Then by (2), we have

$$c(G, \phi) = \max_{1 \le j < n} \left|\partial\big(S_j^{\phi}\big)\right|. \tag{3}$$

In other words, if $v_i = \phi^{-1}(i)$ for $1 \le i < n$, then $S_j = \{v_1, v_2, \ldots, v_j\}$ and $\partial\big(S_j^{\phi}\big) = \{v_i v_h \in E(G) : i \le j < h\}$ (also called the cut at $[j, j+1]$). The cutwidth $c(G, \phi)$ is the maximum size of these coboundaries $\partial\big(S_j^{\phi}\big)$. An ϕ-max-coboundary of G is a $\partial\big(S_j^{\phi}\big)$ achieving the maximum in (3).

For a graph G and integer $i \ge 1$, let $D_i(G) = \{v \in V(G) : d_G(v) = i\}$, where $d_G(v)$ is the degree of vertex $v \in V(G)$, and the maximum degree is denoted as $\Delta(G)$. For each $v \in V(G)$, let $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. For $V' \subset V(G), E' \subset E(G)$ and $V' \ne \emptyset, E' \ne \emptyset$, $G[V'], G[E']$ are an induced subgraph and an edge-induced subgraph of G, respectively. The graph obtained from G by adding an edge $v_1v_2 \notin E(G)$ is denoted as $G + v_1v_2$. If G has a vertex $v \in D_2(G)$ with $N_G(v) = \{v_1, v_2\}$ and $v_1v_2 \notin E(G)$, then $G - v + v_1v_2$, the graph obtained from $G - v$ by adding a new edge v_1v_2, is called a *series reduction* of G. A graph G' is homeomorphic to G if G' is obtained by some series reductions of G. Let G_1 and G_2 be two disjoint graphs with $u \in V(G_1), v \in V(G_2)$. To identify u and v, denoted as $G_1 \odot_{u,v} G_2$, is to replace u, v by a single vertex z (i.e., $u = v = z$) incident to all the edges which were incident to u and v, where z is called the identified vertex. If graph $G = G_1 \odot_{u,v} G_2$, then G is also called the *series composition* of G_1 and G_2. To contract an edge v_1v_2 of graph G is to delete the edge and then identify its ends v_1, v_2. A graph G' is called a *minor* of G if G' is obtained by implementing series reductions and contracting edges from G. Two xy-paths P and Q in G are internally disjoint if they have no internal vertices in common, that is, $V(P) \cap V(Q) = \{x, y\}$. Recall the definition of the bridge of a cycle C [1]. For a connected graph G with cycle C, let N be a component of $G - C$, the graph $G[E(N)]$ is referred to as a *bridge* of C in G together with any edge connecting C with N, denoted as $\mathbb{B}$. For a bridge $\mathbb{B}$ of C, the elements of $V(\mathbb{B}) \cap V(C)$ are called its *vertices of attachment* to C. A bridge with t vertices of attachment is called a *t-bridge.* Let $\{x_1, x_2\} \subset V(C)$ and $\{y_1, y_2\} \subset V(C)$, the two pairs *skew* if and only if they are disjoint and the x-vertices alternate with the y-vertices. Two bridges $\mathbb{B}_1$ and $\mathbb{B}_2$ *skew* if and only if their vertices of attachment skew. $\mathbb{B}_1$ and $\mathbb{B}_2$ *avoid* each other if all the vertices of attachment of $\mathbb{B}_1$ lie in a single segment of $\mathbb{B}_2$; otherwise they *overlap.* A bridge is *simple* if and only if it is a path P. A connected graph that has no cut vertices is called a *block.* Every block with at least three vertices is 2-connected. A block B of a connected graph G is a subgraph that is a block and is maximal with respect to this property. A *block graph* $\mathcal{B}$ of G is the graph whose vertices are the blocks $B_1, B_2, \ldots, B_r$ of G, with B_i, B_j joined if and only if B_i, B_j have a common cut vertex for $1 \le i, j \le r$ (see an example in Figure 1 in which B_3, B_5, B_6 and B_7 are all K_2). From the definition, the following property of the block graph $\mathcal{B}$ is straightforward.



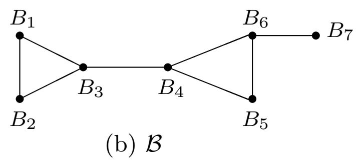


Figure 1. A graph G and its block graph $\mathcal{B}$.

Lemma 2.1. *For an integer $\beta \ge 3$, if the blocks $B_1, B_2, \ldots, B_\beta$ of G have a common cut vertex, then there exists a complete subgraph K_β in the block graph $\mathcal{B}$ of G.*

Definition 1. For a graph G with $|V(G)| = n$ and $c(G) = k$ $(k \ge 1)$, if $c(G + uv) > k$ for any edge $uv \in \{v_iv_j : v_i, v_j \in V(G)$ and $v_iv_j \notin E(G)\}$, then G is called *k-cutwidth edge-maximal.* We denote the set of the class of graphs by $\mathcal{MG}_{n,k}$ $(k \le n)$.

To split a vertex v is to replace v by two adjacent vertices, v' and v'', and to replace each edge incident to v by an edge incident to either v' or v'' (but not both, unless it is a loop at v), the other end of the edge remaining unchanged (Figure 2(a)). To triangulate a vertex v is to split a vertex v by two vertices v' and v'' first, and then to add a new vertex u only connecting v' and v'', respectively (Figure 2(b)).

From the previous definition, the following lemma is trivial.

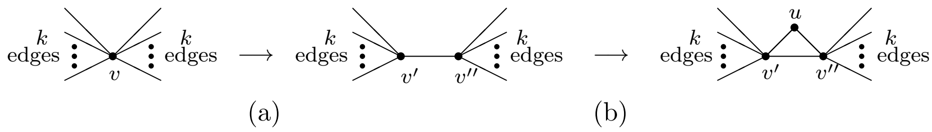


Figure 2. (a) To split a vertex v. (b) To triangulate a vertex v.

Lemma 2.2. *Let G and G' be graphs. Each of the following holds.*

(i) *If G' is a subgraph or minor of G, then $c(G') \leq c(G)$.*

(ii) *If G' is homeomorphic to G, then $c(G') = c(G)$.*

Lemma 2.3. *For a graph $G \in \mathcal{MG}_{n,k}$, let ϕ be an optimal labeling of G with $\phi(v_i) = i$ for $1 \leq i < n$. Each of the following holds.*

(i) *If $|\partial(S_j^\phi)| \leq k-1$ for $1 \leq j < n$, then $v_jv_{j+1} \in E(G)$.*

(ii) *If $v_jv_{j+1} \notin E(G)$, then $|\partial(S_j^\phi)| = k$ for $1 \leq j < n$.*

Proof. For (i), if $v_jv_{j+1} \notin E(G)$, then $\left|\partial\left(S_j^\phi\right) \cup \{v_jv_{j+1}\}\right| \leq (k-1)+1 = k$, contrary to $G \in \mathcal{MG}_{n,k}$. Likewise, since $v_jv_{j+1} \notin E(G)$, if $\left|\partial\left(S_j^\phi\right)\right| < k$, then $\left|\partial\left(S_j^\phi\right) \cup \{v_jv_{j+1}\}\right| \leq (k-1)+1 = k$ for (ii), also contrary to $G \in \mathcal{MG}_{n,k}$. ■

Theorem 2.4. *Let G be a k-cutwidth graph with $|V(G)| = n$, ϕ be an optimal labeling with $\phi(v_i) = i$ for $1 \leq i < n$. Then $G \in \mathcal{MG}_{n,k}$ if and only if there are no two vertices v_j, v_{j+1} with $v_{j-1}v_{j+1} \notin E(G)$ and $v_jv_{j+2} \notin E(G)$ such that $\left|\partial\left(S_j^\phi\right)\right| \leq k-1$ and $\left|\partial\left(S_{j+1}^\phi\right)\right| \leq k-1$, where $v_jv_{j+1} \in E(G), v_{j+1}v_{j+2} \in E(G)$ with $1 \leq j < n-1$.*

Proof. *Sufficiency.* By assumption, for v_j and v_{j+1} with $v_{j-1}v_{j+1} \notin E(G)$ and $v_jv_{j+2} \notin E(G)$, one of the three cases holds under ϕ:

(i) $\left|\partial\left(S_{j-1}^\phi\right)\right| = \left|\partial\left(S_j^\phi\right)\right| = \left|\partial\left(S_{j+1}^\phi\right)\right| = k$;

(ii) $\left|\partial\left(S_{j-1}^\phi\right)\right| = k$, $\left|\partial\left(S_j^\phi\right)\right| \leq k-1$, $\left|\partial\left(S_{j+1}^\phi\right)\right| = k$;

(iii) $\left|\partial\left(S_{j-1}^\phi\right)\right| = \left|\partial\left(S_j^\phi\right)\right| = k$, $\left|\partial\left(S_{j+1}^\phi\right)\right| \leq k-1$.

Assume towards a contradiction that $G \notin \mathcal{MG}_{n,k}$. Then $c(G+uv) = k$ for some $uv \notin E(G)$ because otherwise $c(G+uv) \geq c(G)$. Let ϕ' be an optimal labeling of $G+uv$ such that $c(G+uv, \phi') = k$. Then ϕ' must be an optimal labeling of G (as otherwise $c(G, \phi') \leq k-1$, a contradiction), say $\phi = \phi'$. However, let $u = v_j$, $v \in V(G) \backslash \{v_j\}$. Then, for each case above, $c(G+uv, \phi') = c(G+uv, \phi) = k+1$, a contradiction. So $G \in \mathcal{MG}_{n,k}$.

