

THE PETERSEN AND HEAWOOD GRAPHS MAKE UP GRAPHICAL TWINS VIA INDUCED MATCHINGS

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Abstract

Inspired by the Isaacs remark (published in 1975), we show that the Petersen and Heawood graphs (Pg and Hg) make up a bijectively linked pair of graphs. Another related new result is that Pg is uniquely decomposable into five induced 3-matchings. It shows a kind of the structural rigidity of Pg . Information on maximal matchings with sizes 3, 4 and 5 in Pg is recalled. Constructive proofs confirm that the strong chromatic index $sq(Pg) = 5$ and $sq(Hg) = 7$. The three numerical edge coloring partitions for Pg are also determined.

Keywords: Heawood graph, induced matchings, Petersen graph, strong chromatic index.

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1. PRELIMINARIES

A matching in a graph is called an induced matching if no two edges in the matching are adjacent to another edge of the graph. Strong chromatic index (in symbols sq) of a graph is the smallest number of parts among decompositions of the graph into induced matchings. In both our graphs maximal induced matchings comprise three edges. It is claimed in Faudree *et al.* [4] that $sq(Pg) = 5$ and $sq(Hg) = 7$. We prove that Pg is uniquely decomposable into five induced 3-matchings. We construct next a decomposition of Hg into seven induced 3-matchings.

Theorem 1 (Isaacs [8]). *If a normal map on a closed surface of any genus is 4-region colorable, then its graph G is 3-edge colorable.*

Isaacs' 'combinatorial' proof [8, p. 223] deserves recalling: "Let A, B, C, D be colors of the countries. Color an edge of G 1 if the adjoining countries are A, B or C, D ; 2, if they are A, C or B, D ; 3, if they are A, D or B, C ."

Isaacs explains that the converse theorem is not true. He refers to a map on the torus (wrongly named Heffter's map) which is 7-region colorable and its graph is 3-edge colorable. Actually, it is a toroidal dual of the Heawood [6] toroidal triangulation which is an embedding of the complete graph K_7 in the torus, see Figures in [7, p. 302]. The Heawood graph, Hg , (which is cubic and 3-edge colorable) is the graph of that dual map.

2. ON THE HEAWOOD AND PETERSEN GRAPHS

Isaacs is the first who noticed a close relation between Hg and Pg [8, p. 225].

"If any vertex and its three incident edges are removed from Hg , Pg results". That observation is implied by the following elegant description due to Isaacs. The graph Hg "is a 14-gon with two vertices — i and j under consecutive numeration — also connected when $i - j \equiv 5 \pmod{14}$ ", i.e., $i - j = 5$ if j is odd and $j \leq 9$, otherwise $j = 11, 13$ and $i = 2, 4$, respectively.

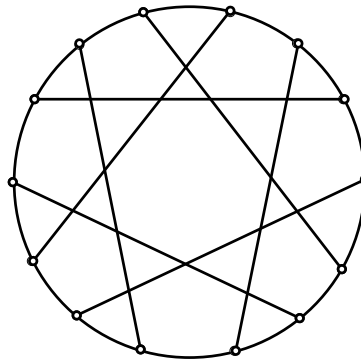


Figure 1. The Heawood graph.

Isaacs' observation is correct if his operation on Hg is seen as a claw **anihilation** (or an annihilation of a claw). Otherwise, it should have the conclusion that what results is: either a subdivision of Pg (as in [7, p. 209]) or Pg with three subdivided edges. We have noticed that those three edges make up an induced 3-matching in Pg , see Figure 2.

Proposition 2. *Deleting a vertex from the Heawood graph Hg gives the Petersen graph Pg in which an induced 3-matching is subdivided.*

Notice that the Petersen and Heawood graphs are both highly symmetric. Namely they are cubic, distance transitive and s -transitive with the largest possible s , $s = \lfloor (g+2)/2 \rfloor$ where g is the girth, $(g, s) = (5, 3), (6, 4)$, respectively. This is listed in [7, Table 3, p. 208]. That list can be slightly modified by including a multigraph. One can see that the following chain of claw annihilations involves a smaller cubic graph $K_{3,3}$ from that table and next a cubic multigraph, $K_2^{(3)}$, on two vertices.

$$Hg \mapsto Pg \mapsto K_{3,3} \mapsto K_2^{(3)}.$$

The converse chain of transformations

$$Hg \leftarrow Pg \leftarrow K_{3,3} \leftarrow K_2^{(3)}$$

is defined as follows. In each step a new vertex of degree 3 is attached to 3 edges so that order and size increase by 4 and 6, respectively, and girth also increases. The girth of Hg cannot be increased this way. This converse chain is the chain of converse transformations to claw annihilations. Consequently, Hg cannot be obtained by applying a claw annihilation.

Theorem 3. *When an induced 3-matching in Pg is subdivided and the three subdividing vertices are joined to a new vertex, what results is the graph Hg .*

Proof is implied by the symmetry of Pg which is stated above. However, conclusive is the following property which shows that Pg is a kind of a gem.

