

## MORE ON SIGNED GRAPHS WITH AT MOST THREE EIGENVALUES

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### Abstract

We consider signed graphs with just 2 or 3 distinct eigenvalues, in particular (i) those with at least one simple eigenvalue, and (ii) those with vertex-deleted subgraphs which themselves have at most 3 distinct eigenvalues. We also construct new examples using weighing matrices and symmetric 3-class association schemes.

**Keywords:** adjacency matrix, simple eigenvalue, strongly regular signed graph, vertex-deleted subgraph, weighing matrix, association scheme.

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## 1. INTRODUCTION

A *signed graph*  $\dot{G}$  is defined to be a pair  $(G, \sigma)$ , in which  $G = (V, E)$  is an unsigned graph, called the *underlying graph*, and  $\sigma: E \rightarrow \{1, -1\}$  is the *sign function*, also known as the *signature*. The *order* of a signed graph, denoted by  $n$ , is the number of its vertices. The edge set of  $\dot{G}$  consists of the subsets of positive and negative edges. We interpret an unsigned graph as a signed graph in which all edges are positive.

The *adjacency matrix*  $A_{\dot{G}}$  of  $\dot{G}$  is the  $n \times n$   $(0, 1, -1)$ -matrix which is obtained from the adjacency matrix of its underlying graph by reversing the sign of all 1s which correspond to negative edges. By the *spectrum* of  $\dot{G}$ , we mean the spectrum of  $A_{\dot{G}}$ . An eigenvalue of  $\dot{G}$  is called a *main* eigenvalue if the corresponding eigenspace is not orthogonal to the all-1 vector. Throughout the paper, by the statement ' $\dot{G}$  has  $k$  eigenvalues' we mean that  $\dot{G}$  has exactly  $k$  distinct eigenvalues.

In Section 2 we give some terminology and notation, and prove some auxiliary results. The problem of classifying graphs with a comparatively small number of eigenvalues has attracted a great deal of attention in the last 70 years; some recent results can be found in [4, 5, 7, 16, 17]. In [14] we considered regular and non-regular signed graphs with at most 3 eigenvalues; here we continue this research and pay more attention to non-regular signed graphs. In Section 3, we consider connected signed graphs with 3 eigenvalues at least one of which is simple. The number of simple eigenvalues governs our investigation in Section 4 of vertex-deleted subgraphs which themselves have 3 eigenvalues. In Section 5 we construct some signed graphs with 2 or 3 eigenvalues using weighing matrices or symmetric 3-class association schemes, and note the implications for vertex-deleted subgraphs.

## 2. PRELIMINARIES

We write  $I$ ,  $O$ ,  $J$ ,  $\mathbf{0}$  and  $\mathbf{j}$  for an identity matrix, an all-0 matrix, an all-1 matrix, an all-0 vector and an all-1 vector, respectively. Subscripts indicate size as necessary.

A signed graph  $\dot{G}$  is said to be connected, complete, regular or bipartite if the same holds for its underlying graph. The degree of a vertex in  $\dot{G}$  is the degree of the same vertex in  $G$ . The *net-degree* of a vertex  $i$ , denoted by  $d_i^\pm$ , is the difference between the numbers of positive and negative edges incident with  $i$ . A signed graph in which vertex net-degrees are equal is called *net-regular*. Similarly, a *net-biregular* signed graph is a signed graph which has 2 distinct net-degrees. It is known that  $\dot{G}$  is net-regular if and only if  $\mathbf{j}$  is an eigenvector of  $\dot{G}$ , and then  $\mathbf{j}$  belongs to the eigenspace of the net-degree [20].

A signed graph is said to be *homogeneous* if all its edges have the same sign

(in particular, if its edge set is empty). Otherwise, it is said to be *inhomogeneous*. The *negation*  $-\dot{G}$  is obtained by reversing the sign of every edge of  $\dot{G}$ .

We say that signed graphs  $\dot{G}$  and  $\dot{H}$  are *isomorphic* if there is a permutation matrix  $P$  such that  $A_{\dot{H}} = P^{-1}A_{\dot{G}}P$ . In this case we write  $\dot{G} \cong \dot{H}$ . We say that  $\dot{G}$  and  $\dot{H}$  are *switching equivalent* if there is a vertex subset  $S \subseteq V(\dot{G})$ , such that  $\dot{H}$  is obtained by reversing the sign of every edge with one vertex in  $S$  and the other in  $V(\dot{G}) \setminus S$ .