Necessity. Suppose to the contrary that there exist two vertices v_j and v_{j+1} with $v_{j-1}v_{j+1} \notin E(G)$ and $v_jv_{j+2} \notin E(G)$ such that $\left|\partial\left(S_j^\phi\right)\right| \leq k-1$ and $\left|\partial\left(S_{j+1}^\phi\right)\right| \leq k-1$ under ϕ and $v_jv_{j+1} \in E(G)$, $v_{j+1}v_{j+2} \in E(G)$ by Lemma 2.3. As $v_jv_{j+2} \notin E(G)$, $\left|\partial\left(S_j^\phi\right) \cup \{v_jv_{j+2}\}\right| \leq (k-1)+1 = k$ and $\left|\partial\left(S_{j+1}^\phi\right) \cup\right.$

$\{v_jv_{j+2}\}\big| \le (k-1)+1 = k$. By $|V(G+v_jv_{j+2})| = |V(G)|$ again, similar to Sufficiency, let ϕ' be a labeling of $G+v_jv_{j+2}$ and $\phi' = \phi$. Then, by $c(G,\phi) = c(G) = k$, $\big|\partial\big(S_{j'}^{\phi'}\big)\big| = \big|\partial\big(S_{j'}^{\phi}\big)\big| \le k$ when $v_{j'} \ne v_j, v_{j+1}$. Thus $\big|\partial\big(S_j^{\phi'}\big)\big| \le k$ for each $1 \le j < n$. By (3), $c(G+v_jv_{j+2},\phi') = k$ implying $c(G+v_jv_{j+2}) \le k$ by (2). So, by $c(G+v_jv_{j+2}) \ge k$, $c(G+v_jv_{j+2}) = k$ contradicting $G \in \mathcal{MG}_{n,k}$. ■

3. Two-Connected Forbidden Subgraphs

In this section, we give eleven 2-connected forbidden subgraphs of 3-cutwidth graphs in Figure 3, where the empty dots in R_{10} imply that two corresponding edges either intersect or not.

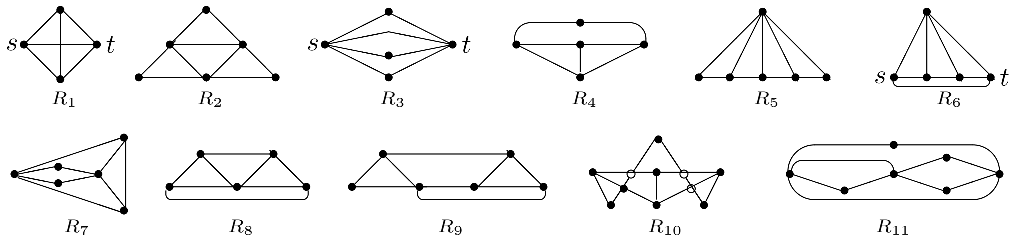


Figure 3. 2-connected forbidden subgraphs of 3-cutwidth graphs.

Lemma 3.1 [10]**.** *For a bipartite graph $K_{m,n}$, $c(K_{m,n}) = \lfloor \frac{m}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor + \lceil \frac{m}{2} \rceil \times \lceil \frac{n}{2} \rceil$.*

Lemma 3.2. *Let $G(s,t)$ be a k-paths graph comprised of k internally-disjoint paths $P_i = sv_{i1}v_{i2}\cdots v_{it_i}t$ $(1 < i \le k, 1 \le t_i < n)$. Then $c(G(s,t)) = k$. In particular, if $k \in \{2,3\}$ and $t_i = 1$ for each $1 < i \le k$, then $G(s,t) \in \mathcal{MG}_{k+2,k}$.*

Proof. Let $G'(s,t)$ be a k-paths graph in which the length of each P_i is two, i.e., $t_i = 1$ for each $1 < i \le k$. Then $G'(s,t)$ is homeomorphic to $G(s,t)$. Since $G'(s,t)$ can be thought of as a bipartite graph $K_{k,2}$ with bipartition (X,Y), where $X = \{v_{11}, v_{21}, \ldots, v_{k1}\}$ and $Y = \{s,t\}$, $c(G'(s,t)) = \lfloor \frac{k}{2} \rfloor + \lceil \frac{k}{2} \rceil = k$ by Lemma 3.1. Thus $c(G(s,t)) = c(G'(s,t)) = k$ by homeomorphism. For $k = 2$ and $t_i = 1$ $(1 < i \le 2)$, $G(s,t) = C_4$ and $C_4 \in \mathcal{MG}_{4,2}$ clearly. For $k = 3$ and $t_i = 1$ $(1 < i \le 3)$, $G(s,t) = R_3 - st$, which can be easily verified that $G(s,t) \in \mathcal{MG}_{5,3}$. This completes the proof. ■

Lemma 3.3 [1]**.** *A graph G with $|V(G)| \ge 3$ is 2-connected if and only if any two vertices $x, y \in V(G)$ are connected by at least two internally-disjoint paths $P_1(x,y)$ and $P_2(x,y)$.*

Theorem 3.4. *For a 2-connected graph G, $c(G) \le 3$ if and only if G does not contain any subgraph R_i $(1 \le i \le 11)$ in Figure 3 as its minor.*

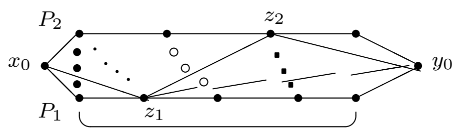


Figure 4. Illustration of Case 3(a).

Proof. Necessity is straightforward. We now show sufficiency by contradiction. Suppose that G is a minimum 2-connected graph with $c(G) = k$ for $k \geq 4$. By Lemma 3.3, there are at least two internally-disjoint paths between x_0 and y_0 in G. Since G contains no R_3 and no R_{11}, there are at most three internally-disjoint paths between x_0 and y_0. Hence three are three cases that need to be considered.

Case 1. There are only two internally-disjoint paths $P_1(x_0, y_0) = x_0x_1x_2\cdots x_py_0$ and $P_2(x_0, y_0) = x_0y_1y_2\cdots y_qy_0$ between x_0 and y_0. In this case, $P_1(x_0, y_0)\cup P_2(x_0, y_0)$ is a $(p+q+2)$-cycle C_{p+q+2} with cutwidth two. By the assumption that G is minimum 2-connected, without loss of generality, we can let every bridge of C_{p+q+2} be simple. So, from the structure of G and the assumption of $c(G) \geq 4$, it suffices to consider the following three subcases.

Subcase 1.1. For a vertex $x_{i'}$ $(1 \leq i' \leq p)$, there are at most three vertices in $V(C_{p+q+2})\setminus\{x_{i'-1}, x_{i'+1}\}$, say y_1, y_2 and y_3, such that $(x_{i'}, y_1)$-path, $(x_{i'}, y_2)$-path and $(x_{i'}, y_3)$-path are avoiding 2-bridges of C_{p+q+2}. In this case, $3 \leq d_G(v_{i'}) \leq 5$. Respectively, if $(x_{i'}, y_1)$-path is a unique 2-bridge then $d_G(v_{i'}) = 3$. If $(x_{i'}, y_1)$-path and $(x_{i'}, y_2)$-path are only two 2-bridges then $d_G(v_{i'}) = 4$. If, for each $1 \leq j \leq 3$, $(x_{i'}, y_j)$-path is a 2-bridge, then $d_G(v_{i'}) = 5$. Thus, if $d_G(v_{i'}) = 3$ then there are at least two vertices x_{i_1} with $i_1 < i'$ and x_{i_2} with $i_2 > i'$ such that $x_{i_1}x_{i_2} \in E(G)$ (because $c(G) = 3$, otherwise), where $x_{p+1} = y_0$. This means that R_4 is a minor. Similarly, if $d_G(v_{i'}) = 4$ then R_6, R_8 are minors, because there are also at least two vertices, say $x_{i'-1}$ and $x_{i'+1}$, such that $x_{i'-1}x_{i'+1} \in E(G)$. If $d_G(v_{i'}) = 5$, then R_5 is a minor. All are contrary to assumption.

Subcase 1.2. There are at least two avoiding 2-bridges which have no common vertices of attachment. Let $x_{i_1}y_{j_1}, x_{i_2}y_{j_2}$ be such two 2-bridges, where $1 \leq i_1 < i_2 \leq p, 1 \leq j_1 < j_2 \leq q$. Since $c(G) \geq 4$, then there are at least two vertices, say y_{j_1} and y_{j_2+1}, such that $y_{j_1}y_{j_2+1} \in E(G)$ (as otherwise $c(G) = 3$), where $y_{q+1} = y_0$. This shows that R_9 is a minor in G, a contradiction.

Subcase 1.3. There are at least two skewing 2-bridges in G. That is to say, there are at least four vertices, say $x_{i'_1}, x_{i'_2}$ $(1 \leq i'_1 < i'_2 \leq p)$ and two of $\{x_i : 1 \leq i \leq p \text{ and } i \neq i'_1 - 1, i'_1, i'_1 + 1, i'_2 - 1, i'_2, i'_2 + 1\} \cup \{y_i : 1 \leq i \leq q\}$, say y_1 and y_2, such that $(x_{i'_1}, y_2)$-path and $(x_{i'_2}, y_1)$-path are skewing 2-bridges. In this subcase, R_1 is a minor of G, also a contradiction.

Case 2. There are three internally-disjoint paths $P_1(x_0, y_0) = x_0x_1x_2\cdots x_py_0$, $P_2(x_0, y_0) = x_0y_1y_2\cdots y_qy_0$ and $P_3(x_0, y_0) = x_0z_1z_2\cdots z_ly_0$ between x_0 and y_0.

If $G = P_1(x_0, y_0) \cup P_2(x_0, y_0) \cup P_3(x_0, y_0)$ then $c(G) = 3$ by Lemma 3.2. So, by assumption, among three cycles C_{p+q+2}, C_{p+l+2} and C_{q+l+2}, there are at least a cycle, say C_{p+q+2} also, such that C_{p+q+2} has at least a simple 2-bridge whose vertices of attachment are neither x_0 nor y_0, which results in an R_4 minor of G, also a contradiction.

