

## EDGE-MAXIMAL GRAPHS WITH CUTWIDTH AT MOST THREE<sup>1</sup>

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### Abstract

The cutwidth minimization problem consists of finding an arrangement of the vertices of a graph  $G$  on a line  $P_n$  with  $n = |V(G)|$  vertices, in such a way that the maximum number of edges between each pair of consecutive vertices is minimized. A graph  $G$  with cutwidth  $k$  ( $k \geq 1$ ) is edge-maximal if  $c(G + uv) > k$  for any  $uv \in \{uv : u, v \in V(G) \text{ and } uv \notin E(G)\}$ . In this paper, we provide a complete insight to structural properties of edge-maximal graphs with cutwidth at most 3.

**Keywords:** combinatorics, graph labeling, cutwidth, edge-maximal graph.

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### 1. INTRODUCTION

Graphs in this paper are finite and simple with undefined notations following [1]. The cutwidth of a graph  $G$  is the smallest integer  $k$  such that the vertices of  $G$  are arranged in a linear layout  $[v_1, v_2, \dots, v_n]$  in such a way that, for each  $i = 1, 2, \dots, n-1$ , there are at most  $k$  edges with one endpoint in  $\{v_1, v_2, \dots, v_i\}$  and the other in  $\{v_{i+1}, \dots, v_n\}$ . The cutwidth problem for graphs, together with a class of optimal labeling (or embedding) problems, have significant applications in VLSI designs, network communications and others. In particular, the cutwidth is closely related to a basic parameter, called the congestion, in designing microchip circuits [2, 5, 12]. Here, a graph  $G$  may be thought of as a model of the wiring diagram of an electronic circuit, with the vertices representing components and

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the edges representing wires connecting them. When a circuit is laid out on a certain architecture (say a path  $P_n$ ), the maximum number of overlap wires is the congestion, which is one of major parameters determining the electronic performance. This motivates the cutwidth problem in graph theory practically. Theoretically, the cutwidth is closely related to other graph-theoretic parameters such as bandwidth, modified bandwidth, pathwidth and treewidth among other domains [3, 5, 7, 11].

Deciding whether the cutwidth of  $G$  is at most  $k$  for a given graph  $G$  and an integer  $k$  is an NP-complete problem [6], even for graphs with maximum vertex degree 3 [11], but it admits a polynomial algorithm within the family of trees [14]. The cutwidth problem has been extensively examined [5]. First, much work has been done for determining the exact value of the cutwidth of special classes of graphs (see e.g., [5, 8, 10, 13]) and algorithms computing the cutwidth of trees [4, 14]. As to the structure of graphs with cutwidth  $k$  ( $k \geq 1$ ), relatively little work has been done. A graph  $G$  is called  $k$ -cutwidth critical if (1)  $c(G) = k$ , and (2)  $c(G - uv) < k$  for any edge  $uv \in E(G)$ . In 2004, all five 3-cutwidth critical graphs  $H'_1, H'_2, H'_3, H'_4$  and  $H'_5$  were presented in [9], where  $H'_1$  is star  $K_{1,5}$ ,  $H'_2$  is a tree with diameter 4 obtained by identifying a pendant vertex in three copies of star  $K_{1,3}$ ,  $H'_3$  is obtained from  $H'_2$  by replacing a  $K_{1,3}$  by a triangle  $K_3$ ,  $H'_4$  is a 'crown' made of a cycle  $C_3$  with a pendant edge in each vertex of it, and  $H'_5$  is a cycle  $C_4$  with a chord. It was proved that any 2-cutwidth graph contains no one of  $H'_1, H'_2, H'_3, H'_4$  and  $H'_5$  being an induced subgraph. Similarly, the 4-cutwidth acyclic critical graph class has 18 graphs each of which can be decomposed into three 3-cutwidth minimal subtrees [15, 16]. For  $k > 4$ , although the structure of the acyclic critical graphs with cutwidth  $k$  is obtained in [17], the structural characterization of general graphs with cutwidth  $k$  is also a task to study further.

A graph  $G$  is  $k$ -cutwidth edge-maximal for an integer  $k \geq 1$  if (1)  $c(G) = k$ , and (2)  $c(G + uv) > k$  for any edge  $uv \in \{v_i v_j : v_i, v_j \in V(G) \text{ and } v_i v_j \notin E(G)\}$ . For any integer  $k \geq 1$ , the  $k$ -cutwidth edge-maximal graphs have not been previously studied. In this paper, we present a graph structure which precisely characterizes the class of  $k$ -cutwidth edge-maximal graphs for  $k \leq 3$ .

The rest of this paper is as follows. In Section 2, some preliminaries are presented. Section 3 gives 2-connected forbidden subgraphs of 3-cutwidth graphs. The 2-cutwidth edge-maximal graphs are characterized in Section 4. Section 5 is devoted to presenting the structure of 3-cutwidth edge-maximal graphs. A short remark is given in Section 6.

## 2. PRELIMINARIES

Suppose that  $G = (V(G), E(G))$  is a graph with  $|V(G)| = n$ . A *labeling* of a graph  $G$  is a bijection  $\phi : V(G) \rightarrow \{1, 2, \dots, n\}$ , viewed as an embedding of  $G$

into the path  $P_n$  with vertices in  $\{1, 2, \dots, n\}$ , where consecutive integers are the adjacent vertices. The *cutwidth of  $G$  with respect to  $\phi$*  is

$$(1) \quad c(G, \phi) = \max_{1 \leq j < n} |\{uv \in E(G) : \phi(u) \leq j < \phi(v)\}|,$$

which is also the *congestion* of the embedding. If  $k = c(G, \phi)$ , then  $\phi$ , as well as the embedding induced by  $\phi$ , is called a  *$k$ -cutwidth embedding* or *labeling* of  $G$ . The *cutwidth* of  $G$  is defined by

$$(2) \quad c(G) = \min_{\phi} c(G, \phi),$$

where the minimum is taken over all labelings  $\phi$ . A labeling  $\phi$  attaining the minimum in (2) is an optimal labeling. For a graph  $G$ , let  $S \subset V(G)$ ,  $\bar{S} = V(G) \setminus S$ . The edge cut  $E[S, \bar{S}]$ , i.e., the set of edges of  $G$  with one end in  $S$  and the other end in  $\bar{S}$ , is called the *coboundary* of  $S$  and denoted by  $\partial(S)$ , i.e.,  $\partial(S) = E[S, \bar{S}]$ . For a labeling  $\phi$  of  $G$  and each  $1 \leq j < n$ , let  $S_j^\phi = \{v \in V(G) : \phi(v) \leq j\}$ . Then by (2), we have

$$(3) \quad c(G, \phi) = \max_{1 \leq j < n} |\partial(S_j^\phi)|.$$

In other words, if  $v_i = \phi^{-1}(i)$  for  $1 \leq i < n$ , then  $S_j = \{v_1, v_2, \dots, v_j\}$  and  $\partial(S_j^\phi) = \{v_i v_h \in E(G) : i \leq j < h\}$  (also called the cut at  $[j, j+1]$ ). The cutwidth  $c(G, \phi)$  is the maximum size of these coboundaries  $\partial(S_j^\phi)$ . An  $\phi$ -max-coboundary of  $G$  is a  $\partial(S_j^\phi)$  achieving the maximum in (3).

For a graph  $G$  and integer  $i \geq 1$ , let  $D_i(G) = \{v \in V(G) : d_G(v) = i\}$ , where  $d_G(v)$  is the degree of vertex  $v \in V(G)$ , and the maximum degree is denoted as  $\Delta(G)$ . For each  $v \in V(G)$ , let  $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ . For  $V' \subset V(G)$ ,  $E' \subset E(G)$  and  $V' \neq \emptyset, E' \neq \emptyset$ ,  $G[V']$ ,  $G[E']$  are an induced subgraph and an edge-induced subgraph of  $G$ , respectively. The graph obtained from  $G$  by adding an edge  $v_1 v_2 \notin E(G)$  is denoted as  $G + v_1 v_2$ . If  $G$  has a vertex  $v \in D_2(G)$  with  $N_G(v) = \{v_1, v_2\}$  and  $v_1 v_2 \notin E(G)$ , then  $G - v + v_1 v_2$ , the graph obtained from  $G - v$  by adding a new edge  $v_1 v_2$ , is called a *series reduction* of  $G$ . A graph  $G'$  is homeomorphic to  $G$  if  $G'$  is obtained by some series reductions of  $G$ . Let  $G_1$  and  $G_2$  be two disjoint graphs with  $u \in V(G_1), v \in V(G_2)$ . To identify  $u$  and  $v$ , denoted as  $G_1 \odot_{u,v} G_2$ , is to replace  $u, v$  by a single vertex  $z$  (i.e.,  $u = v = z$ ) incident to all the edges which were incident to  $u$  and  $v$ , where  $z$  is called the identified vertex. If graph  $G = G_1 \odot_{u,v} G_2$ , then  $G$  is also called the *series composition* of  $G_1$  and  $G_2$ . To contract an edge  $v_1 v_2$  of graph  $G$  is to delete the edge and then identify its ends  $v_1, v_2$ . A graph  $G'$  is called a *minor* of  $G$  if  $G'$  is obtained by implementing series reductions and contracting edges from  $G$ . Two  $xy$ -paths  $P$  and  $Q$  in  $G$  are internally disjoint if they have no internal vertices in

common, that is,  $V(P) \cap V(Q) = \{x, y\}$ . Recall the definition of the bridge of a cycle  $C$  [1]. For a connected graph  $G$  with cycle  $C$ , let  $N$  be a component of  $G - C$ , the graph  $G[E(N)]$  is referred to as a *bridge* of  $C$  in  $G$  together with any edge connecting  $C$  with  $N$ , denoted as  $\mathbb{B}$ . For a bridge  $\mathbb{B}$  of  $C$ , the elements of  $V(\mathbb{B}) \cap V(C)$  are called its *vertices of attachment* to  $C$ . A bridge with  $t$  vertices of attachment is called a  *$t$ -bridge*. Let  $\{x_1, x_2\} \subset V(C)$  and  $\{y_1, y_2\} \subset V(C)$ , the two pairs *skew* if and only if they are disjoint and the  $x$ -vertices alternate with the  $y$ -vertices. Two bridges  $\mathbb{B}_1$  and  $\mathbb{B}_2$  *skew* if and only if their vertices of attachment skew.  $\mathbb{B}_1$  and  $\mathbb{B}_2$  *avoid* each other if all the vertices of attachment of  $\mathbb{B}_1$  lie in a single segment of  $\mathbb{B}_2$ ; otherwise they *overlap*. A bridge is *simple* if and only if it is a path  $P$ . A connected graph that has no cut vertices is called a *block*. Every block with at least three vertices is 2-connected. A block  $B$  of a connected graph  $G$  is a subgraph that is a block and is maximal with respect to this property. A *block graph*  $\mathcal{B}$  of  $G$  is the graph whose vertices are the blocks  $B_1, B_2, \dots, B_r$  of  $G$ , with  $B_i, B_j$  joined if and only if  $B_i, B_j$  have a common cut vertex for  $1 \leq i, j \leq r$  (see an example in Figure 1 in which  $B_3, B_5, B_6$  and  $B_7$  are all  $K_2$ ). From the definition, the following property of the block graph  $\mathcal{B}$  is straightforward.

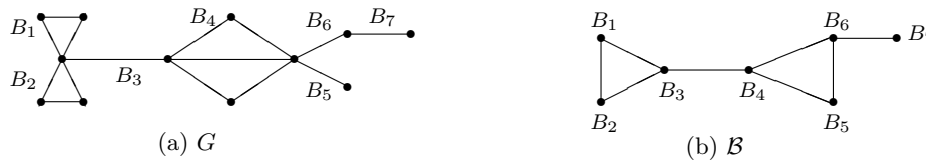


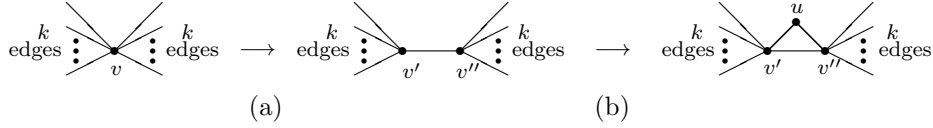
Figure 1. A graph  $G$  and its block graph  $\mathcal{B}$ .