If the vertex labelling is transferred from the underlying graph common to  $\dot{G}$  and  $\dot{H}$ , then  $\dot{G}$  and  $\dot{H}$  are switching equivalent if and only if there is a diagonal matrix  $D$  with  $\pm 1$  on the diagonal such that  $A_{\dot{H}} = D^{-1}A_{\dot{G}}D$ . Clearly, isomorphism and switching equivalence preserve the spectrum.

An *equitable partition* of a signed graph  $\dot{G}$  is a partition of the vertex set  $V(\dot{G})$  into non-empty cells  $C_1, C_2, \dots, C_s$ , such that each cell induces a net-regular signed graph and for  $1 \leq i < j \leq s$  the edges between  $C_i$  and  $C_j$  induce a net-biregular or net-regular signed graph, in which vertices from each of  $C_i, C_j$  are equal in net-degree.

We say that a signed graph  $\dot{G}$  is *strongly regular* (for short,  $\dot{G}$  is a *SRSG*) with parameters  $r, a, b, c$  if the entries of  $A_{\dot{G}}^2$  satisfy

$$a_{ij}^{(2)} = \begin{cases} r & \text{if } i = j, \\ a & \text{if } i \overset{+}{\sim} j, \\ b & \text{if } i \overset{-}{\sim} j, \\ c & \text{if } i \not\sim j \text{ and } i \neq j. \end{cases}$$

Note that  $a_{ij}^{(2)}$  is the difference between the numbers of positive and negative  $i$ - $j$  walks of length 2 in  $\dot{G}$ . Accordingly, this definition generalizes the definition of strongly regular graphs. We mostly deal with SRSGs in Subsection 5.2.

In the forthcoming sections we frequently use the following result.

**Proposition 1** [14]. *A connected signed graph  $\dot{G}$  has exactly one positive eigenvalue if and only if  $\dot{G}$  is switching equivalent to a non-trivial complete multipartite graph. If  $\dot{G}$  has exactly one non-negative eigenvalue, then  $\dot{G}$  is switching equivalent to a complete graph.*

We now transfer the following two results from the domain of unsigned graphs.

**Proposition 2.** *If  $A$  is a real symmetric matrix with distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$  such that  $\lambda_1$  is a simple eigenvalue, then*

$$\prod_{i=2}^k (A - \lambda_i I) = \left( \prod_{i=2}^k (\lambda_1 - \lambda_i) \right) \mathbf{x}\mathbf{x}^T,$$

where  $\mathbf{x}$  is a unit eigenvector associated with  $\lambda_1$ . For  $k = 3$ , there exists an eigenvector  $\mathbf{a}$  for  $\lambda_1$ , such that  $(A - \lambda_2 I)(A - \lambda_3 I) = p\mathbf{a}\mathbf{a}^\top$ , where

$$p = \begin{cases} 1 & \text{if } \lambda_1 \notin (\lambda_2, \lambda_3), \\ -1 & \text{if } \lambda_1 \in (\lambda_2, \lambda_3). \end{cases}$$

**Proof.** Considering the spectral decomposition of  $A$ , we see that there exists an orthogonal matrix  $X$  such that

$$\prod_{i=2}^k (A - \lambda_i I) = X \begin{pmatrix} \prod_{i=2}^k (\lambda_1 - \lambda_i) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} X^\top = \left( \prod_{i=2}^k (\lambda_1 - \lambda_i) \right) \mathbf{x}\mathbf{x}^\top,$$

where  $\mathbf{x}$  is a unit eigenvector of  $\prod_{i=2}^k (A - \lambda_i I)$  afforded by  $\prod_{i=2}^k (\lambda_1 - \lambda_i)$ . The result follows since  $A\mathbf{x} = \lambda_1 \mathbf{x}$ .

For  $k = 3$ , by taking  $\mathbf{a} = \sqrt{p(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}\mathbf{x}$ , we arrive at the desired result.  $\blacksquare$

The previous theorem is a slight extension of the result in which  $A$  is the adjacency matrix of an unsigned graph,  $k = 3$  and  $\lambda_1$  is the largest eigenvalue [4, 7, 16]. Our formulation is more general in order to embrace signed graphs, with the possibility that  $\lambda_1$  is not the largest eigenvalue.