Case 3. There is a path $P_3(x_0, y_0) = x_0 z_1 z_2 \cdots z_h y_0$ such that at least one of $\{V(P_1(x_0, y_0) \cap P_3(x_0, y_0)), V(P_2(x_0, y_0) \cap P_3(x_0, y_0))\}$ is not empty, but $E(P_1(x_0, y_0) \cap P_3(x_0, y_0)) = E(P_2(x_0, y_0) \cap P_3(x_0, y_0)) = \emptyset$. By the assumption that G is minimum, without loss of generality, let $h = 3$, i.e., $P_3(x_0, y_0) = x_0 z_1 z_2 y_0$. Then there are two subcases: (a) $V(P_1(x_0, y_0) \cap P_3(x_0, y_0)) \neq \emptyset$, $V(P_2(x_0, y_0) \cap P_3(x_0, y_0)) \neq \emptyset$ (see Figure 4). Assume that $V(P_1(x_0, y_0) \cap P_3(x_0, y_0)) = \{z_1\}$, $V(P_2(x_0, y_0) \cap P_3(x_0, y_0)) = \{z_2\}$. By $c(G) \geq 4$, $P_1(x_0, y_0) \cup P_2(x_0, y_0)$ must contain at least one simple 2-bridge except $E(P_3(x_0, y_0))$ in G, see the different dotted lines in Figure 4. This implies that one of $\{R_1, R_2, R_4, R_5, R_8, R_{10}\}$ must be a minor of G, a contradiction. (b) $V(P_1(x_0, y_0) \cap P_3(x_0, y_0)) \neq \emptyset$ but $V(P_2(x_0, y_0) \cap P_3(x_0, y_0)) = \emptyset$. In this case, similar to (a), one of $\{R_3, R_4, R_5, R_{10}, R_{11}\}$ is a minor in G, also a contradiction. So $c(G) \leq 3$. The proof is completed. ∎

4. Edge-Maximal Graphs with Cutwidth at Most 2

For any cycle $C_{\mu+2}$ ($\mu \geq 1$) with $v_l, v_r \in V(C_{\mu+2})$, if $P_1 = v_l v_1 v_2 \cdots v_i v_r$, $P_2 = v_l v_{i+1} v_{i+2} \cdots v_\mu v_r$ are the internally-disjoint paths forming $C_{\mu+2}$, then v_l, v_r are viewed as the left terminal and the right terminal, respectively. Since $c(K_{1,5}) = 3, c(R_1 - st) = 3$ (see Figure 3), there are no $K_{1,5}$ or no $R_1 - st$ induced subgraph or minor in any graph with cutwidth 2. First, Lemmas 4.1 and 4.2 are straightforward.

Lemma 4.1. *A graph $G \in \mathcal{MG}_{n,1}$ if and only if G is a path P_n with n vertices for $n \geq 2$.*

Lemma 4.2. *For cycle $C_{\mu+2}$ with $\mu \geq 1$, $C_{\mu+2} \in \mathcal{MG}_{\mu+2,2}$.*

Now, for $1 \leq j \leq \beta$, let $\mu = \mu_j$, $l = l_j, r = r_j, i = i_j$ with $1 \leq i_j < \mu_j$, and let $P_1^j = v_{l_j} v_1^j v_2^j \cdots v_{i_j}^j v_{r_j}$, $P_2^j = v_{l_j} v_{i_j+1}^j v_{i_j+2}^j \cdots v_{\mu_j}^j v_{r_j}$ be two paths forming C_{μ_j+2}, where v_{l_j} and v_{r_j} are the left terminal and the right terminal respectively. By identifying v_{r_j} of C_{μ_j+2} and $v_{l_{j+1}}$ of $C_{\mu_{j+1}+2}$ for each $1 \leq j \leq \beta - 1$ (i.e., $v_{r_j} = v_{l_{j+1}} = z_j$) consecutively, one can obtain the series composition H_0 of $C_{\mu_1+2}, C_{\mu_2+2}, \ldots, C_{\mu_\beta+2}$ with the left terminal v_{l_1} ($= z_0$) and the right terminal v_{r_β} ($= z_\beta$) and $|V(H_0)| = \sum_{i=1}^{\beta} \mu_j + \beta + 1$ (see H_0 with 4 cycles in Figure 5(a)).

Clearly, $c(H_0) \geq 2$ by Lemma 4.2. For each $1 \leq j \leq \beta$, C_{μ_j+2} is a block B_j with the left terminal z_{j-1} and the right terminal z_j in H_0, and the block graph $\mathcal{B}$ of H_0 is a path P_β, where $z_0 = v_{l_1}, z_\beta = v_{r_\beta}$. Suppose that $\phi : V(H_0) \mapsto \{1, 2, \ldots, |V(H_0)|\}$ is an optimal labeling of H_0 such that its sublabeling ϕ_j restricted to C_{μ_j+2} is

$$\phi_j(v) = \begin{cases} \sum_{i=0}^{j-1} \mu_i + j & \text{if } v = z_{j-1}, \\ \sum_{i=0}^{j-1} \mu_i + j + i_j & \text{if } v \in V\big(P_1^j\big) \cup V\big(P_2^j\big), \\ \sum_{i=1}^{j} \mu_i + j + 1 & \text{if } v = z_j\,, \end{cases}$$

for $1 \le j \le \beta$ and $1 \le i_j \le \mu_j$, where $\mu_0 = 0$. Since $c(C_{\mu_j+2}, \phi_j) = 2$ and $V(C_{\mu_j}) \cap V(C_{\mu_{j+1}}) = z_j$ with $\phi_j(z_j) = \max\{\phi_j(v) : v \in V(C_{\mu_j})\} = \min\{\phi_{j+1}(v) : v \in V(C_{\mu_{j+1}})\}$, $c(H_0, \phi) = 2$. Hence $c(H_0) = 2$. On the other hand, for any $x, y \in V(H_0)$ with $xy \notin E(H_0)$, if $x, y \in V(C_{\mu_j+2})$ for some j, then by Lemma 4.2 $c(C_{\mu_j+2}+xy) = 3$; if $x \in V(C_{\mu_{j_1}+2}), y \in V(C_{\mu_{j_2}+2})$ with $j_1 < j_2$, then, for any labeling ϕ' of H_0+xy with $\phi'(z_{j_1}) = \rho$, $|\partial(S_\rho^{\phi'})| \ge 3$. So, by (3), $c(H_0+xy, \phi') \ge 3$ resulting in $c(H_0 + xy) \ge 3$ too. So $H_0 \in \mathcal{MG}_{n,2}$ with $n = |V(H_0)|$.

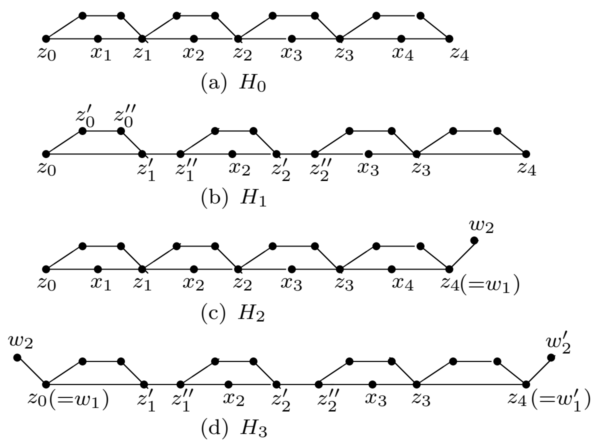


Figure 5. Four 2-cutwidth edge-maximal graphs.

There are $\beta - 1$ cut vertices $z_1, z_2, \ldots, z_{\beta-1}$ with degree 4 in H_0. For an integer ξ $(1 \le \xi \le \beta - 1)$ and $\{z_{l_1}, z_{l_2}, \ldots, z_{l_\xi}\} \subseteq \{z_1, z_2, \ldots, z_{\beta-1}\}$, we carry out three operations in H_0 at the same time: (1) splitting $z_{l_1}, z_{l_2}, \ldots, z_{l_\xi}$ respectively; (2) choosing ξ vertices $x_1, x_2, \ldots, x_\xi$ with degree 2 arbitrarily; and (3) for $x_1, x_2, \ldots, x_\xi$, implementing ξ series reductions consecutively in order to keep $|V(H_0)|$ constant. Let H_ξ be the graph obtained by carrying out the above operations for z_{l_i} and x_i $(1 \le i \le \xi)$, and $\mathcal{H}_\xi^\beta = \{H_\xi : 0 \le \xi \le \beta - 1\}$, $\dot{\mathcal{H}}_\xi^\beta = \{H' : H' = H \odot_{z_0,w_1} K_2, H \in \mathcal{H}_\xi^\beta\}$, $\ddot{\mathcal{H}}_\xi^\beta = \{H'' : H'' = H' \odot_{z_\beta,w_1'} K_2', H' \in \dot{\mathcal{H}}_\xi^\beta\}$, where $K_2 = w_1w_2, K_2' = w_1'w_2'$. For example, H_1 in Figure 5(b) is obtained by splitting z_1, z_2 and implementing 2 series reductions for x_1, x_4 in H_0 at the same time,

$H_2 = H_0 \odot_{z_4,w_1} K_2$ in Figure 5(c), and $H_3 = (H_1 \odot_{z_0,w_1} K_2) \odot_{z_4,w'_1} K'_2$ in Figure 5(d). So $H_1 \in \mathcal{H}^4_2$, $H_2 \in \dot{\mathcal{H}}^4_0$, $H_3 - z_4 w'_2 \in \dot{\mathcal{H}}^4_2$ and $H_3 \in \ddot{\mathcal{H}}^4_2$. Let $\mathcal{H} = \bigcup_{\xi=0}^{\beta-1} \mathcal{H}^\beta_\xi$, $\dot{\mathcal{H}} = \bigcup_{\xi=0}^{\beta-1} \dot{\mathcal{H}}^\beta_\xi$ and $\ddot{\mathcal{H}} = \bigcup_{\xi=0}^{\beta-1} \ddot{\mathcal{H}}^\beta_\xi$. For $G \in \mathcal{H} \cup \dot{\mathcal{H}} \cup \ddot{\mathcal{H}}$, its block B_i is either a cycle C_μ or a K_2, the block graph $\mathcal{B}$ is a path P_h with h vertices, where

$$h = \begin{cases} \beta & \text{if } G = H_0, \\ \beta+1 & \text{if } G = H_0 \odot_{z_0,w_1} K_2, \\ \beta+2 & \text{if } G = (H_0 \odot_{z_0,w_1} K_2) \odot_{z_\beta,w'_1} K'_2, \\ \beta+\xi & \text{if } G \in \mathcal{H}^\beta_\xi \text{ but } G \neq H_0, \\ \beta+\xi+1 & \text{if } G \in \dot{\mathcal{H}}^\beta_\xi \text{ but } G \neq H_0 \odot_{z_0,w_1} K_2, \\ \beta+\xi+2 & \text{if } G \in \ddot{\mathcal{H}}^\beta_\xi \text{ but } G \neq (H_0 \odot_{z_0,w_1} K_2) \odot_{z_\beta,w'_1} K'_2. \end{cases}$$

Using a similar argument to that of the proof of H_0, we can get the following lemma.