**Lemma 2.1.** *For an integer  $\beta \geq 3$ , if the blocks  $B_1, B_2, \dots, B_\beta$  of  $G$  have a common cut vertex, then there exists a complete subgraph  $K_\beta$  in the block graph  $\mathcal{B}$  of  $G$ .*

**Definition 1.** For a graph  $G$  with  $|V(G)| = n$  and  $c(G) = k$  ( $k \geq 1$ ), if  $c(G + uv) > k$  for any edge  $uv \in \{v_i v_j : v_i, v_j \in V(G) \text{ and } v_i v_j \notin E(G)\}$ , then  $G$  is called  *$k$ -cutwidth edge-maximal*. We denote the set of the class of graphs by  $\mathcal{MG}_{n,k}$  ( $k \leq n$ ).

To split a vertex  $v$  is to replace  $v$  by two adjacent vertices,  $v'$  and  $v''$ , and to replace each edge incident to  $v$  by an edge incident to either  $v'$  or  $v''$  (but not both, unless it is a loop at  $v$ ), the other end of the edge remaining unchanged (Figure 2(a)). To triangulate a vertex  $v$  is to split a vertex  $v$  by two vertices  $v'$  and  $v''$  first, and then to add a new vertex  $u$  only connecting  $v'$  and  $v''$ , respectively (Figure 2(b)).

From the previous definition, the following lemma is trivial.

Figure 2. (a) To split a vertex  $v$ . (b) To triangulate a vertex  $v$ .

**Lemma 2.2.** *Let  $G$  and  $G'$  be graphs. Each of the following holds.*

- (i) *If  $G'$  is a subgraph or minor of  $G$ , then  $c(G') \leq c(G)$ .*
- (ii) *If  $G'$  is homeomorphic to  $G$ , then  $c(G') = c(G)$ .*

**Lemma 2.3.** *For a graph  $G \in \mathcal{MG}_{n,k}$ , let  $\phi$  be an optimal labeling of  $G$  with  $\phi(v_i) = i$  for  $1 \leq i < n$ . Each of the following holds.*

- (i) *If  $|\partial(S_j^\phi)| \leq k - 1$  for  $1 \leq j < n$ , then  $v_j v_{j+1} \in E(G)$ .*
- (ii) *If  $v_j v_{j+1} \notin E(G)$ , then  $|\partial(S_j^\phi)| = k$  for  $1 \leq j < n$ .*

**Proof.** For (i), if  $v_j v_{j+1} \notin E(G)$ , then  $|\partial(S_j^\phi) \cup \{v_j v_{j+1}\}| \leq (k - 1) + 1 = k$ , contrary to  $G \in \mathcal{MG}_{n,k}$ . Likewise, since  $v_j v_{j+1} \notin E(G)$ , if  $|\partial(S_j^\phi)| < k$ , then  $|\partial(S_j^\phi) \cup \{v_j v_{j+1}\}| \leq (k - 1) + 1 = k$  for (ii), also contrary to  $G \in \mathcal{MG}_{n,k}$ . ■

**Theorem 2.4.** *Let  $G$  be a  $k$ -cutwidth graph with  $|V(G)| = n$ ,  $\phi$  be an optimal labeling with  $\phi(v_i) = i$  for  $1 \leq i < n$ . Then  $G \in \mathcal{MG}_{n,k}$  if and only if there are no two vertices  $v_j, v_{j+1}$  with  $v_{j-1} v_{j+1} \notin E(G)$  and  $v_j v_{j+2} \notin E(G)$  such that  $|\partial(S_j^\phi)| \leq k - 1$  and  $|\partial(S_{j+1}^\phi)| \leq k - 1$ , where  $v_j v_{j+1} \in E(G), v_{j+1} v_{j+2} \in E(G)$  with  $1 \leq j < n - 1$ .*

**Proof.** *Sufficiency.* By assumption, for  $v_j$  and  $v_{j+1}$  with  $v_{j-1} v_{j+1} \notin E(G)$  and  $v_j v_{j+2} \notin E(G)$ , one of the three cases holds under  $\phi$ :

- (i)  $|\partial(S_{j-1}^\phi)| = |\partial(S_j^\phi)| = |\partial(S_{j+1}^\phi)| = k$ ;
- (ii)  $|\partial(S_{j-1}^\phi)| = k, |\partial(S_j^\phi)| \leq k - 1, |\partial(S_{j+1}^\phi)| = k$ ;
- (iii)  $|\partial(S_{j-1}^\phi)| = |\partial(S_j^\phi)| = k, |\partial(S_{j+1}^\phi)| \leq k - 1$ .

Assume towards a contradiction that  $G \notin \mathcal{MG}_{n,k}$ . Then  $c(G + uv) = k$  for some  $uv \notin E(G)$  because otherwise  $c(G + uv) \geq c(G)$ . Let  $\phi'$  be an optimal labeling of  $G + uv$  such that  $c(G + uv, \phi') = k$ . Then  $\phi'$  must be an optimal labeling of  $G$  (as otherwise  $c(G, \phi') \leq k - 1$ , a contradiction), say  $\phi = \phi'$ . However, let  $u = v_j, v \in V(G) \setminus \{v_j\}$ . Then, for each case above,  $c(G + uv, \phi') = c(G + uv, \phi) = k + 1$ , a contradiction. So  $G \in \mathcal{MG}_{n,k}$ .

*Necessity.* Suppose to the contrary that there exist two vertices  $v_j$  and  $v_{j+1}$  with  $v_{j-1} v_{j+1} \notin E(G)$  and  $v_j v_{j+2} \notin E(G)$  such that  $|\partial(S_j^\phi)| \leq k - 1$  and  $|\partial(S_{j+1}^\phi)| \leq k - 1$  under  $\phi$  and  $v_j v_{j+1} \in E(G), v_{j+1} v_{j+2} \in E(G)$  by Lemma 2.3. As  $v_j v_{j+2} \notin E(G)$ ,  $|\partial(S_j^\phi) \cup \{v_j v_{j+2}\}| \leq (k - 1) + 1 = k$  and  $|\partial(S_{j+1}^\phi) \cup \{v_{j+1} v_{j+2}\}| \leq (k - 1) + 1 = k$ , contrary to  $G \in \mathcal{MG}_{n,k}$ .

$\{v_j v_{j+2}\} \leq (k-1) + 1 = k$ . By  $|V(G + v_j v_{j+2})| = |V(G)|$  again, similar to Sufficiency, let  $\phi'$  be a labeling of  $G + v_j v_{j+2}$  and  $\phi' = \phi$ . Then, by  $c(G, \phi) = c(G) = k$ ,  $|\partial(S_{j'}^{\phi'})| = |\partial(S_{j'}^{\phi})| \leq k$  when  $v_{j'} \neq v_j, v_{j+1}$ . Thus  $|\partial(S_j^{\phi'})| \leq k$  for each  $1 \leq j < n$ . By (3),  $c(G + v_j v_{j+2}, \phi') = k$  implying  $c(G + v_j v_{j+2}) \leq k$  by (2). So, by  $c(G + v_j v_{j+2}) \geq k$ ,  $c(G + v_j v_{j+2}) = k$  contradicting  $G \in \mathcal{MG}_{n,k}$ . ■

### 3. TWO-CONNECTED FORBIDDEN SUBGRAPHS

In this section, we give eleven 2-connected forbidden subgraphs of 3-cutwidth graphs in Figure 3, where the empty dots in  $R_{10}$  imply that two corresponding edges either intersect or not.

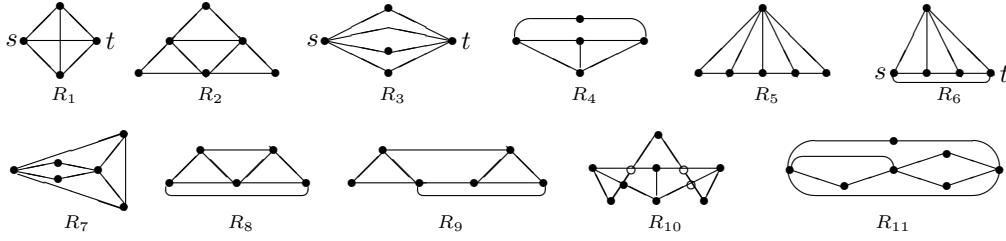


Figure 3. 2-connected forbidden subgraphs of 3-cutwidth graphs.

**Lemma 3.1** [10]. For a bipartite graph  $K_{m,n}$ ,  $c(K_{m,n}) = \lfloor \frac{m}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor + \lceil \frac{m}{2} \rceil \times \lceil \frac{n}{2} \rceil$ .

**Lemma 3.2.** Let  $G(s, t)$  be a  $k$ -paths graph comprised of  $k$  internally-disjoint paths  $P_i = sv_{i1}v_{i2} \cdots v_{it_i}t$  ( $1 < i \leq k, 1 \leq t_i < n$ ). Then  $c(G(s, t)) = k$ . In particular, if  $k \in \{2, 3\}$  and  $t_i = 1$  for each  $1 < i \leq k$ , then  $G(s, t) \in \mathcal{MG}_{k+2,k}$ .

**Proof.** Let  $G'(s, t)$  be a  $k$ -paths graph in which the length of each  $P_i$  is two, i.e.,  $t_i = 1$  for each  $1 < i \leq k$ . Then  $G'(s, t)$  is homeomorphic to  $G(s, t)$ . Since  $G'(s, t)$  can be thought of as a bipartite graph  $K_{k,2}$  with bipartition  $(X, Y)$ , where  $X = \{v_{11}, v_{21}, \dots, v_{k1}\}$  and  $Y = \{s, t\}$ ,  $c(G'(s, t)) = \lfloor \frac{k}{2} \rfloor + \lceil \frac{k}{2} \rceil = k$  by Lemma 3.1. Thus  $c(G(s, t)) = c(G'(s, t)) = k$  by homeomorphism. For  $k = 2$  and  $t_i = 1$  ( $1 < i \leq 2$ ),  $G(s, t) = C_4$  and  $C_4 \in \mathcal{MG}_{4,2}$  clearly. For  $k = 3$  and  $t_i = 1$  ( $1 < i \leq 3$ ),  $G(s, t) = R_3 - st$ , which can be easily verified that  $G(s, t) \in \mathcal{MG}_{5,3}$ . This completes the proof. ■

**Lemma 3.3** [1]. A graph  $G$  with  $|V(G)| \geq 3$  is 2-connected if and only if any two vertices  $x, y \in V(G)$  are connected by at least two internally-disjoint paths  $P_1(x, y)$  and  $P_2(x, y)$ .

**Theorem 3.4.** For a 2-connected graph  $G$ ,  $c(G) \leq 3$  if and only if  $G$  does not contain any subgraph  $R_i$  ( $1 \leq i \leq 11$ ) in Figure 3 as its minor.

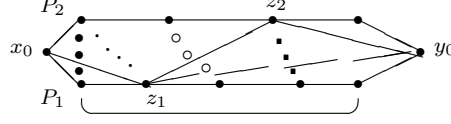


Figure 4. Illustration of Case 3(a).

**Proof.** Necessity is straightforward. We now show sufficiency by contradiction. Suppose that  $G$  is a minimum 2-connected graph with  $c(G) = k$  for  $k \geq 4$ . By Lemma 3.3, there are at least two internally-disjoint paths between  $x_0$  and  $y_0$  in  $G$ . Since  $G$  contains no  $R_3$  and no  $R_{11}$ , there are at most three internally-disjoint paths between  $x_0$  and  $y_0$ . Hence there are three cases that need to be considered.

*Case 1.* There are only two internally-disjoint paths  $P_1(x_0, y_0) = x_0x_1x_2 \cdots x_py_0$  and  $P_2(x_0, y_0) = x_0y_1y_2 \cdots y_qy_0$  between  $x_0$  and  $y_0$ . In this case,  $P_1(x_0, y_0) \cup P_2(x_0, y_0)$  is a  $(p+q+2)$ -cycle  $C_{p+q+2}$  with cutwidth two. By the assumption that  $G$  is minimum 2-connected, without loss of generality, we can let every bridge of  $C_{p+q+2}$  be simple. So, from the structure of  $G$  and the assumption of  $c(G) \geq 4$ , it suffices to consider the following three subcases.