**Proposition 3.** Let  $\dot{G}$  be obtained from a signed graph  $\dot{H}$  of order  $n$  by adding a new vertex whose neighbourhood in  $\dot{H}$  is determined by the characteristic  $(0, 1, -1)$ -vector  $\mathbf{r}$ . The characteristic polynomial of  $\dot{G}$  is given by

$$(1) \quad P_{\dot{G}}(x) = P_{\dot{H}}(x) \left( x - \sum_{i=1}^m \frac{\|Q_i \mathbf{r}\|^2}{x - \mu_i} \right),$$

where  $\mu_1, \mu_2, \dots, \mu_m$  are the distinct eigenvalues of  $\dot{H}$  and  $Q_1, Q_2, \dots, Q_m$  are the matrices of the orthogonal projections of  $\mathbb{R}^n$  onto the eigenspaces of  $\dot{H}$  with respect to the canonical basis.

**Proof.** Using the Schur matrix decomposition in conjunction with the known identity  $\text{adj}(xI - A_{\dot{H}}) = \det(xI - A_{\dot{H}})(xI - A_{\dot{H}})^{-1}$ , we obtain

$$\begin{aligned} P_{\dot{G}}(x) &= \det \begin{pmatrix} x & -\mathbf{r}^\top \\ -\mathbf{r} & xI - A_{\dot{H}} \end{pmatrix} = xP_{\dot{H}}(x) - \mathbf{r}^\top \text{adj}(xI - A_{\dot{H}}) \mathbf{r} \\ &= P_{\dot{H}}(x) (x - \mathbf{r}^\top (xI - A_{\dot{H}})^{-1} \mathbf{r}). \end{aligned}$$

Since  $(xI - A_{\dot{H}})^{-1}$  has spectral decomposition  $\sum_{i=1}^m \frac{1}{x - \mu_i} Q_i$ , we have

$$P_{\dot{G}}(x) = P_{\dot{H}}(x) \left( x - \mathbf{r}^\top \left( \sum_{i=1}^m \frac{1}{x - \mu_i} Q_i \right) \mathbf{r} \right).$$

Now (1) follows since  $\mathbf{r}^\top Q_i \mathbf{r} = \mathbf{r}^\top Q_i Q_i \mathbf{r} = \mathbf{r}^\top Q_i^\top Q_i \mathbf{r} = (Q_i \mathbf{r})^\top Q_i \mathbf{r} = \|Q_i \mathbf{r}\|^2$ . ■

The ‘unsigned’ version of the previous theorem is well-known, see [6, Theorem 2.2.8]. The *cone* over a signed graph  $\dot{G}$  is obtained by adding a vertex  $v$  along with positive edges between  $v$  and every vertex of  $\dot{G}$ . We denote this cone by  $K_1 \nabla \dot{G}$ . The following result is a direct consequence of the previous one.

**Corollary 4.** *The cone over  $\dot{H}$  has the characteristic polynomial*

$$P_{K_1 \nabla \dot{H}}(x) = P_{\dot{H}}(x) \left( x - \sum_{i=1}^m \frac{n \beta_i^2}{x - \mu_i} \right),$$

where  $\mu_1, \mu_2, \dots, \mu_m$  are distinct eigenvalues of  $\dot{H}$  and  $\beta_1, \beta_2, \dots, \beta_m$  are the corresponding main angles defined by  $\beta_i = \|Q_i \mathbf{j}\| / \sqrt{n}$ .

### 3. SIGNED GRAPHS WITH 3 EIGENVALUES, AT LEAST ONE OF WHICH IS SIMPLE

In this section we give some characterizations of signed graphs described in the section title. We start with the following lemma.

**Lemma 5.** *If  $\dot{G}$  is a connected signed graph with 3 eigenvalues such that at least 2 of them are simple, then  $\dot{G}$  is switching equivalent to a complete bipartite graph.*

**Proof.** If every eigenvalue of  $\dot{G}$  is simple, then  $\dot{G}$  is switching equivalent to (the complete bipartite graph)  $K_{1,2}$ . Now suppose that  $\lambda$  is the unique non-simple eigenvalue. If  $\lambda = 0$  then  $\dot{G}$  has exactly one positive eigenvalue, hence is switching equivalent to a complete multipartite graph by Proposition 2.1. Moreover, since its spectrum has the form  $[-\rho, 0^{n-2}, \rho]$ ,  $\dot{G}$  is switching equivalent to a complete bipartite graph [6, p. 47].