Lemma 4.3. *Assume that graphs $H \in \mathcal{H}$, $H' \in \dot{\mathcal{H}}$ and $H'' \in \ddot{\mathcal{H}}$. Then $H \in \mathcal{MG}_{n,2}$, $H' \in \mathcal{MG}_{n+1,2}$ and $H'' \in \mathcal{MG}_{n+2,2}$.*

Theorem 4.4. *For a graph G with $|V(G)| = n$ and block B_i $(1 \leq i \leq \beta)$, $G \in \mathcal{MG}_{n,2}$ if and only if each of the following holds.*

(i) *For each block B_i of G, either $B_i = C_{\mu_i}$ with $\mu_i \geq 3$ or $B_i = K_2$.*

(ii) *For each $1 \leq i \leq \beta - 1$, at least one member of $\{B_i, B_{i+1}\}$ is not K_2.*

(iii) *The block graph $\mathcal{B}$ of G is a path P_β.*

Proof. By Lemma 4.3, it suffices to show its necessity by contradiction. First assume that there is at least a block B_{i_0} such that $B_{i_0} \neq C_{\mu_{i_0}}$ and $B_{i_0} \neq K_2$ in G, which implies that some C'_μs and K'_2s must be the proper subgraphs of B_{i_0}. For instance, $B_{i_0} = H_1 + z_0 z_4$ containing two C'_4s, two C'_5s and two K'_2s (see H_1 in Figure 5). Without loss of generality, let B_{i_0} be a minimum block that contains these C'_μs and K'_2s. Then, without considering the vertex number of each C'_μ by homeomorphism, we have

Claim 1. *B_{i_0} must be homeomorphic to one of the six graphs in Figure 6, where any of $\{u, v\}$ can be viewed as a cut vertex of G by homeomorphism.*

In fact, by the minimality of B_{i_0}, there is an edge $uv \in E(G)$ such that (i) holds in $G - uv$, in which either (1) $u, v \in V(C_{\mu_{i_0}})$ or (2) $u \in V(C_{\mu_{i_0}})$ and $v \notin V(C_{\mu_{i_0}})$, in which any of u and v may be either a cut vertex or not in G. For case (1), it is clear that $B_{i_0} - uv$ is Figure 6(a). For case (2), there are two subcases to consider: (a) $B_{i_0} - uv$ contains two blocks which are either a C_{μ_1} and a C_{μ_2} or a C_{μ_1} and a K_2. In this case, B_{i_0} must be one of Figure 6(b) and Figure 6(c). (b) $B_{i_0} - uv$ contains at least three blocks $B^{(1)}_{i_0}, B^{(2)}_{i_0}, \ldots, B^{(\rho)}_{i_0}$ $(\rho \geq 3)$. If $B^{(1)}_{i_0} = C_{\mu_1}$ and $B^{(\rho)}_{i_0} = C_{\mu_2}$ then B_{i_0} must be Figure 6(d); if $B^{(1)}_{i_0} = C_{\mu_1}$ and

$B_{i_0}^{(\rho)} = K_2$ (or $B_{i_0}^{(1)} = K_2$ and $B_{i_0}^{(\rho)} = C_{\mu_\rho}$) then B_{i_0} must be Figure 6(e); if $B_{i_0}^{(1)} = K_2$ and $B_{i_0}^{(\rho)} = K_2$ then B_{i_0} must be Figure 6(f), where $B_{i_0}^{(r)}$ is either a C_{μ_r} or a K_2 for every $1 < r < \rho$. Thus Claim 1 holds.

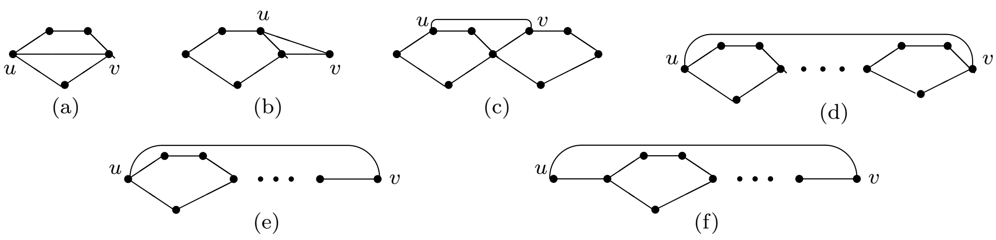


Figure 6. Six possible graphs homeomorphic to B_{i_0}.

Hence, by Claim 1, B_{i_0} is homeomorphic to Figure 6(a) for case (1), and B_{i_0} is homeomorphic to one of the graphs Figure 6(b)–(f) for case (2). Since the cutwidth of each of the graphs in Figure 6 is three, i.e., $c(B_{i_0}) = 3$ by Lemma 2.2(ii), $c(G) \geq 3$ by Lemma 2.2(i), contrary to $c(G) = 2$. Hence $B_i = C_{\mu_i}$ or $B_i = K_2$ for each block B_i of G, which shows that (i) holds.

Second, assume that B_i and B_{i+1} are both K_2 for $1 \leq i \leq \beta - 1$, and $B_i = z_1z_2, B_{i+1} = z_2z_3$. Then $c(G + z_1z_3) = 2$, contradicting $G \in \mathcal{MG}_{n,2}$. So (ii) holds.

For (iii), assume to the contrary that the block graph $\mathcal{B}$ of G is not a path P_β, then $\mathcal{B}$ contains at least a complete graph K_r ($r \geq 3$) by Lemma 2.1, say $K_3 = B_1B_2B'B_1$, and let P_h be the path with maximum length h in $\mathcal{B}$, where the common cut vertex of B_1, B_2, B' is z_{i_0} and $\{B_1, B_2\} \subset V(P_h)$. There are four cases to consider by (ii): (1) $B_1 = C_{\mu_1}, B_2 = C_{\mu_2}, B' = C_{\mu_3}$; (2) $B_1 = C_{\mu_1}, B_2 = C_{\mu_2}, B' = K_2$; (3) $B_1 = C_{\mu_1}, B_2 = K_2, B' = C_{\mu_3}$; (4) $B_1 = C_{\mu_1}, B_2 = K_2, B' = K_2$. It is easy to verify that cases (1), (2) and (3) are not possible, as $K_{1,5}$ is a subgraph in G for each of them, which leads to $c(G) \geq 3$, a contradiction. For case (4), let $B_2 = z_{i_0}z_{i_0+1}, B' = z_{i_0}w$, then $c(G + wz_{i_0+1}) = 2$ contradicting $G \in \mathcal{MG}_{n,2}$. So case (4) is not possible. Thus B' does not exist, and $\mathcal{B}$ is a path P_β. ■

Corollary 4.5. *Let $G \in \mathcal{MG}_{n,2}$ with block B_i, $\phi_i : V(B_i) \mapsto \{1, 2, \ldots, |V(B_i)|\}$ be an optimal labeling of B_i $(0 \leq i \leq \beta)$. Then $\phi_{i,i+1}$ is an optimal labeling of $B_i \cup B_{i+1}$ for each $0 \leq i \leq \beta - 1$, where*

$$\phi_{i,i+1}(v) = \begin{cases} \phi_i(v) & \textit{if } v \in V(B_i), \\ \phi_{i+1}(v) + |V(B_i)| - 1 & \textit{if } v \in V(B_{i+1}) \setminus \{z_i\}, \end{cases}$$

and $V(B_i) \cap V(B_{i+1}) = \{z_i\}$, $B_0 = z_0$.

5. Edge-Maximal Graphs with Cutwidth at Most 3

In this section, by Theorem 3.4, 3-cutwidth edge-maximal graphs are investigated carefully. For convenience, $P_{x,y}$ will be a path between x and y instead of $P(x,y)$. The following is Kuratowski's Theorem, which can be seen in [1].

Theorem 5.1 (Kuratowski)**.** *A graph is planar if and only if it contains no subdivision of either K_5 or $K_{3,3}$.*

A 2-tree T is recursively defined as follows: (1) K_3 is a 2-tree; (2) If T is a 2-tree, the graph obtained from T by joining a new vertex to two vertices of a K_3 in T is also a 2-tree. Clearly, 2-tree T with m ($m \geq 2$) inner faces is planar. The dual T^* of T is defined as follows: corresponding to each face f of T there is a vertex f^* of T^*, and corresponding to each edge e of T there is an edge e^* of T^*; two vertices f^* and g^* are joined by e^* in T^* if and only if the corresponding faces f and g are separated by e in T. If f_0^* of T^* is the vertex corresponding to the outer face f_0 of T with $\Delta(T) = 4$, and $T^* - f_0^*$ is a path P_m, then we call T *linear*. In a linear 2-tree T, except two 3 degree vertices x, y and two 2 degree vertices x', y', $d_T(v) = 4$ for every $v \in V(T) \setminus \{x, y, x', y'\}$. Such a 2-tree T is denoted by $LT(x,y)$, and we can easily obtain

Lemma 5.2. *For each $LT(x,y)$, $LT(x,y) \in \mathcal{MG}_{\mu,3}$, where $|V(LT(x,y))| = \mu$.*

A 3-paths graph $G(s,t)$ is a graph formed by three internally-disjoint paths $P_{s,t}^{(1)}$, $P_{s,t}^{(2)}$, $P_{s,t}^{(3)}$ with two common vertices s, t. Any 3-paths graph $G(s,t)$ is planar with two inner faces f_1 and f_2. By Lemma 3.2, $c(G(s,t)) = 3$, and $G(s,t) \in \mathcal{MG}_{\mu,3}$ if and only if $G \in \{R_1 - st, R_3 - st\}$ (see R_1, R_3 in Figure 3).

Definition 2. A graph $G(x,y)$ is a simple graph consisting of three edge-disjoint paths $P_{x,y}^{(1)}, P_{x,y}^{(2)}$, $P_{x,y}^{(3)}$ with vertices x, y in common. If $P_{x,y}^{(i)} \cup P_{x,y}^{(3)} \in \mathcal{MG}_{\mu_{i3},2}$ for $i = 1, 2$, then $G(x,y)$ is said to be *linear*, denoted as $LG(x,y)$, where $\mu_{i3} = \left|V\left(P_{x,y}^{(i)} \cup P_{x,y}^{(3)}\right)\right|$, x and y are called the 3-*degree gluing points* in $LG(x,y)$.

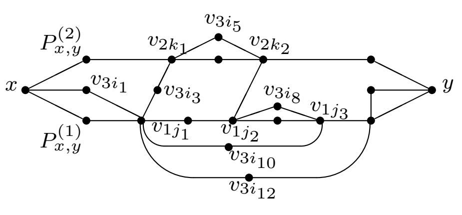
From Definition 2, $LG(x,y) = P_{x,y}^{(1)} \cup P_{x,y}^{(2)} \cup P_{x,y}^{(3)}$ and is 2-connected, in which $P_{x,y}^{(i)} \cup P_{x,y}^{(3)}$ ($i = 1, 2$) either has a configuration as H_0 in Figure 5(a) or is a single cycle C. A linear 2-tree $LT(x,y)$ and a 3-paths graph $G(s,t)$ with $s = x, t = y$ are two special cases of $LG(x,y)$. In the following statements, we will let $V\left(P_{x,y}^{(1)} \cap P_{x,y}^{(3)}\right) = \{v_{1j_1}, v_{1j_2}, \ldots, v_{1j_{m_0}}\}$ and $V\left(P_{x,y}^{(2)} \cap P_{x,y}^{(3)}\right) = \{v_{2k_1}, v_{2k_2}, \ldots, v_{2k_{r_0}}\}$ (possibly empty) except x and y, and let $P_{x,y}^{(1)}, P_{x,y}^{(2)}$ be internally-disjoint by Lemma 3.3.