*Subcase 1.1.* For a vertex  $x_{i'}$  ( $1 \leq i' \leq p$ ), there are at most three vertices in  $V(C_{p+q+2}) \setminus \{x_{i'-1}, x_{i'+1}\}$ , say  $y_1, y_2$  and  $y_3$ , such that  $(x_{i'}, y_1)$ -path,  $(x_{i'}, y_2)$ -path and  $(x_{i'}, y_3)$ -path are avoiding 2-bridges of  $C_{p+q+2}$ . In this case,  $3 \leq d_G(v_{i'}) \leq 5$ . Respectively, if  $(x_{i'}, y_1)$ -path is a unique 2-bridge then  $d_G(v_{i'}) = 3$ . If  $(x_{i'}, y_1)$ -path and  $(x_{i'}, y_2)$ -path are only two 2-bridges then  $d_G(v_{i'}) = 4$ . If, for each  $1 \leq j \leq 3$ ,  $(x_{i'}, y_j)$ -path is a 2-bridge, then  $d_G(v_{i'}) = 5$ . Thus, if  $d_G(v_{i'}) = 3$  then there are at least two vertices  $x_{i_1}$  with  $i_1 < i'$  and  $x_{i_2}$  with  $i_2 > i'$  such that  $x_{i_1}x_{i_2} \in E(G)$  (because  $c(G) = 3$ , otherwise), where  $x_{p+1} = y_0$ . This means that  $R_4$  is a minor. Similarly, if  $d_G(v_{i'}) = 4$  then  $R_6, R_8$  are minors, because there are also at least two vertices, say  $x_{i'-1}$  and  $x_{i'+1}$ , such that  $x_{i'-1}x_{i'+1} \in E(G)$ . If  $d_G(v_{i'}) = 5$ , then  $R_5$  is a minor. All are contrary to assumption.

*Subcase 1.2.* There are at least two avoiding 2-bridges which have no common vertices of attachment. Let  $x_{i_1}y_{j_1}, x_{i_2}y_{j_2}$  be such two 2-bridges, where  $1 \leq i_1 < i_2 \leq p, 1 \leq j_1 < j_2 \leq q$ . Since  $c(G) \geq 4$ , then there are at least two vertices, say  $y_{j_1}$  and  $y_{j_2+1}$ , such that  $y_{j_1}y_{j_2+1} \in E(G)$  (as otherwise  $c(G) = 3$ ), where  $y_{q+1} = y_0$ . This shows that  $R_9$  is a minor in  $G$ , a contradiction.

*Subcase 1.3.* There are at least two skewing 2-bridges in  $G$ . That is to say, there are at least four vertices, say  $x_{i'_1}, x_{i'_2}$  ( $1 \leq i'_1 < i'_2 \leq p$ ) and two of  $\{x_i : 1 \leq i \leq p \text{ and } i \neq i'_1 - 1, i'_1, i'_1 + 1, i'_2 - 1, i'_2, i'_2 + 1\} \cup \{y_i : 1 \leq i \leq q\}$ , say  $y_1$  and  $y_2$ , such that  $(x_{i'_1}, y_2)$ -path and  $(x_{i'_2}, y_1)$ -path are skewing 2-bridges. In this subcase,  $R_1$  is a minor of  $G$ , also a contradiction.

*Case 2.* There are three internally-disjoint paths  $P_1(x_0, y_0) = x_0x_1x_2 \cdots x_py_0$ ,  $P_2(x_0, y_0) = x_0y_1y_2 \cdots y_qy_0$  and  $P_3(x_0, y_0) = x_0z_1z_2 \cdots z_ly_0$  between  $x_0$  and  $y_0$ .

If  $G = P_1(x_0, y_0) \cup P_2(x_0, y_0) \cup P_3(x_0, y_0)$  then  $c(G) = 3$  by Lemma 3.2. So, by assumption, among three cycles  $C_{p+q+2}, C_{p+l+2}$  and  $C_{q+l+2}$ , there are at least a cycle, say  $C_{p+q+2}$  also, such that  $C_{p+q+2}$  has at least a simple 2-bridge whose vertices of attachment are neither  $x_0$  nor  $y_0$ , which results in an  $R_4$  minor of  $G$ , also a contradiction.

*Case 3.* There is a path  $P_3(x_0, y_0) = x_0 z_1 z_2 \cdots z_h y_0$  such that at least one of  $\{V(P_1(x_0, y_0) \cap P_3(x_0, y_0)), V(P_2(x_0, y_0) \cap P_3(x_0, y_0))\}$  is not empty, but  $E(P_1(x_0, y_0) \cap P_3(x_0, y_0)) = E(P_2(x_0, y_0) \cap P_3(x_0, y_0)) = \emptyset$ . By the assumption that  $G$  is minimum, without loss of generality, let  $h = 3$ , i.e.,  $P_3(x_0, y_0) = x_0 z_1 z_2 y_0$ . Then there are two subcases: (a)  $V(P_1(x_0, y_0) \cap P_3(x_0, y_0)) \neq \emptyset$ ,  $V(P_2(x_0, y_0) \cap P_3(x_0, y_0)) \neq \emptyset$  (see Figure 4). Assume that  $V(P_1(x_0, y_0) \cap P_3(x_0, y_0)) = \{z_1\}$ ,  $V(P_2(x_0, y_0) \cap P_3(x_0, y_0)) = \{z_2\}$ . By  $c(G) \geq 4$ ,  $P_1(x_0, y_0) \cup P_2(x_0, y_0)$  must contain at least one simple 2-bridge except  $E(P_3(x_0, y_0))$  in  $G$ , see the different dotted lines in Figure 4. This implies that one of  $\{R_1, R_2, R_4, R_5, R_8, R_{10}\}$  must be a minor of  $G$ , a contradiction. (b)  $V(P_1(x_0, y_0) \cap P_3(x_0, y_0)) \neq \emptyset$  but  $V(P_2(x_0, y_0) \cap P_3(x_0, y_0)) = \emptyset$ . In this case, similar to (a), one of  $\{R_3, R_4, R_5, R_{10}, R_{11}\}$  is a minor in  $G$ , also a contradiction. So  $c(G) \leq 3$ . The proof is completed. ■

#### 4. EDGE-MAXIMAL GRAPHS WITH CUTWIDTH AT MOST 2

For any cycle  $C_{\mu+2}$  ( $\mu \geq 1$ ) with  $v_l, v_r \in V(C_{\mu+2})$ , if  $P_1 = v_l v_1 v_2 \cdots v_i v_r$ ,  $P_2 = v_l v_{i+1} v_{i+2} \cdots v_\mu v_r$  are the internally-disjoint paths forming  $C_{\mu+2}$ , then  $v_l, v_r$  are viewed as the left terminal and the right terminal, respectively. Since  $c(K_{1,5}) = 3, c(R_1 - st) = 3$  (see Figure 3), there are no  $K_{1,5}$  or no  $R_1 - st$  induced subgraph or minor in any graph with cutwidth 2. First, Lemmas 4.1 and 4.2 are straightforward.

**Lemma 4.1.** *A graph  $G \in \mathcal{MG}_{n,1}$  if and only if  $G$  is a path  $P_n$  with  $n$  vertices for  $n \geq 2$ .*

**Lemma 4.2.** *For cycle  $C_{\mu+2}$  with  $\mu \geq 1$ ,  $C_{\mu+2} \in \mathcal{MG}_{\mu+2,2}$ .*

Now, for  $1 \leq j \leq \beta$ , let  $\mu = \mu_j, l = l_j, r = r_j, i = i_j$  with  $1 \leq i_j < \mu_j$ , and let  $P_1^j = v_{l_j} v_1^j v_2^j \cdots v_{i_j}^j v_{r_j}$ ,  $P_2^j = v_{l_j} v_{i_j+1}^j v_{i_j+2}^j \cdots v_{\mu_j}^j v_{r_j}$  be two paths forming  $C_{\mu_j+2}$ , where  $v_{l_j}$  and  $v_{r_j}$  are the left terminal and the right terminal respectively. By identifying  $v_{r_j}$  of  $C_{\mu_j+2}$  and  $v_{l_{j+1}}$  of  $C_{\mu_{j+1}+2}$  for each  $1 \leq j \leq \beta - 1$  (i.e.,  $v_{r_j} = v_{l_{j+1}} = z_j$ ) consecutively, one can obtain the series composition  $H_0$  of  $C_{\mu_1+2}, C_{\mu_2+2}, \dots, C_{\mu_\beta+2}$  with the left terminal  $v_{l_1} (= z_0)$  and the right terminal  $v_{r_\beta} (= z_\beta)$  and  $|V(H_0)| = \sum_{i=1}^{\beta} \mu_j + \beta + 1$  (see  $H_0$  with 4 cycles in Figure 5(a)).

Clearly,  $c(H_0) \geq 2$  by Lemma 4.2. For each  $1 \leq j \leq \beta$ ,  $C_{\mu_j+2}$  is a block  $B_j$  with the left terminal  $z_{j-1}$  and the right terminal  $z_j$  in  $H_0$ , and the



block graph  $\mathcal{B}$  of  $H_0$  is a path  $P_\beta$ , where  $z_0 = v_{l_1}, z_\beta = v_{r_\beta}$ . Suppose that  $\phi : V(H_0) \mapsto \{1, 2, \dots, |V(H_0)|\}$  is an optimal labeling of  $H_0$  such that its sublabeling  $\phi_j$  restricted to  $C_{\mu_j+2}$  is

$$\phi_j(v) = \begin{cases} \sum_{i=0}^{j-1} \mu_i + j & \text{if } v = z_{j-1}, \\ \sum_{i=0}^{j-1} \mu_i + j + i_j & \text{if } v \in V(P_1^j) \cup V(P_2^j), \\ \sum_{i=1}^j \mu_i + j + 1 & \text{if } v = z_j, \end{cases}$$

for  $1 \leq j \leq \beta$  and  $1 \leq i_j \leq \mu_j$ , where  $\mu_0 = 0$ . Since  $c(C_{\mu_j+2}, \phi_j) = 2$  and  $V(C_{\mu_j}) \cap V(C_{\mu_{j+1}}) = z_j$  with  $\phi_j(z_j) = \max\{\phi_j(v) : v \in V(C_{\mu_j})\} = \min\{\phi_{j+1}(v) : v \in V(C_{\mu_{j+1}})\}$ ,  $c(H_0, \phi) = 2$ . Hence  $c(H_0) = 2$ . On the other hand, for any  $x, y \in V(H_0)$  with  $xy \notin E(H_0)$ , if  $x, y \in V(C_{\mu_j+2})$  for some  $j$ , then by Lemma 4.2  $c(C_{\mu_j+2} + xy) = 3$ ; if  $x \in V(C_{\mu_{j_1}+2}), y \in V(C_{\mu_{j_2}+2})$  with  $j_1 < j_2$ , then, for any labeling  $\phi'$  of  $H_0 + xy$  with  $\phi'(z_{j_1}) = \rho$ ,  $|\partial(S_\rho^{\phi'})| \geq 3$ . So, by (3),  $c(H_0 + xy, \phi') \geq 3$  resulting in  $c(H_0 + xy) \geq 3$  too. So  $H_0 \in \mathcal{MG}_{n,2}$  with  $n = |V(H_0)|$ .

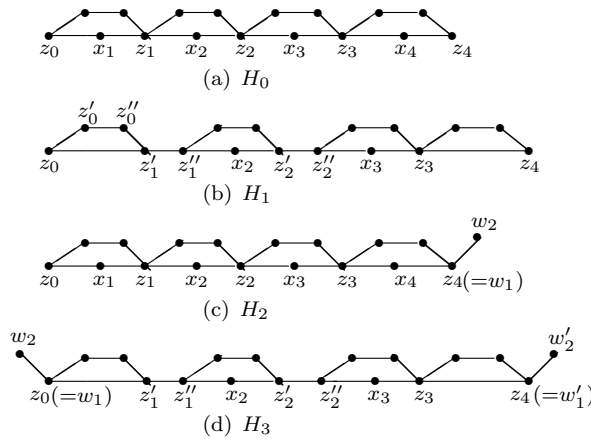


Figure 5. Four 2-cutwidth edge-maximal graphs.