Lemma 4. *For any edge e of Pg there is exactly one maximal induced matching in Pg containing the edge e and this matching comprises three edges.*

Proof. Let e be a fixed edge of Pg . Then there are four length-3 paths in Pg containing e as the central edge and there are altogether five edges covered by those paths. Each of those four paths has a private 2-edge extension to a pentagon in Pg . Therefore the number of edges in Pg covered by those pentagons is $5 + 4 \cdot 2$ only. Hence there are two edges of Pg , say e' and e'' , which are not covered and either of them joins such two of the pentagons which have e as the only edge in common. Consequently, the edges e, e', e'' make up the unique induced 3-matching containing the edge e , see Figure 2. ■

Comment. That proof shows a structural rigidity of Pg .

Five mutually disjoint induced 3-matchings which make up a decomposition of Pg are obtainable in Figure 2 by rotating the given 3-matching.

Theorem 5. *Pg is uniquely decomposable into five induced 3-matchings.*

Thus we have proved a claim in [4] that the strong chromatic index $sq(Pg) = 5$.

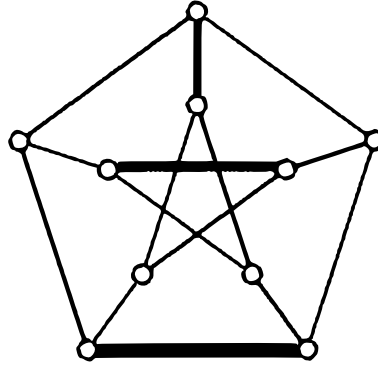


Figure 2. The Petersen graph with an induced 3-matching.

On continuing the proof of Theorem 3 we note that induced 3-matchings are mutually similar in Pg and subdividing any of them increases the girth to 6. The girth increases because removal of any induced 3-matching from Pg gives a bipartite subgraph, the subdivided complete graph K_4 , without any pentagon. In the standard ‘pentagonal drawing’ of the Petersen graph (Figure 2) each induced 3-matching comprises two parallel edges and one perpendicular to them.

Corollary 6. *In the Petersen graph Pg*

- (i) *each edge is uniquely extendable to an induced 3-matching, and each induced 3-matching comprises three induced 2-matchings,*
- (ii) *the number of induced k -matching is five if $k = 3$ and 15 if $k = 2$,*
- (iii) *any two induced k -matchings are similar, $k = 2, 3$,*
- (iv) *removal of an induced 3-matching from Pg gives the subdivided square C_4 with subdivided diagonals (i.e., the subdivided K_4); conversely, adding three edges which join a pair of degree-2 vertices on diagonals and both pairs on the opposite sides of C_4 gives back Pg .*

We are going to use Theorem 3 in order to construct a decomposition of Hg into seven induced 3-matchings. We consider Hg as obtained by attaching a claw to an induced 3-matching in Pg . Now we put labels 1, 2, 3 in order to differentiate between rays of the claw and next between each of rays and the two adjacent non-rays. Then edges with the same label make up one of three induced 3-matching in Hg . Hence Hg is decomposable into seven induced 3-matchings because four more come from Pg . Moreover, 3 is the largest size among induced 3-matchings in Hg . Thus we have proved a claim in [4] that the strong chromatic index $sq(Hg) = 7$.

3. PETERSEN'S MATCHINGS

A matching which is not a proper submatching is called a maximal matching. In the Petersen graph maximal are 5-matchings (perfectness), induced 3-matchings, and also 4-matchings each of which is obtained from an induced 3-matching by replacing its edge, say the edge e , with a 2-matching which covers both endvertices of e .

3.1. Edge coloring classes

Remarks in the paper of Adel'son-Vel'skii and Titov [1, p. 12] can be read as follows.

Proposition 7. *Deleting any 2-matching (either induced or not) from the Petersen graph gives a subgraph with chromatic index 3.*

Theorem 8. *There are the three partitions of 15, namely 5, 4, 4, 2; 5, 4, 3, 3; and 4, 4, 4, 3, each of which is the sequence of sizes of edge coloring classes of Pg .*

Proof. These are sizes of matchings which are being removed from Pg while producing an edge 4-coloring of Pg . The term 5 arises after removing a 5-matching. The next terms after 5 are sizes in the subgraph $2C_5$ which comprises two disjoint pentagons. On the other hand the term 3 in the last partition is for an induced 3-matching. ■

Corollary 9. *Deleting an induced 3-matching from Pg gives a subcubic subgraph decomposable into three 4-matchings.*

3.2. Petersen's matchings in homogeneous traceability

Graphs with a Hamiltonian path are called traceable. Homogeneously traceable is a graph in which every vertex is an end-vertex of a Hamiltonian path. This notion was introduced in 1975 in [15] and a typewritten preprint [10] which was submitted to Ann. New York Acad. Sci. and not published. Nevertheless, several graph-theorists were inspired by the preprint, see two teams: Bermond *et al.* [2] (on digraphs) and Chartrand *et al.* [3], see next [5, 16] and also the present author's several publications, e.g. [11, 12]. The articles [13, 14] were influenced by the respective team's works.

The following result is proved in [14, p. 9].

Theorem 10. *Let E_1 be a subset of edges in Pg and let G be a graph obtained by subdividing once or twice each edge in E_1 . Then G is a homogeneously traceable graph if and only if E_1 is a matching and E_1 is not an induced 3-matching.*

Crucial for a proof is the fact that the four vertices which are not covered by an induced 3-matching in Pg are mutually nonadjacent.

4. CONCLUDING REMARKS

Both graphs, Hg and Pg , are milestones in the history of graph theory. It is rather surprising that they are as related as presented in this paper.

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