Lemma 5.3. *Each $LG(x,y)$ with maximum degree 4 is planar.*

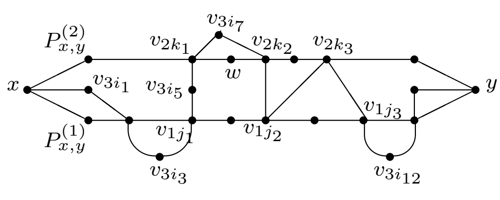
Proof. By contradiction. Suppose that $LG(x,y)$ contains a subdivision of K_5 by Theorem 5.1. By the assumption that $LG(x,y)$ is simple, $LG(x,y)$ does not contain double edges. So $|V(LG(x,y))| \geq 5$. Since $d_{K_5}(v) = 4$ for $v \in V(K_5)$ but $d_{LG(x,y)}(x) = d_{LG(x,y)}(y) = 3$, x and y are not the 4-degree vertices of the subdivision of K_5. Now let $P^{(1)}_{x,y} = xv_{11}v_{12}\cdots v_{1t_1}y, P^{(2)}_{x,y} = xv_{21}v_{22}\cdots v_{2t_2}y$ and $P^{(3)}_{x,y} = xv_{31}v_{32}\cdots v_{3t_3}y$ be the three edge-disjoint paths consisting of $LG(x,y)$ respectively, where $P^{(1)}_{x,y}, P^{(2)}_{x,y}$ are also internally-disjoint by Lemma 3.3. Then there are three cases to consider.

Case 1. $P^{(1)}_{x,y}, P^{(2)}_{x,y}$ and $P^{(3)}_{x,y}$ are internally-disjoint one another. In this case, $LG(x,y)$ is a 3-paths graph in which $d_{LG(x,y)}(x) = d_{LG(x,y)}(y) = 3$, $d_{LG(x,y)}(v) = 2$ for any $v \in V(LG(x,y))\backslash\{x,y\}$. So, by $d_{K_5}(v) = 4$ for any $v \in V(K_5)$, $LG(x,y)$ can not contain a subdivision of K_5.

Case 2. $P^{(1)}_{x,y}$ and $P^{(2)}_{x,y}$, $P^{(1)}_{x,y}$ and $P^{(3)}_{x,y}$ are internally-disjoint, respectively. If there are at most 4 common vertices except x,y in $V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right)$, then the subdivision of K_5 does not exist in $LG(x,y)$. So, without loss of generality, assume that $V(K_5) = \{v_{2k_1}, v_{2k_2}, v_{2k_3}, v_{2k_4}, v_{2k_5}\} \subset V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right)$ and $v_{2k_j} \neq x,y$ for $1 \leq j \leq 5$. Then there is at least a vertex, say v_{2k_1}, such that either $v_{2k_1}v_{2k_5} \in E(LG(x,y))$ or there are at least three subpaths between v_{2k_1} and v_{2k_5} in $P^{(2)}_{x,y} \cup P^{(3)}_{x,y}$, i.e., $d_{LG(x,y)}(v_{2k_1}) \geq 5$ by $d_{K_5}(v_{2k_1}) = 4$ (because otherwise $P^{(2)}_{x,y} \cup P^{(3)}_{x,y}$ is homeomorphic to H_0 in Figure 5, which leads to that the subdivision of K_5 does not exist in $LG(x,y)$). In fact, for the latter, a subpath $P_{v_{2k_1},v_{2k_5}}\left(\subset P^{(3)}_{x,y}\right)$ between v_{2k_1} and v_{2k_5} can be thought of as a subdivision of edge $v_{2k_1}v_{2k_5}$, say $P_{v_{2k_1},v_{2k_5}} = v_{2k_1}v_{3i_6}v_{2k_5}$. Thus $P^{(3)}_{x,y} = x\cdots v_{3i_1}v_{2k_1}v_{3i_2}v_{2k_2}v_{3i_3}v_{2k_3}v_{3i_4}v_{2k_4}v_{3i_5}v_{2k_5}v_{3i_6}v_{2k_1}\cdots y$, which results in $d_{LG(x,y)}$ $(v_{2k_1}) \geq 6$, contrary to $\Delta(LG(x,y)) = 4$ as well as the linearity of $P^{(2)}_{x,y} \cap P^{(3)}_{x,y}$.



(a) Illustration of Case 3.



(b) Illustration of Case 6.

Figure 7. Proof of Lemma 5.4.

Case 3. Only $P^{(1)}_{x,y}$ and $P^{(2)}_{x,y}$ are internally-disjoint. By Case 2, without loss of generality, let $V(K_5) \subseteq V\left(P^{(1)}_{x,y} \cap P^{(3)}_{x,y}\right) \cup V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right)$. Then we can let $v_{1j_1}, v_{1j_2}, v_{1j_3}$, v_{2k_1}, v_{2k_2} form a subdivision of K_5 with $\{v_{1j_1}, v_{1j_2}, v_{1j_3}\} \subset V\left(P^{(1)}_{x,y} \cap P^{(3)}_{x,y}\right)$, $\{v_{2k_1}, v_{2k_2}\} \subset V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right)$. Similar to Case 2, $P^{(3)}_{x,y} = x \cdots v_{3i_1} v_{1j_1} v_{3i_3} v_{2k_1} v_{3i_5}\ v_{2k_2} v_{1j_2}\ v_{3i_8} v_{1j_3} v_{3i_{10}} v_{1j_1} v_{3i_{12}} \cdots y$ (see Figure 7(a)), and a subpath $P_{v_{1j_1}, v_{1j_3}} = v_{1j_1} v_{3i_{10}} v_{1j_3}$ is a subdivision of edge $v_{1j_1}v_{1j_3}$. However $d_{LG(x,y)}(v_{1j_1}) \geq 6$ in this case, contrary to $\Delta(LG(x,y)) = 4$ as well as the linearity of $P^{(1)}_{x,y} \cap P^{(3)}_{x,y}$. Hence $LG(x,y)$ contains no subdivision of K_5.

Now assume that $LG(x,y)$ contains a subdivision of $K_{3,3}$ with bipartition (V_1, V_2) by Theorem 5.1. From the structure of $LG(x,y)$, $x, y \notin V_1 \cup V_2$, so we do not consider x, y in the following statements.

Case 4. $P^{(1)}_{x,y}, P^{(2)}_{x,y}$ and $P^{(3)}_{x,y}$ are internally-disjoint one another. Similar to Case 1, $d_{LG(x,y)}(v) = 2$ for each $v \in V(LG(x,y)) \setminus \{x,y\}$. So, by $d_{K_{3,3}}(v) = 3$ for each $v \in V(K_{3,3})$, $LG(x,y)$ does not contain a subdivision of $K_{3,3}$.

Case 5. $P^{(1)}_{x,y}$ and $P^{(2)}_{x,y}$, $P^{(1)}_{x,y}$ and $P^{(3)}_{x,y}$ are internally-disjoint, respectively. Since $LG(x,y)$ is linear, by Definition 2 and Theorem 4.4, if C', C'' are two cycles in $P^{(2)}_{x,y} \cup P^{(3)}_{x,y}$ then $|V(C') \cap V(C'')| = 0$ or 1. However, if C', C'' are two cycles in $K_{3,3}$ then $|V(C') \cap V(C'')| \geq 2$, a contradiction.

Case 6. Only $P^{(1)}_{x,y}$ and $P^{(2)}_{x,y}$ are internally-disjoint (see an example in Figure 7(b)). By Case 5, $\left|V\left(P^{(1)}_{x,y} \cap P^{(3)}_{x,y}\right)\right| \geq 1$ and $\left|V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right)\right| \geq 1$. Let $\left|V\left(P^{(1)}_{x,y} \cap P^{(3)}_{x,y}\right)\right| + \left|V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right)\right| \geq 6$ because otherwise $LG(x,y)$ contains no a subdivision of $K_{3,3}$. Without loss of generality, let $V_1 = \{v_{1j_1}, v_{1j_2}, v_{1j_3}\}$, $V_2 = \{v_{2k_1}, v_{2k_2}, v_{2k_3}\}$, and $1 \leq \left|V_2 \cap V\left(P^{(1)}_{x,y} \cap P^{(3)}_{x,y}\right)\right| \leq 3$ by $|V_2| = 3$. Then, similar to Case 3, $V_1 \cup V_2 \subseteq V\left(P^{(1)}_{x,y} \cap P^{(3)}_{x,y}\right) \cup V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right)$. And, by the linearity of $LG(x,y)$, for a vertex $v_{2k_3} \in V_2 \cap V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right)$, there are at most two vertices $v_{1j_2}, v_{1j_3} \in V_1 \cap V\left(P^{(1)}_{x,y}\right)$ connecting with v_{2k_3}. So there are at least a vertex, say v_{1j_1}, such that the subdivision path $P_{v_{1j_1}, v_{2k_3}}$ between v_{1j_1} and v_{2k_3} does not exist, a contradiction to assumption. Thus $LG(x,y)$ contains no subdivision of $K_{3,3}$ too. ■

Lemma 5.4. *For $LG(x,y)$ with maximum degree* 4, *$c(LG(x,y)) = 3$.*

Proof. We first show that $c(LG(x,y)) \leq 3$. Let $P^{(1)}_{x,y}, P^{(2)}_{x,y}$ be two internally-disjoint paths in $LG(x,y)$. Consider the subgraph $G^{1,3}_{x,y}$ formed by $P^{(1)}_{x,y}$ and $P^{(3)}_{x,y}$. Since $G^{1,3}_{x,y}$ is a configuration as graph H_0 in Figure 5(a), $c\left(G^{1,3}_{x,y}\right) = 2$ by Theorem 4.4 and there is an optimal labeling $\phi' : V\left(LG^{1,3}_{x,y}\right) \mapsto \left\{1, 2, \ldots, \left|V\left(G^{1,3}_{x,y}\right)\right|\right\}$ in which, for each coboundary $S^{\phi'}_j$ with $1 \leq j \leq \left|V\left(G^{1,3}_{x,y}\right)\right|$, $\left|S^{\phi'}_j\right| \leq 2$. Now let