There are  $\beta - 1$  cut vertices  $z_1, z_2, \dots, z_{\beta-1}$  with degree 4 in  $H_0$ . For an integer  $\xi$  ( $1 \leq \xi \leq \beta - 1$ ) and  $\{z_{l_1}, z_{l_2}, \dots, z_{l_\xi}\} \subseteq \{z_1, z_2, \dots, z_{\beta-1}\}$ , we carry out three operations in  $H_0$  at the same time: (1) splitting  $z_{l_1}, z_{l_2}, \dots, z_{l_\xi}$  respectively; (2) choosing  $\xi$  vertices  $x_1, x_2, \dots, x_\xi$  with degree 2 arbitrarily; and (3) for  $x_1, x_2, \dots, x_\xi$ , implementing  $\xi$  series reductions consecutively in order to keep  $|V(H_0)|$  constant. Let  $H_\xi$  be the graph obtained by carrying out the above operations for  $z_{l_i}$  and  $x_i$  ( $1 \leq i \leq \xi$ ), and  $\mathcal{H}_\xi^\beta = \{H_\xi : 0 \leq \xi \leq \beta - 1\}$ ,  $\dot{\mathcal{H}}_\xi^\beta = \{H' : H' = H \odot_{z_0, w_1} K_2, H \in \mathcal{H}_\xi^\beta\}$ ,  $\ddot{\mathcal{H}}_\xi^\beta = \{H'' : H'' = H' \odot_{z_\beta, w'_1} K'_2, H' \in \dot{\mathcal{H}}_\xi^\beta\}$ , where  $K_2 = w_1 w_2, K'_2 = w'_1 w'_2$ . For example,  $H_1$  in Figure 5(b) is obtained by splitting  $z_1, z_2$  and implementing 2 series reductions for  $x_1, x_4$  in  $H_0$  at the same time,

$H_2 = H_0 \odot_{z_4, w_1} K_2$  in Figure 5(c), and  $H_3 = (H_1 \odot_{z_0, w_1} K_2) \odot_{z_4, w'_1} K'_2$  in Figure 5(d). So  $H_1 \in \mathcal{H}_2^4$ ,  $H_2 \in \dot{\mathcal{H}}_0^4$ ,  $H_3 - z_4 w'_2 \in \dot{\mathcal{H}}_2^4$  and  $H_3 \in \ddot{\mathcal{H}}_2^4$ . Let  $\mathcal{H} = \bigcup_{\xi=0}^{\beta-1} \mathcal{H}_\xi^\beta$ ,  $\dot{\mathcal{H}} = \bigcup_{\xi=0}^{\beta-1} \dot{\mathcal{H}}_\xi^\beta$  and  $\ddot{\mathcal{H}} = \bigcup_{\xi=0}^{\beta-1} \ddot{\mathcal{H}}_\xi^\beta$ . For  $G \in \mathcal{H} \cup \dot{\mathcal{H}} \cup \ddot{\mathcal{H}}$ , its block  $B_i$  is either a cycle  $C_\mu$  or a  $K_2$ , the block graph  $\mathcal{B}$  is a path  $P_h$  with  $h$  vertices, where

$$h = \begin{cases} \beta & \text{if } G = H_0, \\ \beta + 1 & \text{if } G = H_0 \odot_{z_0, w_1} K_2, \\ \beta + 2 & \text{if } G = (H_0 \odot_{z_0, w_1} K_2) \odot_{z_\beta, w'_1} K'_2, \\ \beta + \xi & \text{if } G \in \mathcal{H}_\xi^\beta \text{ but } G \neq H_0, \\ \beta + \xi + 1 & \text{if } G \in \dot{\mathcal{H}}_\xi^\beta \text{ but } G \neq H_0 \odot_{z_0, w_1} K_2, \\ \beta + \xi + 2 & \text{if } G \in \ddot{\mathcal{H}}_\xi^\beta \text{ but } G \neq (H_0 \odot_{z_0, w_1} K_2) \odot_{z_\beta, w'_1} K'_2. \end{cases}$$

Using a similar argument to that of the proof of  $H_0$ , we can get the following lemma.

**Lemma 4.3.** *Assume that graphs  $H \in \mathcal{H}$ ,  $H' \in \dot{\mathcal{H}}$  and  $H'' \in \ddot{\mathcal{H}}$ . Then  $H \in \mathcal{MG}_{n,2}$ ,  $H' \in \mathcal{MG}_{n+1,2}$  and  $H'' \in \mathcal{MG}_{n+2,2}$ .*

**Theorem 4.4.** *For a graph  $G$  with  $|V(G)| = n$  and block  $B_i$  ( $1 \leq i \leq \beta$ ),  $G \in \mathcal{MG}_{n,2}$  if and only if each of the following holds.*

- (i) *For each block  $B_i$  of  $G$ , either  $B_i = C_{\mu_i}$  with  $\mu_i \geq 3$  or  $B_i = K_2$ .*
- (ii) *For each  $1 \leq i \leq \beta - 1$ , at least one member of  $\{B_i, B_{i+1}\}$  is not  $K_2$ .*
- (iii) *The block graph  $\mathcal{B}$  of  $G$  is a path  $P_\beta$ .*

**Proof.** By Lemma 4.3, it suffices to show its necessity by contradiction. First assume that there is at least a block  $B_{i_0}$  such that  $B_{i_0} \neq C_{\mu_{i_0}}$  and  $B_{i_0} \neq K_2$  in  $G$ , which implies that some  $C'_\mu$ s and  $K'_2$ s must be the proper subgraphs of  $B_{i_0}$ . For instance,  $B_{i_0} = H_1 + z_0 z_4$  containing two  $C'_4$ s, two  $C'_5$ s and two  $K'_2$ s (see  $H_1$  in Figure 5). Without loss of generality, let  $B_{i_0}$  be a minimum block that contains these  $C'_\mu$ s and  $K'_2$ s. Then, without considering the vertex number of each  $C'_\mu$  by homeomorphism, we have

**Claim 1.**  *$B_{i_0}$  must be homeomorphic to one of the six graphs in Figure 6, where any of  $\{u, v\}$  can be viewed as a cut vertex of  $G$  by homeomorphism.*

In fact, by the minimality of  $B_{i_0}$ , there is an edge  $uv \in E(G)$  such that (i) holds in  $G - uv$ , in which either (1)  $u, v \in V(C_{\mu_{i_0}})$  or (2)  $u \in V(C_{\mu_{i_0}})$  and  $v \notin V(C_{\mu_{i_0}})$ , in which any of  $u$  and  $v$  may be either a cut vertex or not in  $G$ . For case (1), it is clear that  $B_{i_0} - uv$  is Figure 6(a). For case (2), there are two subcases to consider: (a)  $B_{i_0} - uv$  contains two blocks which are either a  $C_{\mu_1}$  and a  $C_{\mu_2}$  or a  $C_{\mu_1}$  and a  $K_2$ . In this case,  $B_{i_0}$  must be one of Figure 6(b) and Figure 6(c). (b)  $B_{i_0} - uv$  contains at least three blocks  $B_{i_0}^{(1)}, B_{i_0}^{(2)}, \dots, B_{i_0}^{(\rho)}$  ( $\rho \geq 3$ ). If  $B_{i_0}^{(1)} = C_{\mu_1}$  and  $B_{i_0}^{(\rho)} = C_{\mu_2}$  then  $B_{i_0}$  must be Figure 6(d); if  $B_{i_0}^{(1)} = C_{\mu_1}$  and

$B_{i_0}^{(\rho)} = K_2$  (or  $B_{i_0}^{(1)} = K_2$  and  $B_{i_0}^{(\rho)} = C_{\mu_\rho}$ ) then  $B_{i_0}$  must be Figure 6(e); if  $B_{i_0}^{(1)} = K_2$  and  $B_{i_0}^{(\rho)} = K_2$  then  $B_{i_0}$  must be Figure 6(f), where  $B_{i_0}^{(r)}$  is either a  $C_{\mu_r}$  or a  $K_2$  for every  $1 < r < \rho$ . Thus Claim 1 holds.

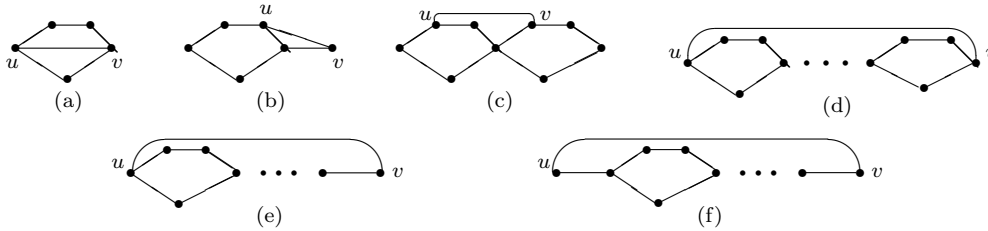


Figure 6. Six possible graphs homeomorphic to  $B_{i_0}$ .

Hence, by Claim 1,  $B_{i_0}$  is homeomorphic to Figure 6(a) for case (1), and  $B_{i_0}$  is homeomorphic to one of the graphs Figure 6(b)–(f) for case (2). Since the cutwidth of each of the graphs in Figure 6 is three, i.e.,  $c(B_{i_0}) = 3$  by Lemma 2.2(ii),  $c(G) \geq 3$  by Lemma 2.2(i), contrary to  $c(G) = 2$ . Hence  $B_i = C_{\mu_i}$  or  $B_i = K_2$  for each block  $B_i$  of  $G$ , which shows that (i) holds.

Second, assume that  $B_i$  and  $B_{i+1}$  are both  $K_2$  for  $1 \leq i \leq \beta - 1$ , and  $B_i = z_1 z_2, B_{i+1} = z_2 z_3$ . Then  $c(G + z_1 z_3) = 2$ , contradicting  $G \in \mathcal{MG}_{n,2}$ . So (ii) holds.

For (iii), assume to the contrary that the block graph  $\mathcal{B}$  of  $G$  is not a path  $P_\beta$ , then  $\mathcal{B}$  contains at least a complete graph  $K_r$  ( $r \geq 3$ ) by Lemma 2.1, say  $K_3 = B_1 B_2 B' B_1$ , and let  $P_h$  be the path with maximum length  $h$  in  $\mathcal{B}$ , where the common cut vertex of  $B_1, B_2, B'$  is  $z_{i_0}$  and  $\{B_1, B_2\} \subset V(P_h)$ . There are four cases to consider by (ii): (1)  $B_1 = C_{\mu_1}, B_2 = C_{\mu_2}, B' = C_{\mu_3}$ ; (2)  $B_1 = C_{\mu_1}, B_2 = C_{\mu_2}, B' = K_2$ ; (3)  $B_1 = C_{\mu_1}, B_2 = K_2, B' = C_{\mu_3}$ ; (4)  $B_1 = C_{\mu_1}, B_2 = K_2, B' = K_2$ . It is easy to verify that cases (1), (2) and (3) are not possible, as  $K_{1,5}$  is a subgraph in  $G$  for each of them, which leads to  $c(G) \geq 3$ , a contradiction. For case (4), let  $B_2 = z_{i_0} z_{i_0+1}, B' = z_{i_0} w$ , then  $c(G + w z_{i_0+1}) = 2$  contradicting  $G \in \mathcal{MG}_{n,2}$ . So case (4) is not possible. Thus  $B'$  does not exist, and  $\mathcal{B}$  is a path  $P_\beta$ . ■

**Corollary 4.5.** *Let  $G \in \mathcal{MG}_{n,2}$  with block  $B_i$ ,  $\phi_i : V(B_i) \mapsto \{1, 2, \dots, |V(B_i)|\}$  be an optimal labeling of  $B_i$  ( $0 \leq i \leq \beta$ ). Then  $\phi_{i,i+1}$  is an optimal labeling of  $B_i \cup B_{i+1}$  for each  $0 \leq i \leq \beta - 1$ , where*

$$\phi_{i,i+1}(v) = \begin{cases} \phi_i(v) & \text{if } v \in V(B_i), \\ \phi_{i+1}(v) + |V(B_i)| - 1 & \text{if } v \in V(B_{i+1}) \setminus \{z_i\}, \end{cases}$$

and  $V(B_i) \cap V(B_{i+1}) = \{z_i\}$ ,  $B_0 = z_0$ .

## 5. EDGE-MAXIMAL GRAPHS WITH CUTWIDTH AT MOST 3

In this section, by Theorem 3.4, 3-cutwidth edge-maximal graphs are investigated carefully. For convenience,  $P_{x,y}$  will be a path between  $x$  and  $y$  instead of  $P(x,y)$ . The following is Kuratowski's Theorem, which can be seen in [1].

**Theorem 5.1** (Kuratowski). *A graph is planar if and only if it contains no subdivision of either  $K_5$  or  $K_{3,3}$ .*

A 2-tree  $T$  is recursively defined as follows: (1)  $K_3$  is a 2-tree; (2) If  $T$  is a 2-tree, the graph obtained from  $T$  by joining a new vertex to two vertices of a  $K_3$  in  $T$  is also a 2-tree. Clearly, 2-tree  $T$  with  $m$  ( $m \geq 2$ ) inner faces is planar. The dual  $T^*$  of  $T$  is defined as follows: corresponding to each face  $f$  of  $T$  there is a vertex  $f^*$  of  $T^*$ , and corresponding to each edge  $e$  of  $T$  there is an edge  $e^*$  of  $T^*$ ; two vertices  $f^*$  and  $g^*$  are joined by  $e^*$  in  $T^*$  if and only if the corresponding faces  $f$  and  $g$  are separated by  $e$  in  $T$ . If  $f_0^*$  of  $T^*$  is the vertex corresponding to the outer face  $f_0$  of  $T$  with  $\Delta(T) = 4$ , and  $T^* - f_0^*$  is a path  $P_m$ , then we call  $T$  *linear*. In a linear 2-tree  $T$ , except two 3 degree vertices  $x, y$  and two 2 degree vertices  $x', y'$ ,  $d_T(v) = 4$  for every  $v \in V(T) \setminus \{x, y, x', y'\}$ . Such a 2-tree  $T$  is denoted by  $LT(x, y)$ , and we can easily obtain

**Lemma 5.2.** *For each  $LT(x, y)$ ,  $LT(x, y) \in \mathcal{MG}_{\mu,3}$ , where  $|V(LT(x, y))| = \mu$ .*

A 3-paths graph  $G(s, t)$  is a graph formed by three internally-disjoint paths  $P_{s,t}^{(1)}, P_{s,t}^{(2)}, P_{s,t}^{(3)}$  with two common vertices  $s, t$ . Any 3-paths graph  $G(s, t)$  is planar with two inner faces  $f_1$  and  $f_2$ . By Lemma 3.2,  $c(G(s, t)) = 3$ , and  $G(s, t) \in \mathcal{MG}_{\mu,3}$  if and only if  $G \in \{R_1 - st, R_3 - st\}$  (see  $R_1, R_3$  in Figure 3).

**Definition 2.** A graph  $G(x, y)$  is a simple graph consisting of three edge-disjoint paths  $P_{x,y}^{(1)}, P_{x,y}^{(2)}, P_{x,y}^{(3)}$  with vertices  $x, y$  in common. If  $P_{x,y}^{(i)} \cup P_{x,y}^{(3)} \in \mathcal{MG}_{\mu_{i3},2}$  for  $i = 1, 2$ , then  $G(x, y)$  is said to be *linear*, denoted as  $LG(x, y)$ , where  $\mu_{i3} = |V(P_{x,y}^{(i)} \cup P_{x,y}^{(3)})|$ ,  $x$  and  $y$  are called the *3-degree gluing points* in  $LG(x, y)$ .

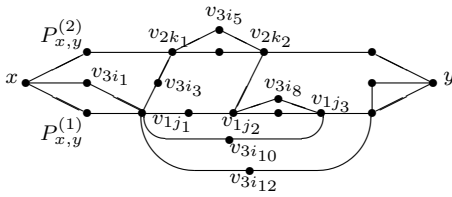
From Definition 2,  $LG(x, y) = P_{x,y}^{(1)} \cup P_{x,y}^{(2)} \cup P_{x,y}^{(3)}$  and is 2-connected, in which  $P_{x,y}^{(i)} \cup P_{x,y}^{(3)}$  ( $i = 1, 2$ ) either has a configuration as  $H_0$  in Figure 5(a) or is a single cycle  $C$ . A linear 2-tree  $LT(x, y)$  and a 3-paths graph  $G(s, t)$  with  $s = x, t = y$  are two special cases of  $LG(x, y)$ . In the following statements, we will let  $V(P_{x,y}^{(1)} \cap P_{x,y}^{(3)}) = \{v_{1j_1}, v_{1j_2}, \dots, v_{1j_{m_0}}\}$  and  $V(P_{x,y}^{(2)} \cap P_{x,y}^{(3)}) = \{v_{2k_1}, v_{2k_2}, \dots, v_{2k_{r_0}}\}$  (possibly empty) except  $x$  and  $y$ , and let  $P_{x,y}^{(1)}, P_{x,y}^{(2)}$  be internally-disjoint by Lemma 3.3.

**Lemma 5.3.** *Each  $LG(x, y)$  with maximum degree 4 is planar.*

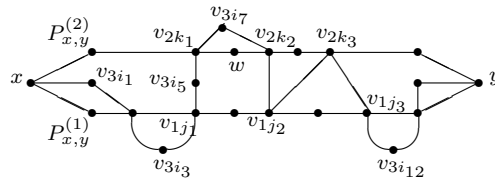
**Proof.** By contradiction. Suppose that  $LG(x, y)$  contains a subdivision of  $K_5$  by Theorem 5.1. By the assumption that  $LG(x, y)$  is simple,  $LG(x, y)$  does not contain double edges. So  $|V(LG(x, y))| \geq 5$ . Since  $d_{K_5}(v) = 4$  for  $v \in V(K_5)$  but  $d_{LG(x, y)}(x) = d_{LG(x, y)}(y) = 3$ ,  $x$  and  $y$  are not the 4-degree vertices of the subdivision of  $K_5$ . Now let  $P_{x, y}^{(1)} = xv_{11}v_{12} \cdots v_{1t_1}y$ ,  $P_{x, y}^{(2)} = xv_{21}v_{22} \cdots v_{2t_2}y$  and  $P_{x, y}^{(3)} = xv_{31}v_{32} \cdots v_{3t_3}y$  be the three edge-disjoint paths consisting of  $LG(x, y)$  respectively, where  $P_{x, y}^{(1)}, P_{x, y}^{(2)}$  are also internally-disjoint by Lemma 3.3. Then there are three cases to consider.

*Case 1.*  $P_{x, y}^{(1)}, P_{x, y}^{(2)}$  and  $P_{x, y}^{(3)}$  are internally-disjoint one another. In this case,  $LG(x, y)$  is a 3-paths graph in which  $d_{LG(x, y)}(x) = d_{LG(x, y)}(y) = 3$ ,  $d_{LG(x, y)}(v) = 2$  for any  $v \in V(LG(x, y)) \setminus \{x, y\}$ . So, by  $d_{K_5}(v) = 4$  for any  $v \in V(K_5)$ ,  $LG(x, y)$  can not contain a subdivision of  $K_5$ .

*Case 2.*  $P_{x, y}^{(1)}$  and  $P_{x, y}^{(2)}, P_{x, y}^{(1)}$  and  $P_{x, y}^{(3)}$  are internally-disjoint, respectively. If there are at most 4 common vertices except  $x, y$  in  $V(P_{x, y}^{(2)} \cap P_{x, y}^{(3)})$ , then the subdivision of  $K_5$  does not exist in  $LG(x, y)$ . So, without loss of generality, assume that  $V(K_5) = \{v_{2k_1}, v_{2k_2}, v_{2k_3}, v_{2k_4}, v_{2k_5}\} \subset V(P_{x, y}^{(2)} \cap P_{x, y}^{(3)})$  and  $v_{2k_j} \neq x, y$  for  $1 \leq j \leq 5$ . Then there is at least a vertex, say  $v_{2k_1}$ , such that either  $v_{2k_1}v_{2k_5} \in E(LG(x, y))$  or there are at least three subpaths between  $v_{2k_1}$  and  $v_{2k_5}$  in  $P_{x, y}^{(2)} \cup P_{x, y}^{(3)}$ , i.e.,  $d_{LG(x, y)}(v_{2k_1}) \geq 5$  by  $d_{K_5}(v_{2k_1}) = 4$  (because otherwise  $P_{x, y}^{(2)} \cup P_{x, y}^{(3)}$  is homeomorphic to  $H_0$  in Figure 5, which leads to that the subdivision of  $K_5$  does not exist in  $LG(x, y)$ ). In fact, for the latter, a subpath  $P_{v_{2k_1}, v_{2k_5}} \left( \subset P_{x, y}^{(3)} \right)$  between  $v_{2k_1}$  and  $v_{2k_5}$  can be thought of as a subdivision of edge  $v_{2k_1}v_{2k_5}$ , say  $P_{v_{2k_1}, v_{2k_5}} = v_{2k_1}v_{3i_6}v_{2k_5}$ . Thus  $P_{x, y}^{(3)} = x \cdots v_{3i_1}v_{2k_1}v_{3i_2}v_{2k_2}v_{3i_3}v_{2k_3}v_{3i_4}v_{2k_4}v_{3i_5}v_{2k_5}v_{3i_6}v_{2k_1} \cdots y$ , which results in  $d_{LG(x, y)}(v_{2k_1}) \geq 6$ , contrary to  $\Delta(LG(x, y)) = 4$  as well as the linearity of  $P_{x, y}^{(2)} \cap P_{x, y}^{(3)}$ .



(a) Illustration of Case 3.



(b) Illustration of Case 6.

Figure 7. Proof of Lemma 5.4.

*Case 3.* Only  $P_{x, y}^{(1)}$  and  $P_{x, y}^{(2)}$  are internally-disjoint. By Case 2, without loss of generality, let  $V(K_5) \subseteq V(P_{x, y}^{(1)} \cap P_{x, y}^{(3)}) \cup V(P_{x, y}^{(2)} \cap P_{x, y}^{(3)})$ . Then we can

let  $v_{1j_1}, v_{1j_2}, v_{1j_3}, v_{2k_1}, v_{2k_2}$  form a subdivision of  $K_5$  with  $\{v_{1j_1}, v_{1j_2}, v_{1j_3}\} \subset V(P_{x,y}^{(1)} \cap P_{x,y}^{(3)})$ ,  $\{v_{2k_1}, v_{2k_2}\} \subset V(P_{x,y}^{(2)} \cap P_{x,y}^{(3)})$ . Similar to Case 2,  $P_{x,y}^{(3)} = x \cdots v_{3i_1} v_{1j_1} v_{3i_3} v_{2k_1} v_{3i_5} v_{2k_2} v_{1j_2} v_{3i_8} v_{1j_3} v_{3i_{10}} v_{1j_1} v_{3i_{12}} \cdots y$  (see Figure 7(a)), and a subpath  $P_{v_{1j_1}, v_{1j_3}} = v_{1j_1} v_{3i_{10}} v_{1j_3}$  is a subdivision of edge  $v_{1j_1} v_{1j_3}$ . However  $d_{LG(x,y)}(v_{1j_1}) \geq 6$  in this case, contrary to  $\Delta(LG(x,y)) = 4$  as well as the linearity of  $P_{x,y}^{(1)} \cap P_{x,y}^{(3)}$ . Hence  $LG(x,y)$  contains no subdivision of  $K_5$ .

Now assume that  $LG(x,y)$  contains a subdivision of  $K_{3,3}$  with bipartition  $(V_1, V_2)$  by Theorem 5.1. From the structure of  $LG(x,y)$ ,  $x, y \notin V_1 \cup V_2$ , so we do not consider  $x, y$  in the following statements.

*Case 4.*  $P_{x,y}^{(1)}, P_{x,y}^{(2)}$  and  $P_{x,y}^{(3)}$  are internally-disjoint one another. Similar to Case 1,  $d_{LG(x,y)}(v) = 2$  for each  $v \in V(LG(x,y)) \setminus \{x, y\}$ . So, by  $d_{K_{3,3}}(v) = 3$  for each  $v \in V(K_{3,3})$ ,  $LG(x,y)$  does not contain a subdivision of  $K_{3,3}$ .

*Case 5.*  $P_{x,y}^{(1)}$  and  $P_{x,y}^{(2)}, P_{x,y}^{(1)}$  and  $P_{x,y}^{(3)}$  are internally-disjoint, respectively. Since  $LG(x,y)$  is linear, by Definition 2 and Theorem 4.4, if  $C', C''$  are two cycles in  $P_{x,y}^{(2)} \cup P_{x,y}^{(3)}$  then  $|V(C') \cap V(C'')| = 0$  or 1. However, if  $C', C''$  are two cycles in  $K_{3,3}$  then  $|V(C') \cap V(C'')| \geq 2$ , a contradiction.