$V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right) = \left\{v_{2k_1}, v_{2k_2}, \ldots, v_{2k_{r_0}}\right\}$ except x and y. Then it is possible that there are two subpaths $P^{(2)}_{r,r+1}$ and $P^{(3)}_{r,r+1}$ between v_{2k_r} and $v_{2k_{r+1}}$ for $0 \le r \le r_0$, where $v_{2k_0} = x$, $v_{2k_{r_0+1}} = y$, $P^{(2)}_{r,r+1}$ is a subpath of $P^{(2)}_{x,y}$, and $P^{(3)}_{r,r+1}$ is a subpath of $P^{(3)}_{x,y}$. Note that $P^{(3)}_{r,r+1}$ is possibly a vertex (see $P^{(3)}_{1,2}$ and vertex v_{2k_3} in Figure 7(b) respectively, for instance), in which case $P^{(2)}_{r,r+1} \cup P^{(3)}_{r,r+1}$ is either a cycle $C^{(r+1)}$ (for example, 4-cycle $C^{(2)} = v_{2k_1} w v_{2k_2} v_{3i_7} v_{2k_1}$ in Figure 7(b)) or path $P^{(2)}_{r,r+1}$ itself only (for example, $P^{(3)}_{0,1}$, $P^{(3)}_{2,3}$ in Figure 7(b)) and $d_{LG(x,y)}(v_{2k_r}) = 4$ for each r except $d_{LG(x,y)}(x) = d_{LG(x,y)}(y) = 3$. Now, for all vertices with degree two of $P^{(2)}_{x,y}$, we carry out the series reduction operations in $LG(x,y)$ continuously until there are no vertices with degree two in $P^{(2)}_{x,y}$, and denote the resulting graph by $LG'(x,y)$. By Lemma 2.2(ii), $c(LG(x,y)) = c(LG'(x,y))$. Thus, using ϕ' of $G^{1,3}_{x,y}$, the cutwidth of $LG(x,y)$ is equivalent to putting edge $v_{2k_r}v_{2k_{r+1}}$ back to the embedding ϕ' for each $0 \le r \le r_0$ in $LG'(x,y)$. Since $|V(LG'(x,y))| = \left|V\left(G^{1,3}_{x,y}\right)\right|$, ϕ' is also a labeling of $LG'(x,y)$ in which the congestion was increased at most one. Thus we get a labeling ϕ' of $LG'(x,y)$ with cutwidth at most three. So $c(LG'(x,y)) \le 3$ leading to $c(LG(x,y)) \le 3$. On the other hand, $c(LG(x,y)) \ge 3$ is obvious, since R_3-st in Figure 3 with cutwidth 3 is a minor of $LG(x,y)$. Hence $c(LG(x,y)) = 3$. ■

From Theorem 3.4 and Lemma 5.4, $LG(x,y)$ contains no subgraph R_i ($1 \le i \le 11$) (see Figure 3) as its minor. By Lemma 5.3, $LG(x,y)$ is planar with $V\left(P^{(1)}_{x,y} \cap P^{(3)}_{x,y}\right) = \left\{v_{1j_1}, v_{1j_2}, \ldots, v_{1j_{m_0}}\right\}$, and $V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right) = \left\{v_{2k_1}, v_{2k_2}, \ldots, v_{2k_{r_0}}\right\}$ except x and y. Let $P^{(1)}_{x,v_{1j_1}}$ and $P^{(3)}_{x,v_{1j_1}}$ be the subpaths of $P^{(1)}_{x,y}, P^{(3)}_{x,y}$ between x and v_{1j_1} respectively, and $P^{(2)}_{x,v_{2k_1}}$ be the subpath of $P^{(2)}_{x,y}$ between x and v_{2k_1} (see Figure 8). Likewise, let $P^{(2)}_{y,v_{2k_{r_0}}}$, $P^{(3)}_{y,v_{2k_{r_0}}}$ be the subpaths of $P^{(2)}_{x,y}, P^{(3)}_{x,y}$ between y and $v_{2k_{r_0}}$ respectively, and $P^{(1)}_{y,v_{1j_{m_0}}}$ be the subpath of $P^{(1)}_{x,y}$ between y and $v_{1j_{m_0}}$ (see Figure 8).

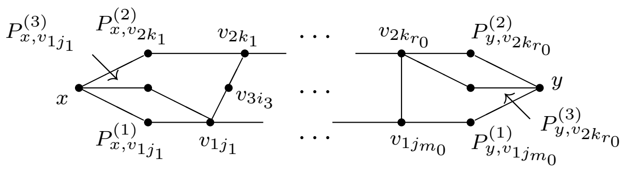


Figure 8. Six subpaths.

Now let $\mathcal{P} = \left\{P^{(1)}_{x,v_{1j_1}}, P^{(2)}_{x,v_{1k_1}}, P^{(3)}_{x,v_{1j_1}}, P^{(1)}_{y,v_{1j_{m_0}}}, P^{(2)}_{y,v_{2k_{r_0}}}, P^{(3)}_{y,v_{2k_{r_0}}}\right\}$. Then we have

Lemma 5.5. *For any $LG(x,y)$ with maximum degree 4, $LG(x,y) \in \mathcal{MG}_{\mu,3}$ if and only if the length of each element of $\mathcal{P}$ is at most two, where $\mu = |V(LG(x,y))|$.*

Proof. *Sufficiency.* It suffices to verify that $c(LG(x,y)+uv) \geq 4$ for any $uv \notin E(LG(x,y))$ by Lemma 5.4. There are four cases to consider as follows.

Case 1. $u = x, v = y$. $LG(x,y)+uv$ contains four edge-disjoint paths with the common vertices x, y and maximum degree 4. Without loss of generality, assume that these four paths are also internally disjoint, then R_3 (see Figure 3) is its minor. So $c(LG(x,y)+uv) \geq 4$, a contradiction.

Case 2. $u = x, v \neq y$.

Subcase 2.1. $v \in \{v_{2k_1}, v_{1j_1}\}$. In this case, R_3 or R_{11} is a minor of $LG(x,y)+uv$ resulting in $c(LG(x,y)+uv) \geq 4$, a contradiction.

Subcase 2.2. $v \notin \{v_{2k_1}, v_{1j_1}\}$. If $v \in V(P^{(1)}_{x,y})$ then R_3 is a minor of $LG(x,y)+uv$. If $v \in V(P^{(2)}_{x,y})$ then one of $\{R_6, R_{11}\}$ is a minor of $LG(x,y)+uv$. If $v \in V(P^{(3)}_{x,y})$, say v_{3i_3}, then R_4 is a minor of $LG(x,y)+uv$. So, there is always a subgraph $R_i \in \{R_3, R_4, R_6, R_{11}\}$ such that R_i is a minor of $LG(x,y)+uv$, a contradiction. This case is not possible.

Case 3. $u \neq x, v = y$. There are two subcases which are either $u \in \left\{v_{1j_{m_0}}, v_{2k_{r_0}}\right\}$ or $u \notin \left\{v_{1j_{m_0}}, v_{2k_{r_0}}\right\}$. Similar to Case 2.

Case 4. $u \neq x, v \neq y$.

Subcase 4.1. $u, v \in V\left(P^{(1)}_{x,y} \cap P^{(3)}_{x,y}\right) \cup V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right)$. If $u, v \in V\left(P^{(1)}_{x,y} \cap P^{(3)}_{x,y}\right)$ or $u, v \in V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right)$ then one of $\{R_3, R_8, R_{11}\}$ is a minor in $LG(x,y)+uv$. If $u \in V\left(P^{(1)}_{x,y} \cap P^{(3)}_{x,y}\right)$ and $v \in V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right)$ then one of $\{R_1, R_5\}$ is a minor in $LG(x,y)+uv$. These lead to $c(LG(x,y)+uv) \geq 4$. So this case is not possible.

Subcase 4.2. $u \in V\left(P^{(1)}_{x,y} \cap P^{(3)}_{x,y}\right) \cup V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right)$, but $v \notin V\left(P^{(1)}_{x,y} \cap P^{(3)}_{x,y}\right) \cup V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right)$. In this case, R_5 must be a minor in $LG(x,y)+uv$, leading to $c(LG(x,y)=uv) \geq 4$, a contradiction.

Subcase 4.3. $u, v \notin V\left(P^{(1)}_{x,y} \cap P^{(3)}_{x,y}\right) \cup V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right)$. If $u, v \in V\left(P^{(1)}_{xy}\right)$ or $V\left(P^{(2)}_{xy}\right)$ then one of $\{R_1, R_3, R_{11}\}$ is a minor in $LG(x,y)+uv$. If $u, v \in V\left(P^{(3)}_{xy}\right)$, then R_3 is a minor in $LG(x,y)+uv$. If $u \in V\left(P^{(1)}_{xy}\right)$ or $V\left(P^{(1)}_{xy}\right)$, then $v \in V\left(P^{(3)}_{xy}\right)$ and one of $\{R_1, R_4, R_{10}\}$ is a minor in $LG(x,y)+uv$. So $c(LG(x,y)+uv) \geq 4$, a contradiction.