*Case 6.* Only  $P_{x,y}^{(1)}$  and  $P_{x,y}^{(2)}$  are internally-disjoint (see an example in Figure 7(b)). By Case 5,  $|V(P_{x,y}^{(1)} \cap P_{x,y}^{(3)})| \geq 1$  and  $|V(P_{x,y}^{(2)} \cap P_{x,y}^{(3)})| \geq 1$ . Let  $|V(P_{x,y}^{(1)} \cap P_{x,y}^{(3)})| + |V(P_{x,y}^{(2)} \cap P_{x,y}^{(3)})| \geq 6$  because otherwise  $LG(x,y)$  contains no a subdivision of  $K_{3,3}$ . Without loss of generality, let  $V_1 = \{v_{1j_1}, v_{1j_2}, v_{1j_3}\}$ ,  $V_2 = \{v_{2k_1}, v_{2k_2}, v_{2k_3}\}$ , and  $1 \leq |V_2 \cap V(P_{x,y}^{(1)} \cap P_{x,y}^{(3)})| \leq 3$  by  $|V_2| = 3$ . Then, similar to Case 3,  $V_1 \cup V_2 \subseteq V(P_{x,y}^{(1)} \cap P_{x,y}^{(3)}) \cup V(P_{x,y}^{(2)} \cap P_{x,y}^{(3)})$ . And, by the linearity of  $LG(x,y)$ , for a vertex  $v_{2k_3} \in V_2 \cap V(P_{x,y}^{(2)} \cap P_{x,y}^{(3)})$ , there are at most two vertices  $v_{1j_2}, v_{1j_3} \in V_1 \cap V(P_{x,y}^{(1)})$  connecting with  $v_{2k_3}$ . So there are at least a vertex, say  $v_{1j_1}$ , such that the subdivision path  $P_{v_{1j_1}, v_{2k_3}}$  between  $v_{1j_1}$  and  $v_{2k_3}$  does not exist, a contradiction to assumption. Thus  $LG(x,y)$  contains no subdivision of  $K_{3,3}$  too. ■

**Lemma 5.4.** For  $LG(x,y)$  with maximum degree 4,  $c(LG(x,y)) = 3$ .

**Proof.** We first show that  $c(LG(x,y)) \leq 3$ . Let  $P_{x,y}^{(1)}, P_{x,y}^{(2)}$  be two internally-disjoint paths in  $LG(x,y)$ . Consider the subgraph  $G_{x,y}^{1,3}$  formed by  $P_{x,y}^{(1)}$  and  $P_{x,y}^{(3)}$ . Since  $G_{x,y}^{1,3}$  is a configuration as graph  $H_0$  in Figure 5(a),  $c(G_{x,y}^{1,3}) = 2$  by Theorem 4.4 and there is an optimal labeling  $\phi' : V(LG_{x,y}^{1,3}) \mapsto \{1, 2, \dots, |V(G_{x,y}^{1,3})|\}$  in which, for each coboundary  $S_j^{\phi'}$  with  $1 \leq j \leq |V(G_{x,y}^{1,3})|$ ,  $|S_j^{\phi'}| \leq 2$ . Now let

$V(P_{x,y}^{(2)} \cap P_{x,y}^{(3)}) = \{v_{2k_1}, v_{2k_2}, \dots, v_{2k_{r_0}}\}$  except  $x$  and  $y$ . Then it is possible that there are two subpaths  $P_{r,r+1}^{(2)}$  and  $P_{r,r+1}^{(3)}$  between  $v_{2k_r}$  and  $v_{2k_{r+1}}$  for  $0 \leq r \leq r_0$ , where  $v_{2k_0} = x$ ,  $v_{2k_{r_0+1}} = y$ ,  $P_{r,r+1}^{(2)}$  is a subpath of  $P_{x,y}^{(2)}$ , and  $P_{r,r+1}^{(3)}$  is a subpath of  $P_{x,y}^{(3)}$ . Note that  $P_{r,r+1}^{(3)}$  is possibly a vertex (see  $P_{1,2}^{(3)}$  and vertex  $v_{2k_3}$  in Figure 7(b) respectively, for instance), in which case  $P_{r,r+1}^{(2)} \cup P_{r,r+1}^{(3)}$  is either a cycle  $C^{(r+1)}$  (for example, 4-cycle  $C^{(2)} = v_{2k_1} w v_{2k_2} v_{3i_7} v_{2k_1}$  in Figure 7(b)) or path  $P_{r,r+1}^{(2)}$  itself only (for example,  $P_{0,1}^{(3)}$ ,  $P_{2,3}^{(3)}$  in Figure 7(b)) and  $d_{LG(x,y)}(v_{2k_r}) = 4$  for each  $r$  except  $d_{LG(x,y)}(x) = d_{LG(x,y)}(y) = 3$ . Now, for all vertices with degree two of  $P_{x,y}^{(2)}$ , we carry out the series reduction operations in  $LG(x,y)$  continuously until there are no vertices with degree two in  $P_{x,y}^{(2)}$ , and denote the resulting graph by  $LG'(x,y)$ . By Lemma 2.2(ii),  $c(LG(x,y)) = c(LG'(x,y))$ . Thus, using  $\phi'$  of  $G_{x,y}^{1,3}$ , the cutwidth of  $LG(x,y)$  is equivalent to putting edge  $v_{2k_r} v_{2k_{r+1}}$  back to the embedding  $\phi'$  for each  $0 \leq r \leq r_0$  in  $LG'(x,y)$ . Since  $|V(LG'(x,y))| = |V(G_{x,y}^{1,3})|$ ,  $\phi'$  is also a labeling of  $LG'(x,y)$  in which the congestion was increased at most one. Thus we get a labeling  $\phi'$  of  $LG'(x,y)$  with cutwidth at most three. So  $c(LG'(x,y)) \leq 3$  leading to  $c(LG(x,y)) \leq 3$ . On the other hand,  $c(LG(x,y)) \geq 3$  is obvious, since  $R_3$ -st in Figure 3 with cutwidth 3 is a minor of  $LG(x,y)$ . Hence  $c(LG(x,y)) = 3$ . ■

From Theorem 3.4 and Lemma 5.4,  $LG(x,y)$  contains no subgraph  $R_i$  ( $1 \leq i \leq 11$ ) (see Figure 3) as its minor. By Lemma 5.3,  $LG(x,y)$  is planar with  $V(P_{x,y}^{(1)} \cap P_{x,y}^{(3)}) = \{v_{1j_1}, v_{1j_2}, \dots, v_{1j_{m_0}}\}$ , and  $V(P_{x,y}^{(2)} \cap P_{x,y}^{(3)}) = \{v_{2k_1}, v_{2k_2}, \dots, v_{2k_{r_0}}\}$  except  $x$  and  $y$ . Let  $P_{x,v_{1j_1}}^{(1)}$  and  $P_{x,v_{1j_1}}^{(3)}$  be the subpaths of  $P_{x,y}^{(1)}$ ,  $P_{x,y}^{(3)}$  between  $x$  and  $v_{1j_1}$  respectively, and  $P_{x,v_{2k_1}}^{(2)}$  be the subpath of  $P_{x,y}^{(2)}$  between  $x$  and  $v_{2k_1}$  (see Figure 8). Likewise, let  $P_{y,v_{2k_{r_0}}}^{(2)}$ ,  $P_{y,v_{2k_{r_0}}}^{(3)}$  be the subpaths of  $P_{x,y}^{(2)}$ ,  $P_{x,y}^{(3)}$  between  $y$  and  $v_{2k_{r_0}}$  respectively, and  $P_{y,v_{1j_{m_0}}}^{(1)}$  be the subpath of  $P_{x,y}^{(1)}$  between  $y$  and  $v_{1j_{m_0}}$  (see Figure 8).

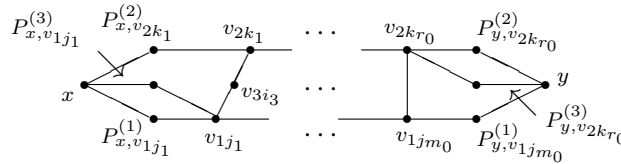


Figure 8. Six subpaths.

Now let  $\mathcal{P} = \{P_{x,v_{1j_1}}^{(1)}, P_{x,v_{1j_1}}^{(2)}, P_{x,v_{1j_1}}^{(3)}, P_{x,v_{1j_{m_0}}}^{(1)}, P_{y,v_{2k_{r_0}}}^{(2)}, P_{y,v_{2k_{r_0}}}^{(3)}\}$ . Then we have

**Lemma 5.5.** *For any  $LG(x, y)$  with maximum degree 4,  $LG(x, y) \in \mathcal{MG}_{\mu, 3}$  if and only if the length of each element of  $\mathcal{P}$  is at most two, where  $\mu = |V(LG(x, y))|$ .*

**Proof.** *Sufficiency.* It suffices to verify that  $c(LG(x, y) + uv) \geq 4$  for any  $uv \notin E(LG(x, y))$  by Lemma 5.4. There are four cases to consider as follows.

*Case 1.*  $u = x, v = y$ .  $LG(x, y) + uv$  contains four edge-disjoint paths with the common vertices  $x, y$  and maximum degree 4. Without loss of generality, assume that these four paths are also internally disjoint, then  $R_3$  (see Figure 3) is its minor. So  $c(LG(x, y) + uv) \geq 4$ , a contradiction.

*Case 2.*  $u = x, v \neq y$ .

*Subcase 2.1.*  $v \in \{v_{2k_1}, v_{1j_1}\}$ . In this case,  $R_3$  or  $R_{11}$  is a minor of  $LG(x, y) + uv$  resulting in  $c(LG(x, y) + uv) \geq 4$ , a contradiction.

*Subcase 2.2.*  $v \notin \{v_{2k_1}, v_{1j_1}\}$ . If  $v \in V(P_{x,y}^{(1)})$  then  $R_3$  is a minor of  $LG(x, y) + uv$ . If  $v \in V(P_{x,y}^{(2)})$  then one of  $\{R_6, R_{11}\}$  is a minor of  $LG(x, y) + uv$ . If  $v \in V(P_{x,y}^{(3)})$ , say  $v_{3i_3}$ , then  $R_4$  is a minor of  $LG(x, y) + uv$ . So, there is always a subgraph  $R_i \in \{R_3, R_4, R_6, R_{11}\}$  such that  $R_i$  is a minor of  $LG(x, y) + uv$ , a contradiction. This case is not possible.

*Case 3.*  $u \neq x, v = y$ . There are two subcases which are either  $u \in \{v_{1j_{m_0}}, v_{2k_{r_0}}\}$  or  $u \notin \{v_{1j_{m_0}}, v_{2k_{r_0}}\}$ . Similar to Case 2.

*Case 4.*  $u \neq x, v \neq y$ .

*Subcase 4.1.*  $u, v \in V(P_{x,y}^{(1)} \cap P_{x,y}^{(3)}) \cup V(P_{x,y}^{(2)} \cap P_{x,y}^{(3)})$ . If  $u, v \in V(P_{x,y}^{(1)} \cap P_{x,y}^{(3)})$  or  $u, v \in V(P_{x,y}^{(2)} \cap P_{x,y}^{(3)})$  then one of  $\{R_3, R_8, R_{11}\}$  is a minor in  $LG(x, y) + uv$ . If  $u \in V(P_{x,y}^{(1)} \cap P_{x,y}^{(3)})$  and  $v \in V(P_{x,y}^{(2)} \cap P_{x,y}^{(3)})$  then one of  $\{R_1, R_5\}$  is a minor in  $LG(x, y) + uv$ . These lead to  $c(LG(x, y) + uv) \geq 4$ . So this case is not possible.

*Subcase 4.2.*  $u \in V(P_{x,y}^{(1)} \cap P_{x,y}^{(3)}) \cup V(P_{x,y}^{(2)} \cap P_{x,y}^{(3)})$ , but  $v \notin V(P_{x,y}^{(1)} \cap P_{x,y}^{(3)}) \cup V(P_{x,y}^{(2)} \cap P_{x,y}^{(3)})$ . In this case,  $R_5$  must be a minor in  $LG(x, y) + uv$ , leading to  $c(LG(x, y) + uv) \geq 4$ , a contradiction.

*Subcase 4.3.*  $u, v \notin V(P_{x,y}^{(1)} \cap P_{x,y}^{(3)}) \cup V(P_{x,y}^{(2)} \cap P_{x,y}^{(3)})$ . If  $u, v \in V(P_{xy}^{(1)})$  or  $V(P_{xy}^{(2)})$  then one of  $\{R_1, R_3, R_{11}\}$  is a minor in  $LG(x, y) + uv$ . If  $u, v \in V(P_{xy}^{(3)})$ , then  $R_3$  is a minor in  $LG(x, y) + uv$ . If  $u \in V(P_{xy}^{(1)})$  or  $V(P_{xy}^{(1)})$ , then  $v \in V(P_{xy}^{(3)})$  and one of  $\{R_1, R_4, R_{10}\}$  is a minor in  $LG(x, y) + uv$ . So  $c(LG(x, y) + uv) \geq 4$ , a contradiction.



*Necessity.* Suppose to the contrary that the length of at least one element of  $\mathcal{P}$ , say  $P_{x,v_{1j_1}}^{(1)}$ , is at least three. Let  $P_{x,v_{1j_1}}^{(1)} = xx_1x_2v_{1j_1}$ . Then there is an optimal 3-cutwidth labeling  $\phi$  of  $LG(x, y)$  with  $\phi(x_1) = 1, \phi(x_2) = 2$  and  $\phi(x) = 3$ , i.e.,  $c(LG(x, y), \phi) = 3$ . But, under  $\phi$ ,  $c(LG(x, y) + xx_2, \phi) = 3$ , contradicting  $LG(x, y) \in \mathcal{MG}_{\mu,3}$ . In summary,  $LG(x, y) \in \mathcal{MG}_{\mu,3}$ , and the proof is complete. ■

Like  $x$  and  $y$ , the neighbors of  $x, y$  satisfying Lemma 5.5 are also called the *2-degree gluing points* of  $LG(x, y)$ . Likewise, for a 3-cycle  $C_3 = z_1z_2z_3z_1$ ,  $z_1, z_2$  and  $z_3$  are also called the gluing points of  $C_3$ . Now let

$$\mathcal{G} = \{G : G \text{ is a } LG(x, y)\} \cup \{C_3\}.$$

For  $H_1, H_2 \in \mathcal{G}$  (not necessarily distinct) with  $H_1 = LG(x_1, y_1)$  and  $H_2 = LG(x_2, y_2)$  or  $C_3$ , define  $G_1 = H_1 \odot_{y_1, x_2} H_2$ ,  $G_2 = H_1 \odot_{y_1, w} H_2$  with  $w \in N_{H_2}(x_2)$  and  $d_{H_2}(w) = 2$  or  $w \in V(C_3)$ . Then we have

**Lemma 5.6.** *Let  $\phi_1, \phi_2$  be optimal labelings of  $H_1$  and  $H_2$ , respectively. Then two labelings  $\phi : V(G_1) \mapsto \{1, 2, \dots, |V(G_1)|\}$ ,  $\psi : V(G_2) \mapsto \{1, 2, \dots, |V(G_2)|\}$  are optimal 3-cutwidth labelings of  $G_1$  and  $G_2$  respectively, where*

$$\phi(v) = \begin{cases} \phi_1(v) & \text{if } v \in V(H_1), \\ \phi_2(v) + |V(H_1)| - 1 & \text{if } v \in V(H_2) \setminus \{x_2\}, \end{cases}$$

and

$$\psi(v) = \begin{cases} \phi_1(v) & \text{if } v \in V(H_1), \\ \phi_2(v) + |V(H_1)| - 1 & \text{if } v \in V(H_2) \setminus \{w\}. \end{cases}$$

**Proof.** For  $\phi_1$ , since  $x_1, y_1$  are the original and the terminal with degree 3 in  $H_1$ , we can conclude that  $\phi_1(x_1), \phi_1(y_1)$  can equal 1 and  $|V(H_1)|$  respectively, and so do the labels  $\phi_2(x_2)$  and  $\phi_2(y_2)$ . So, for  $j = 1, 2$ ,  $\phi_j$  is an optimal sublabeling of  $\phi$  restricted to block  $H_j$  of  $G_1$ , which leads to that  $c(G_1, \phi) = c(H_1, \phi_1) = 3$ . Thus  $\phi$  is an optimal labeling of  $G_1$ .

Now we consider  $\psi$  of  $G_2$ . The proof of the case of  $H_2 = C_3$  is straightforward, so it suffices to consider the case of  $H_2 = LG(x_2, y_2)$ . If  $\phi_2(w) = 1$  then it is trivial. Otherwise, assume that there is an optimal labeling  $\phi'_2$  of  $H_2$  such that  $\phi'_2(w) = \alpha \neq 1$ . By the assumption that  $w \in N_{H_2}(x_2)$  and  $d_{H_2}(w) = 2$ , define  $\phi_2$  as follows: for  $v \in V(H_2)$ ,

$$\phi_2(v) = \begin{cases} 1 & \text{if } v = w, \\ \phi'_2(v) + 1 & \text{if } v \neq w \text{ and } \phi'_2(v) < \alpha, \\ \phi'_2(v) & \text{if } v \neq w \text{ and } \phi'_2(v) > \alpha. \end{cases}$$

Then  $c(H_2, \phi_2) = c(H_2, \phi'_2) = 3$  leading to that  $\phi_2$  is also optimal for  $H_2$ . Thus, similar to the labeling of  $G_1$ , let  $\phi_1(x_1) = 1, \phi_1(y_1) = |V(H_1)|$ . Then  $\phi_j$  is an optimal sublabeling of  $\psi$  restricted to block  $H_j$  of  $G_2$  for  $j = 1, 2$ , which leads to  $c(G_2, \psi) = c(H_1, \phi_1) = c(H_2, \phi_2) = 3$ . So,  $\psi$  is an optimal labeling of  $G_2$ . ■

**Lemma 5.7.** *Each of the following holds.*

- (i) *If the neighbors of  $x_1, y_2$  are 2-degree gluing points of  $H_1, H_2$ , then  $G_1 \in \mathcal{MG}_{n_1,3}$  with  $n_1 = |V(G_1)|$ .*
- (ii) *If  $w$  and other neighbors of  $x_1, y_2$  are 2-degree gluing points of  $H_1, H_2$ , then  $G_2 \in \mathcal{MG}_{n_2,3}$  with  $n_2 = |V(G_2)|$ .*

**Proof.** (i) By Lemmas 5.4–5.6, each  $H_j \in \mathcal{MG}_{\mu_j,3}$  with  $\mu_j = |V(H_j)|$  for  $j = 1, 2$ , so it suffices to show that  $c(G_1 + uv) \geq 4$  for any  $uv \notin E(G_1)$  with  $u \in V(H_1)$  and  $v \in V(H_2)$ . In fact, let the identified vertex of  $y_1$  and  $x_2$  be  $z$  in  $G_1$ , i.e.,  $y_1 = x_2 = z$ , then  $d_{G_1}(z) = 6$ . Thus  $R_5$  (see Figure 3) is a minor in  $G_1 + uv$ , which results in  $c(G_1 + uv) \geq 4$ . So  $G_1 \in \mathcal{MG}_{n_1,3}$  with  $n_1 = |V(G_1)|$ .

(ii) Denote the identified vertex by  $z$  in  $G_2$ , i.e.,  $y_1 = w = z$ . First let  $H_2 = C_3$ . Since  $c(H_1) = 3$  and  $c(C_3) = 2$ . By Lemma 5.6,  $c(G_2) = 3$ . On the other hand, since  $x_1$  and its neighbors are 2-degree gluing points in  $H_1$  and  $H_2 = C_3$ , for any  $uv \notin E(H_1)$  with  $u, v \in V(H_1)$ ,  $c(G_2 + uv) \geq 4$  by Lemma 5.5. For any  $uv \notin E(H_1)$  with  $u \in V(H_1)$  and  $v \in V(C_3)$ , similar to that of (i), we can see that  $R_5$  is a minor of  $G_2 + uv$  because of  $d_{G_2}(z) = 5$ . So  $c(G_1 + uv) \geq 4$ . Next let  $H_2 = LG(x_2, y_2)$ . Since  $x_1, w$  and the neighbors of  $x_1$  are 2-degree gluing points, by Lemma 5.5, it suffices to consider the case of  $u \in V(H_1)$  and  $v \in V(H_2)$  for any  $uv \notin E(G_1)$ . In this case,  $R_5$  is also a minor of  $G_2 + uv$  because of  $d_{G_2}(z) = 5$ . Thus  $c(G + uv) \geq 4$ . ■

**Lemma 5.8.** *For 2-connected graph  $G$  with  $|V(G)| = n$  and  $c(G) = 3$ , if  $G \in \mathcal{MG}_{n,3}$ , then  $G \in \mathcal{G} \setminus \{C_3\}$ .*

**Proof.** Clearly,  $G \neq C_3$ . By assumption that  $G$  is 2-connected, there are at least two internally-disjoint paths  $P_{x,y}^{(1)}, P_{x,y}^{(2)}$  for  $x, y \in V(G)$  by Lemma 3.3. But it is not possible that there are four edge-disjoint paths between  $x$  and  $y$  because of  $c(G) = 3$ . So, there is a path  $P_{x,y}^{(3)}$  between  $x$  and  $y$  such that at most one of  $V(P_{x,y}^{(1)} \cap P_{x,y}^{(3)})$  and  $V(P_{x,y}^{(2)} \cap P_{x,y}^{(3)})$  is the empty set except  $x$  and  $y$ . Now suppose towards contradiction that  $G \notin \mathcal{MG}_{n,3}$ , and  $G$  is a minimum counterexample on  $|V(G)|$ , then we have

**Claim 2.**  *$G$  is either a graph Figure 9(a) or a graph Figure 9(b) only.*

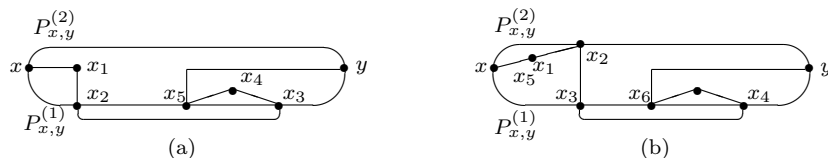


Figure 9. Two cases of minimum counterexamples of  $G$ .

In fact, for the case of  $V(P_{x,y}^{(1)} \cap P_{x,y}^{(3)}) = \emptyset$  and  $V(P_{x,y}^{(2)} \cap P_{x,y}^{(3)}) \neq \emptyset$ , let

$P_{x,y}^{(3)} = x_0x_1x_2 \cdots x_\rho x_{\rho+1}$  with  $x_0 = x, x_{\rho+1} = y$ , and let  $P_{x,y}^{(1)} = K_2 = xy$  by the assumption that  $G$  is minimum on  $|V(G)|$ . First, by  $G \notin \mathcal{MG}_{n,3}$ ,  $P_{x,y}^{(2)} \cap P_{x,y}^{(3)}$  is not a configuration like  $H_0$  (see  $H_0$  in Figure 5), as otherwise  $c(G) = 3$  by Lemma 5.4. So there is at least a vertex in  $P_{x,y}^{(3)}$ , say  $x_{i_0}$ , such that  $x_{i_0}$  must be a subdivision vertex of some edge  $x_{r_i}x_{r_{i+1}}$  of path  $P_{x,y}^{(2)}$ , where  $1 \leq i_0 \leq \rho$  and  $r_{i+1} \leq i_0 - 1$ . Since  $G$  is minimum and simple,  $|V(P_{x,y}^{(2)} \cap P_{x,y}^{(3)}) \setminus \{x, y\}| = 5$ , which results in that  $P_{x,y}^{(3)} = xx_1x_2x_3x_4x_5y$ , and  $G$  must be a graph in Figure 9(a). Similarly, for the case of  $V(P_{x,y}^{(j)} \cap P_{x,y}^{(3)}) \neq \emptyset$  for each  $j = 1, 2$ , by the minimality of  $G$ , we can first let  $V(P_{x,y}^{(1)} \cap P_{x,y}^{(3)}) = \{x_1\}$ . And then, for  $P_{x,y}^{(2)} \cap P_{x,y}^{(3)}$ , with an argument similar to the above case,  $P_{x,y}^{(3)} = xx_1x_2x_3x_4x_5x_6y$ , which results in that  $G$  is a graph in Figure 9(b). Hence Claim 2 holds.