Necessity. Suppose to the contrary that the length of at least one element of $\mathcal{P}$, say $P^{(1)}_{x,v_{1j_1}}$, is at least three. Let $P^{(1)}_{x,v_{1j_1}} = xx_1x_2v_{1j_1}$. Then there is an optimal 3-cutwidth labeling ϕ of $LG(x,y)$ with $\phi(x_1) = 1, \phi(x_2) = 2$ and $\phi(x) = 3$, i.e., $c(LG(x,y),\phi) = 3$. But, under ϕ, $c(LG(x,y) + xx_2, \phi) = 3$, contradicting $LG(x,y) \in \mathcal{MG}_{\mu,3}$. In summary, $LG(x,y) \in \mathcal{MG}_{\mu,3}$, and the proof is complete. ■

Like x and y, the neighbors of x, y satisfying Lemma 5.5 are also called the 2-*degree gluing points* of $LG(x,y)$. Likewise, for a 3-cycle $C_3 = z_1z_2z_3z_1$, z_1, z_2 and z_3 are also called the gluing points of C_3. Now let

$$\mathcal{G} = \{G : G \text{ is a } LG(x,y)\} \cup \{C_3\}.$$

For $H_1, H_2 \in \mathcal{G}$ (not necessarily distinct) with $H_1 = LG(x_1,y_1)$ and $H_2 = LG(x_2,y_2)$ or C_3, define $G_1 = H_1 \odot_{y_1,x_2} H_2$, $G_2 = H_1 \odot_{y_1,w} H_2$ with $w \in N_{H_2}(x_2)$ and $d_{H_2}(w) = 2$ or $w \in V(C_3)$. Then we have

Lemma 5.6. *Let ϕ_1, ϕ_2 be optimal labelings of H_1 and H_2, respectively. Then two labelings $\phi : V(G_1) \mapsto \{1,2,\ldots,|V(G_1)|\}$, $\psi : V(G_2) \mapsto \{1,2,\ldots,|V(G_2)|\}$ are optimal 3-cutwidth labelings of G_1 and G_2 respectively, where*

$$\phi(v) = \begin{cases} \phi_1(v) & \text{if } v \in V(H_1), \\ \phi_2(v) + |V(H_1)| - 1 & \text{if } v \in V(H_2) \setminus \{x_2\}, \end{cases}$$

and

$$\psi(v) = \begin{cases} \phi_1(v) & \text{if } v \in V(H_1), \\ \phi_2(v) + |V(H_1)| - 1 & \text{if } v \in V(H_2) \setminus \{w\}. \end{cases}$$

Proof. For ϕ_1, since x_1, y_1 are the original and the terminal with degree 3 in H_1, we can conclude that $\phi_1(x_1)$, $\phi_1(y_1)$ can equal 1 and $|V(H_1)|$ respectively, and so do the labels $\phi_2(x_2)$ and $\phi_2(y_2)$. So, for $j = 1, 2$, ϕ_j is an optimal sublabeling of ϕ restricted to block H_j of G_1, which leads to that $c(G_1,\phi) = c(H_1,\phi_1) = 3$. Thus ϕ is an optimal labeling of G_1.

Now we consider ψ of G_2. The proof of the case of $H_2 = C_3$ is straightforward, so it suffices to consider the case of $H_2 = LG(x_2,y_2)$. If $\phi_2(w) = 1$ then it is trivial. Otherwise, assume that there is an optimal labeling ϕ'_2 of H_2 such that $\phi'_2(w) = \alpha \neq 1$. By the assumption that $w \in N_{H_2}(x_2)$ and $d_{H_2}(w) = 2$, define ϕ_2 as follows: for $v \in V(H_2)$,

$$\phi_2(v) = \begin{cases} 1 & \text{if } v = w, \\ \phi'_2(v) + 1 & \text{if } v \neq w \text{ and } \phi'_2(v) < \alpha, \\ \phi'_2(v) & \text{if } v \neq w \text{ and } \phi'_2(v) > \alpha. \end{cases}$$

Then $c(H_2,\phi_2) = c(H_2,\phi'_2) = 3$ leading to that ϕ_2 is also optimal for H_2. Thus, similar to the labeling of G_1, let $\phi_1(x_1) = 1$, $\phi_1(y_1) = |V(H_1)|$. Then ϕ_j is an optimal sublabeling of ψ restricted to block H_j of G_2 for $j = 1, 2$, which leads to $c(G_2,\psi) = c(H_1,\phi_1) = c(H_2,\phi_2) = 3$. So, ψ is an optimal labeling of G_2. ■

Lemma 5.7. *Each of the following holds.*

(i) *If the neighbors of x_1, y_2 are 2-degree gluing points of H_1, H_2, then $G_1 \in \mathcal{MG}_{n_1,3}$ with $n_1 = |V(G_1)|$.*

(ii) *If w and other neighbors of x_1, y_2 are 2-degree gluing points of H_1, H_2, then $G_2 \in \mathcal{MG}_{n_2,3}$ with $n_2 = |V(G_2)|$.*

Proof. (i) By Lemmas 5.4–5.6, each $H_j \in \mathcal{MG}_{\mu_j,3}$ with $\mu_j = |V(H_j)|$ for $j = 1, 2$, so it suffices to show that $c(G_1+uv) \geq 4$ for any $uv \notin E(G_1)$ with $u \in V(H_1)$ and $v \in V(H_2)$. In fact, let the identified vertex of y_1 and x_2 be z in G_1, i.e., $y_1 = x_2 = z$, then $d_{G_1}(z) = 6$. Thus R_5 (see Figure 3) is a minor in $G_1 + uv$, which results in $c(G_1 + uv) \geq 4$. So $G_1 \in \mathcal{MG}_{n_1,3}$ with $n_1 = |V(G_1)|$.

(ii) Denote the identified vertex by z in G_2, i.e., $y_1 = w = z$. First let $H_2 = C_3$. Since $c(H_1) = 3$ and $c(C_3) = 2$. By Lemma 5.6, $c(G_2) = 3$. On the other hand, since x_1 and its neighbors are 2-degree gluing points in H_1 and $H_2 = C_3$, for any $uv \notin E(H_1)$ with $u, v \in V(H_1)$, $c(G_2 + uv) \geq 4$ by Lemma 5.5. For any $uv \notin E(H_1)$ with $u \in V(H_1)$ and $v \in V(C_3)$, similar to that of (i), we can see that R_5 is a minor of G_2+uv because of $d_{G_2}(z) = 5$. So $c(G_1+uv) \geq 4$. Next let $H_2 = LG(x_2, y_2)$. Since x_1, w and the neighors of x_1 are 2-degree gluing points, by Lemma 5.5, it suffices to consider the case of $u \in V(H_1)$ and $v \in V(H_2)$ for any $uv \notin E(G_1)$. In this case, R_5 is also a minor of G_2+uv because of $d_{G_2}(z) = 5$. Thus $c(G + uv) \geq 4$. ■

Lemma 5.8. *For 2-connected graph G with $|V(G)| = n$ and $c(G) = 3$, if $G \in \mathcal{MG}_{n,3}$, then $G \in \mathcal{G} \setminus \{C_3\}$.*

Proof. Clearly, $G \neq C_3$. By assumption that G is 2-connected, there are at least two internally-disjoint paths $P^{(1)}_{x,y}, P^{(2)}_{x,y}$ for $x, y \in V(G)$ by Lemma 3.3. But it is not possible that there are four edge-disjoint paths between x and y because of $c(G) = 3$. So, there is a path $P^{(3)}_{x,y}$ between x and y such that at most one of $V\left(P^{(1)}_{x,y} \cap P^{(3)}_{x,y}\right)$ and $V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right)$ is the empty set except x and y. Now suppose towards contradiction that $G \notin \mathcal{MG}_{n,3}$, and G is a minimum counterexample on $|V(G)|$, then we have

Claim 2. *G is either a graph Figure 9(a) or a graph Figure 9(b) only.*

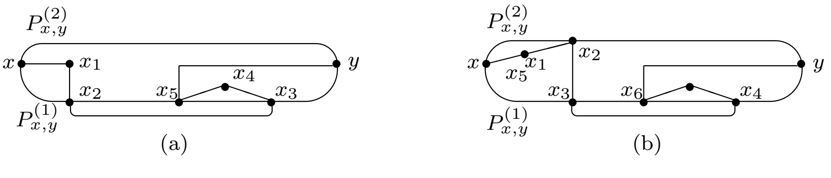


Figure 9. Two cases of minimum counterexamples of G.

In fact, for the case of $V\left(P^{(1)}_{x,y} \cap P^{(3)}_{x,y}\right) = \emptyset$ and $V\left(P^{(2)}_{x,y} \cap P^{(3)}_{x,y}\right) \neq \emptyset$, let

$P_{x,y}^{(3)} = x_0x_1x_2 \cdots x_\rho x_{\rho+1}$ with $x_0 = x, x_{\rho+1} = y$, and let $P_{x,y}^{(1)} = K_2 = xy$ by the assumption that G is minimum on $|V(G)|$. First, by $G \notin \mathcal{MG}_{n,3}$, $P_{x,y}^{(2)} \cap P_{x,y}^{(3)}$ is not a configuration like H_0 (see H_0 in Figure 5), as otherwise $c(G) = 3$ by Lemma 5.4. So there is at least a vertex in $P_{x,y}^{(3)}$, say x_{i_0}, such that x_{i_0} must be a subdivision vertex of some edge $x_{r_i}x_{r_{i+1}}$ of path $P_{x,y}^{(2)}$, where $1 \leq i_0 \leq \rho$ and $r_{i+1} \leq i_0 - 1$. Since G is minimum and simple, $\left|V\left(P_{x,y}^{(2)} \cap P_{x,y}^{(3)}\right) \setminus \{x,y\}\right| = 5$, which results in that $P_{x,y}^{(3)} = xx_1x_2x_3x_4x_5y$, and G must be a graph in Figure 9(a). Similarly, for the case of $V\left(P_{x,y}^{(j)} \cap P_{x,y}^{(3)}\right) \neq \emptyset$ for each $j = 1, 2$, by the minimlality of G, we can first let $V\left(P_{x,y}^{(1)} \cap P_{x,y}^{(3)}\right) = \{x_1\}$. And then, for $P_{x,y}^{(2)} \cap P_{x,y}^{(3)}$, with an argument similar to the above case, $P_{x,y}^{(3)} = xx_1x_2x_3x_4x_5x_6y$, which results in that G is a graph in Figure 9(b). Hence Claim 2 holds.