By Claim 2,  $G$  is one of Figure 9(a) and Figure 9(b). However, in Figure 9(a) and Figure 9(b),  $R_4$  and  $R_9$  (see Figure 3) are minors leading to that the cutwidth of each of them is at least 4, contradicting  $c(G) = 3$ . So  $G \in \mathcal{G} \setminus \{C_3\}$ . This completes the proof. ■

**Lemma 5.9.** *For graph  $G \in \mathcal{MG}_{n,3}$  with  $|V(G)| = n$ , let  $B_1, B_2, \dots, B_\beta$  be blocks of  $G$ . Then  $B_i \in \mathcal{G}$  for each  $1 \leq i \leq \beta$ , and the block graph  $\mathcal{B}$  of  $G$  is a path  $P_\beta$ , where  $\beta \geq 2$ .*

**Proof.** If  $B_i = LG(x, y)$  or  $C_3$  then it is trivial by Lemma 5.8. So let  $B_i \neq LG(x, y)$  or  $C_3$ . We first verify that, for each  $1 \leq i \leq \beta$ ,  $B_i \neq K_2$ . Otherwise, let  $B_i \cap B_{i+1} = \{z_i\}$  and  $B_i = K_2 = z_{i-1}z_i$ . Clearly,  $c(B_{i+1}) \geq 3$  by Lemma 5.4 and  $d_{B_{i+1}}(z_i) \geq 3$  (since otherwise there must exist a vertex  $w \in V(B_{i+1})$  such that  $c(G + z_{i-1}w) = c(G)$ , a contradiction). Since  $G$  is simple, there is at least a vertex, say  $w$ , such that  $z_iw \in E(B_{i+1})$  and  $d_{B_{i+1}}(w) = 2$ . By Lemma 5.6, let  $\phi_{i+1}$  be the optimal sublabeling restricted to  $B_{i+1}$  by an optimal labeling  $\phi$  of  $G$ , in which  $\phi_{i+1}(w) = \min\{\phi_{i+1}(v) : v \in V(B_{i+1})\}$ . Then  $c(G + z_{i-1}w) = c(G)$ , contradicting  $G \in \mathcal{MG}_{n,3}$ . So  $B_i \neq K_2$ . Next we claim  $|V(B_i)| = 3$ . Otherwise, let  $|V(B_i)| \geq 4$ . Since  $c(B_i) \leq 3$ , by Lemmas 5.4, 5.5 and 5.8, for  $r \geq 4$ ,  $B_i$  must be either a cycle  $C_r$  with at least one simple 2-bridge or a cycle  $C_r$  without bridge. For the former, if  $C_r$  has a simple 2-bridge  $\mathbb{B}$  then  $B_i \in \mathcal{G}$ ; if  $C_r$  has at least two simple 2-bridges  $\mathbb{B}_1$  with 2 vertices  $x_1, x_2$  of attachment and  $\mathbb{B}_2$  with 2 vertices  $y_1, y_2$  of attachment, then  $\mathbb{B}_1$  and  $\mathbb{B}_2$  avoid each other (otherwise, if  $\mathbb{B}_1, \mathbb{B}_2$  skew then  $R_1$  is a minor; if  $x_1, y_1$  overlap then  $B_i = R_6 - st \in \mathcal{G}$  (see  $R_6$  in Figure 3)), where  $\mathbb{B}, \mathbb{B}_1$  and  $\mathbb{B}_2$  are all paths by definition. Thus there are two vertices, say  $x_2$  and  $y_1$ , such that  $c(G + x_2y_1) = c(G)$ , a contradiction. For the latter, let  $r = 4$ , i.e.,  $C_4 = x_1x_2x_3x_4x_1$ . In this case, if  $x_1$  and  $x_4$  are gluing vertices then  $c(G + x_1x_3) = 3$ ; if  $x_1$  and  $x_3$  are gluing vertices then  $c(G + x_2x_4) = 3$ , a contradiction. The case of  $r > 4$  is similar. Hence  $B_i = C_3$  resulting in  $B_i \in \mathcal{G}$ .

Now let  $z_i$  be a common vertex of three blocks  $B_i, B_{i+1}$  and  $B'$ , then  $d_{B_i}(z_i) = 2$ ,  $d_{B_{i+1}}(z_i) = 3$  (or  $d_{B_i}(z_i) = 3$ ,  $d_{B_{i+1}}(z_i) = 2$ ) and  $B' = K_2 = z_i z'$  (otherwise  $G$  has a 4-cutwidth subgraph containing  $z_i$  with  $d_G(z_i) \geq 6$ , a contradiction). Thus there is at least a vertex  $u \in N_{B_i}(z_i)$  (or  $u \in N_{B_{i+1}}(z_i)$ ) such that  $c(G + z'u) = c(G)$ , a contradiction. So  $B'$  does not exist, which leads to that  $\mathcal{B}$  is a path  $P_\beta$ . ■

**Theorem 5.10.** *For graph  $G$  with  $|V(G)| = n$ ,  $G \in \mathcal{MG}_{n,3}$  if and only if each of the following holds.*

- (i)  $G$  is planar, and its blocks can be listed as  $B_1, B_2, \dots, B_\beta$  with  $V(B_i) \cap V(B_{i+1}) = \{z_i\}$  ( $1 \leq i \leq \beta - 1$ ) such that the block graph  $\mathcal{B}$  is a path  $P_\beta$ .
- (ii) For each  $1 \leq i \leq \beta$ ,  $B_i \in \mathcal{G}$ , where  $\mathcal{G} = \{G : G \text{ is a } LG(x, y)\} \cup \{C_3\}$ .
- (iii) For each  $1 \leq i \leq \beta - 1$ ,  $d_{B_i}(z_i) \geq 2$ ,  $d_{B_{i+1}}(z_i) \geq 2$ , and at least one of them is 3.
- (iv)  $z_i$  is a gluing point with degree either 3 or 2 of  $B_i$  as well as  $B_{i+1}$ . If  $d_{B_i}(z_{i-1}) = d_{B_i}(z_i) = 2$ , then  $B_i \in \mathcal{MG}_{\mu_i,3}$  with  $\mu_i = |V(B_i)|$  or  $B_i = C_3$ . If  $d_{B_i}(z_i) = 3$  or  $d_{B_{i+1}}(z_i) = 3$ , then  $B_i = LG_i$  or  $B_{i+1} = LG_{i+1}$ , and the 2-degree neighbors of  $z_i$  in  $B_i$  or  $B_{i+1}$  are unnecessary to be the gluing points, where  $LG_i = LG(x_i, y_i)$ .
- (v) If  $B_1 = LG_1$  and  $B_\beta = LG_\beta$ , then the neighbors with degree 2 of  $x_1, y_\beta$  must be 2-degree gluing points in  $B_1, B_\beta$ , respectively.

**Proof.** *Sufficiency.* By Lemmas 5.4–5.7,  $G \in \mathcal{MG}_{n,3}$  is true.

*Necessity.* (i) and (ii) are true by Lemmas 5.3, 5.8 and 5.9.

(iii) Clearly,  $d_{B_i}(z_i) \geq 2$ ,  $d_{B_{i+1}}(z_i) \geq 2$  by Lemma 5.9. Assume now that  $B_i = LG(x_i, y_i)$  and  $d_{B_i}(z_i) = d_{B_{i+1}}(z_i) = 2$  for  $1 \leq i \leq \beta - 1$ . Then there are at least two vertices  $u \in N_{B_i}(z_i)$  and  $v \in N_{B_{i+1}}(z_i)$ , say  $u = y_i$  and  $v = x_{i+1}$ , such that  $c(G + uv) = c(G)$ , a contradiction to  $G \in \mathcal{MG}_{n,3}$ . So one member of  $\{d_{B_i}(z_i), d_{B_{i+1}}(z_i)\}$  is 3.

(iv) The first conclusion is obvious. For the second conclusion,  $B_i = LG(x_i, y_i)$  or  $C_i$  by Lemma 5.9. So it is needed to show that if  $B_i = LG(x_i, y_i)$  then  $B_i \in \mathcal{MG}_{\mu_i,3}$ . In fact, if  $B_i \notin \mathcal{MG}_{\mu_i,3}$ , then  $z_{i-1}$  (or  $z_i$ ) must be adjacent to a 2-degree vertex  $w_{i-1}$  (or  $w_i$ ). Without loss of generality, let  $N_{B_i}(z_{i-1}) = \{x_i, w_{i-1}\}$  and  $N_{B_i}(z_i) = \{y_i, w_i\}$ . Then, by Lemma 5.6, there is a sublabeling  $\phi_i$  restricted to  $B_i$  by an optimal labeling  $\phi$  of  $G$  such that  $\phi_i(z_{i-1}) = \min\{\phi_i(v) : v \in V(B_i)\}$ ,  $\phi_i(w_{i-1}) = \phi_i(z_{i-1}) + 1$  and  $\phi_i(x_i) = \phi_i(z_{i-1}) + 2$ . Thus  $c(G + x_i w_{i-1}, \phi) = c(G, \phi)$ , a contradiction. Likewise, for  $z_i$ , let  $\phi_i(z_i) = \max\{\phi_i(v) : v \in V(B_i)\}$ ,  $\phi_i(w_i) = \phi_i(z_i) - 1$  and  $\phi_i(y_i) = \phi_i(z_i) - 2$ . Then  $c(G + y_i w_i, \phi) = c(G, \phi)$ , also a contradiction. Hence  $B_i \in \mathcal{MG}_{\mu_i,3}$ . For the third conclusion, on the one hand,  $B_i \neq C_3$  or  $B_{i+1} \neq C_3$  (as otherwise  $d_{B_i}(z_i) = 2$  or  $d_{B_{i+1}}(z_i) = 2$ ), so  $B_i = LG_i$  or  $B_{i+1} = LG_{i+1}$  by (ii). On the other hand,  $z_i = y_i = x_{i+1}$ , which is a 3-degree

gluing point of  $B_i$  as well as  $B_{i+1}$ . So the 2-degree neighbors of  $z_i$  in  $B_i$  and  $B_{i+1}$  are not necessarily the gluing points.

(v) Similar to that of (iii), by Lemma 5.7, (v) is also true, omitted here. This completes the proof. ■

Figure 10 is a graph  $G \in \mathcal{MG}_{n,3}$  with  $\beta$  blocks  $B_i$  and their gluing pattern, in which  $w_i, w'_i$  are neighbors with degree two of  $x_i$  and  $y_i$  in  $B_i$  for  $1 \leq i \leq \beta$  respectively, and  $B_1 = LG_1 = LG(x_1, y_1)$ .

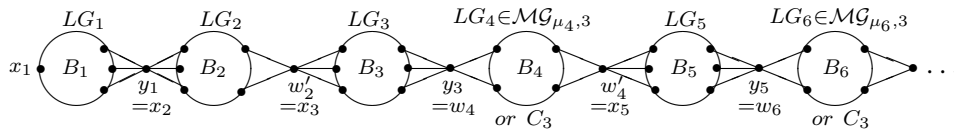


Figure 10. The gluing pattern of blocks of  $G$ .

**Corollary 5.11.** *Suppose that  $G \in \mathcal{MG}_{n,3}$  with blocks  $B_1, B_2, \dots, B_\beta$  ( $\beta \geq 1$ ),  $V(B_i) \cap V(B_{i+1}) = \{z_i\}$ ,  $\{z_{l_1}, z_{l_2}, \dots, z_{l_r}\} \subseteq \{z_i : 1 \leq i \leq \beta-1\}$  for  $1 \leq r \leq \beta-1$ . If  $d_{B_{l_j}}(z_{l_j}) = d_{B_{l_j+1}}(z_{l_j}) = 3$  for each  $1 \leq j \leq r$ ,  $G'$  is obtained by triangulating each  $z_{l_j}$  consecutively, then  $G' \in \mathcal{MG}_{n+2r,3}$ .*

**Proof.** The proof is straightforward by Theorem 5.10(iv). ■

## 6. REMARKS

In this paper, we characterized the structures of the edge-maximal graphs with  $c(G) \leq 3$ , from which we know that any edge-maximal graph with  $c(G) \leq 3$  is decomposable. Regarding the edge-maximal graphs with  $c(G) \geq 4$ , we guess that the structure of each of them is similar to that of 3-cutwidth edge-maximal graphs. But its block  $B_i$  does not necessarily consist of 4 edge-disjoint paths. We achieved some results in this direction and will try to finish those efforts. For instance, let  $G = (R_1 \odot_{s,s_0} G(s_0, t_0)) \odot_{t_0,t'} R'_1$ , where  $G(s_0, t_0)$  is a 4-paths graph with common vertices  $s_0$  and  $t_0$  (see Lemma 3.2),  $R'_1$  and  $t'$  are copies of  $R_1$  and  $t$  in Figure 3, respectively. It is not hard to verify that  $G \in \mathcal{MG}_{n,4}$ . However, it seems that our technique cannot be easily used to examine the structure of the class of graphs. A further task is to detect the structures of such  $k$ -cutwidth edge-maximal graphs for  $k \geq 4$ .

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