By Claim 2, G is one of Figure 9(a) and Figure 9(b). However, in Figure 9(a) and Figure 9(b), R_4 and R_9 (see Figure 3) are minors leading to that the cutwidth of each of them is at least 4, contradicting $c(G) = 3$. So $G \in \mathcal{G} \setminus \{C_3\}$. This completes the proof. ∎

Lemma 5.9. *For graph $G \in \mathcal{MG}_{n,3}$ with $|V(G)| = n$, let $B_1, B_2, \ldots, B_\beta$ be blocks of G. Then $B_i \in \mathcal{G}$ for each $1 \leq i \leq \beta$, and the block graph $\mathcal{B}$ of G is a path P_β, where $\beta \geq 2$.*

Proof. If $B_i = LG(x, y)$ or C_3 then it is trivial by Lemma 5.8. So let $B_i \neq LG(x, y)$ or C_3. We first verify that, for each $1 \leq i \leq \beta$, $B_i \neq K_2$. Otherwise, let $B_i \cap B_{i+1} = \{z_i\}$ and $B_i = K_2 = z_{i-1}z_i$. Clearly, $c(B_{i+1}) \geq 3$ by Lemma 5.4 and $d_{B_{i+1}}(z_i) \geq 3$ (since otherwise there must exist a vertex $w \in V(B_{i+1})$ such that $c(G + z_{i-1}w) = c(G)$, a contradiction). Since G is simple, there is at least a vertex, say w, such that $z_iw \in E(B_{i+1})$ and $d_{B_{i+1}}(w) = 2$. By Lemma 5.6, let ϕ_{i+1} be the optimal sublabeling restricted to B_{i+1} by an optimal labeling ϕ of G, in which $\phi_{i+1}(w) = \min\{\phi_{i+1}(v) : v \in V(B_{i+1})\}$. Then $c(G + z_{i-1}w) = c(G)$, contradicting $G \in \mathcal{MG}_{n,3}$. So $B_i \neq K_2$. Next we claim $|V(B_i)| = 3$. Otherwise, let $|V(B_i)| \geq 4$. Since $c(B_i) \leq 3$, by Lemmas 5.4, 5.5 and 5.8, for $r \geq 4$, B_i must be either a cycle C_r with at least one simple 2-bridge or a cycle C_r without bridge. For the former, if C_r has a simple 2-bridge $\mathbb{B}$ then $B_i \in \mathcal{G}$; if C_r has at least two simple 2-bridges $\mathbb{B}_1$ with 2 vertices x_1, x_2 of attachment and $\mathbb{B}_2$ with 2 vertices y_1, y_2 of attachment, then $\mathbb{B}_1$ and $\mathbb{B}_2$ avoid each other (otherwise, if $\mathbb{B}_1, \mathbb{B}_2$ skew then R_1 is a minor; if x_1, y_1 overlap then $B_i = R_6 - st \in \mathcal{G}$ (see R_6 in Figure 3)), where $\mathbb{B}$, $\mathbb{B}_1$ and $\mathbb{B}_2$ are all paths by definition. Thus there are two vertices, say x_2 and y_1, such that $c(G + x_2y_1) = c(G)$, a contradiction. For the latter, let $r = 4$, i.e., $C_4 = x_1x_2x_3x_4x_1$. In this case, if x_1 and x_4 are gluing vertices then $c(G + x_1x_3) = 3$; if x_1 and x_3 are gluing vertices then $c(G + x_2x_4) = 3$, a contradiction. The case of $r > 4$ is similar. Hence $B_i = C_3$ resulting in $B_i \in \mathcal{G}$.

Now let z_i be a common vertex of three blocks B_i, B_{i+1} and B', then $d_{B_i}(z_i) = 2$, $d_{B_{i+1}}(z_i) = 3$ (or $d_{B_i}(z_i) = 3$, $d_{B_{i+1}}(z_i) = 2$) and $B' = K_2 = z_i z'$ (otherwise G has a 4-cutwidth subgraph containing z_i with $d_G(z_i) \geq 6$, a contradiction). Thus there is at least a vertex $u \in N_{B_i}(z_i)$ (or $u \in N_{B_{i+1}}(z_i)$) such that $c(G + z'u) = c(G)$, a contradiction. So B' does not exist, which leads to that $\mathcal{B}$ is a path P_β. ■

Theorem 5.10. *For graph G with $|V(G)| = n$, $G \in \mathcal{MG}_{n,3}$ if and only if each of the following holds.*

- (i) *G is planar, and its blocks can be listed as $B_1, B_2, \ldots, B_\beta$ with $V(B_i) \cap V(B_{i+1}) = \{z_i\}$ $(1 \leq i \leq \beta - 1)$ such that the block graph $\mathcal{B}$ is a path P_β.*
- (ii) *For each $1 \leq i \leq \beta$, $B_i \in \mathcal{G}$, where $\mathcal{G} = \{G : G \text{ is a } LG(x, y)\} \cup \{C_3\}$.*
- (iii) *For each $1 \leq i \leq \beta - 1$, $d_{B_i}(z_i) \geq 2$, $d_{B_{i+1}}(z_i) \geq 2$, and at least one of them is 3.*
- (iv) *z_i is a gluing point with degree either 3 or 2 of B_i as well as B_{i+1}. If $d_{B_i}(z_{i-1}) = d_{B_i}(z_i) = 2$, then $B_i \in \mathcal{MG}_{\mu_i,3}$ with $\mu_i = |V(B_i)|$ or $B_i = C_3$. If $d_{B_i}(z_i) = 3$ or $d_{B_{i+1}}(z_i) = 3$, then $B_i = LG_i$ or $B_{i+1} = LG_{i+1}$, and the 2-degree neighbors of z_i in B_i or B_{i+1} are unnecessary to be the gluing points, where $LG_i = LG(x_i, y_i)$.*
- (v) *If $B_1 = LG_1$ and $B_\beta = LG_\beta$, then the neighbors with degree 2 of x_1, y_β must be 2-degree gluing points in B_1, B_β, respectively.*

Proof. *Sufficiency.* By Lemmas 5.4–5.7, $G \in \mathcal{MG}_{n,3}$ is true.
Necessity. (i) and (ii) are true by Lemmas 5.3, 5.8 and 5.9.

(iii) Clearly, $d_{B_i}(z_i) \geq 2, d_{B_{i+1}}(z_i) \geq 2$ by Lemma 5.9. Assume now that $B_i = LG(x_i, y_i)$ and $d_{B_i}(z_i) = d_{B_{i+1}}(z_i) = 2$ for $1 \leq i \leq \beta - 1$. Then there are at least two vertices $u \in N_{B_i}(z_i)$ and $v \in N_{B_{i+1}}(z_i)$, say $u = y_i$ and $v = x_{i+1}$, such that $c(G + uv) = c(G)$, a contradiction to $G \in \mathcal{MG}_{n,3}$. So one member of $\{d_{B_i}(z_i), d_{B_{i+1}}(z_i)\}$ is 3.

(iv) The first conclusion is obvious. For the second conclusion, $B_i = LG(x_i, y_i)$ or C_i by Lemma 5.9. So it is needed to show that if $B_i = LG(x_i, y_i)$ then $B_i \in \mathcal{MG}_{\mu_i,3}$. In fact, if $B_i \notin \mathcal{MG}_{\mu_i,3}$, then z_{i-1} (or z_i) must be adjacent to a 2-degree vertex w_{i-1} (or w_i). Without loss of generality, let $N_{B_i}(z_{i-1}) = \{x_i, w_{i-1}\}$ and $N_{B_i}(z_i) = \{y_i, w_i\}$. Then, by Lemma 5.6, there is a sublabeling ϕ_i restricted to B_i by an optimal labeling ϕ of G such that $\phi_i(z_{i-1}) = \min\{\phi_i(v) : v \in V(B_i)\}$, $\phi_i(w_{i-1}) = \phi_i(z_{i-1}) + 1$ and $\phi_i(x_i) = \phi_i(z_{i-1}) + 2$. Thus $c(G + x_i w_{i-1}, \phi) = c(G, \phi)$, a contradiction. Likewise, for z_i, let $\phi_i(z_i) = \max\{\phi_i(v) : v \in V(B_i)\}$, $\phi_i(w_i) = \phi_i(z_i) - 1$ and $\phi_i(y_i) = \phi_i(z_i) - 2$. Then $c(G + y_i w_i, \phi) = c(G, \phi)$, also a contradiction. Hence $B_i \in \mathcal{MG}_{\mu_i,3}$. For the third conclusion, on the one hand, $B_i \neq C_3$ or $B_{i+1} \neq C_3$ (as otherwise $d_{B_i}(z_i) = 2$ or $d_{B_{i+1}}(z_i) = 2$), so $B_i = LG_i$ or $B_{i+1} = LG_{i+1}$ by (ii). On the other hand, $z_i = y_i = x_{i+1}$, which is a 3-degree gluing point of B_i as well as B_{i+1}. So the 2-degree neighbors of z_i in B_i and B_{i+1} are not necessarily the gluing points.

(v) Similar to that of (iii), by Lemma 5.7, (v) is also true, omitted here. This completes the proof. ∎

Figure 10 is a graph $G \in \mathcal{MG}_{n,3}$ with β blocks B_i and their gluing pattern, in which w_i, w_i' are neighbors with degree two of x_i and y_i in B_i for $1 \leq i \leq \beta$ respectively, and $B_1 = LG_1 = LG(x_1, y_1)$.

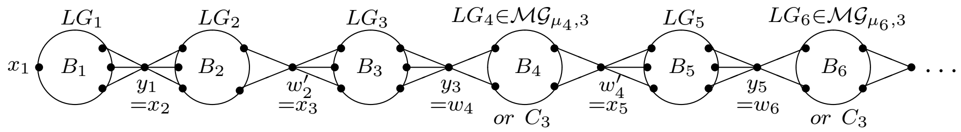


Figure 10. The gluing pattern of blocks of G.

Corollary 5.11. *Suppose that $G \in \mathcal{MG}_{n,3}$ with blocks $B_1, B_2, \ldots, B_\beta$ $(\beta \geq 1)$, $V(B_i) \cap V(B_{i+1}) = \{z_i\}$, $\{z_{l_1}, z_{l_2}, \ldots, z_{l_r}\} \subseteq \{z_i : 1 \leq i \leq \beta - 1\}$ for $1 \leq r \leq \beta - 1$. If $d_{B_{l_j}}(z_{l_j}) = d_{B_{l_j+1}}(z_{l_j}) = 3$ for each $1 \leq j \leq r$, G' is obtained by triangulating each z_{l_j} consecutively, then $G' \in \mathcal{MG}_{n+2r,3}$.*

Proof. The proof is straightforward by Theorem 5.10(iv). ∎

6. Remarks

In this paper, we characterized the structures of the edge-maximal graphs with $c(G) \leq 3$, from which we know that any edge-maximal graph with $c(G) \leq 3$ is decomposable. Regarding the edge-maximal graphs with $c(G) \geq 4$, we guess that the structure of each of them is similar to that of 3-cutwidth edge-maximal graphs. But its block B_i does not necessarily consist of 4 edge-disjoint paths. We achieved some results in this direction and will try to finish those efforts. For instance, let $G = (R_1 \odot_{s,s_0} G(s_0, t_0)) \odot_{t_0,t'} R_1'$, where $G(s_0, t_0)$ is a 4-paths graph with common vertices s_0 and t_0 (see Lemma 3.2), R_1' and t' are copies of R_1 and t in Figure 3, respectively. It is not hard to verify that $G \in \mathcal{MG}_{n,4}$. However, it seems that our technique cannot be easily used to examine the structure of the class of graphs. A further task is to detect the structures of such k-cutwidth edge-maximal graphs for $k \geq 4$.

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