If  $\lambda \neq 0$  then  $\dot{G}$  has a connected subgraph without  $\lambda$  as an eigenvalue, namely  $K_1$ . Since the eigenspace of  $\lambda$  has codimension 2,  $K_1$  can be extended to a connected induced subgraph  $\dot{H}$  of order 2 without  $\lambda$  as an eigenvalue (see [6, Theorem 5.1.6], which can be extended to the framework of signed graphs with slight modifications in the proof). Since  $\dot{H} \cong \pm K_2$  we have  $\lambda \notin \{1, -1\}$ . Since also  $\lambda \neq 0$ , we know from [13, Theorem 3.3] that  $\dot{G}$  has at most 4 vertices, and this case is resolved by inspection. ■

Now we consider signed graphs with 3 eigenvalues, exactly one of which is simple. Accordingly, we assume that a connected signed graph  $\dot{G}$  has spectrum  $[\rho, \mu^m, \lambda^l]$ , with  $m, l \geq 2$  and  $\mu > \lambda$ . By Proposition 2, there is a non-zero vector  $\mathbf{a} = (a_1, a_2, \dots, a_n)^\top$  such that  $A\mathbf{a} = \rho\mathbf{a}$  and

$$(2) \quad (A - \mu I)(A - \lambda I) = p\mathbf{a}\mathbf{a}^\top,$$

where  $A = A_{\dot{G}}$ ,  $p = -1$  if  $\mu > \rho > \lambda$ , and  $p = 1$  otherwise. By equating the diagonal entries of both sides, we get

$$(3) \quad d_i = pa_i^2 - \mu\lambda,$$

where  $d_i$  is degree of the vertex  $i$ .

**Lemma 6.** *If  $\dot{G}$  is a connected net-regular signed graph with spectrum  $[\rho, \mu^m, \lambda^l]$ , where  $\rho$  is its net-degree, then  $\dot{G}$  is regular.*

**Proof.** Every eigenvector afforded by  $\rho$  is constant, which, by (3), means that  $\dot{G}$  is regular. ■

We note in passing that, by [8], the signed graph  $\dot{G}$  mentioned in the previous result is strongly regular. Moreover, we have the following result, which will be used in our last section.

**Lemma 7.** *A connected inhomogeneous non-complete regular signed graph  $\dot{G}$  is net-regular with spectrum  $[\rho, \mu^m, \lambda^l]$  ( $\rho$  being the net-degree) if and only if  $\dot{G}$  is strongly regular and its parameters satisfy  $a + b = 2c \neq 0$ .*

**Proof.** Let  $\dot{G}$  have spectrum as in the statement of the lemma. Since every eigenvector afforded by  $\rho$  is constant, by (2) we have

$$A^2 = (\mu + \lambda)A - \mu\lambda I + kJ, \quad \text{for some } k \neq 0.$$

Comparing the entries of the left and the right hand side, we conclude that  $\dot{G}$  is strongly regular with  $a + b = 2k$  and  $c = k \neq 0$ . The converse follows directly from [8, Theorem 4.2]. ■

Next we deal with the case in which an eigenvalue other than  $\rho$  is the only non-main eigenvalue.

**Theorem 8.** *If  $\dot{G}$  is a connected signed graph with spectrum  $[\rho, \mu^m, \lambda^l]$  ( $m, l \geq 2$ ) such that only  $\lambda$  is non-main, then there is a non-zero constant  $\alpha$  such that*

$$(4) \quad d_i = \alpha(d_i^\pm - \mu)^2 - \mu\lambda,$$

where  $d_i$  and  $d_i^\pm$  are the degree and the net-degree of the vertex  $i$ , respectively.

*In particular, if  $\dot{G}$  is regular then it is net-biregular, the sum of the corresponding net-degrees is  $2\mu$  and  $\dot{G}$  is switching equivalent to a net-regular signed graph. If  $\dot{G}$  is biregular then it is switching equivalent to a net-biregular signed graph.*

**Proof.** Observe first that  $\dot{G}$  is not net-regular, since it has more than one main eigenvalue. We retain the notation introduced in Lemma 6. Since  $\rho$  and  $\mu$  are main, we have

$$(5) \quad (A - \rho I)(A - \mu I)\mathbf{j} = \mathbf{0},$$

as proved in [19] on the basis of the result for unsigned graphs which can be found in [15]. It follows that  $A^2\mathbf{j} \in \text{span}\langle \mathbf{d}, \mathbf{j} \rangle$ , where  $A\mathbf{j} = \mathbf{d} = (d_1^\pm, d_2^\pm, \dots, d_n^\pm)^\top$ . Moreover, since  $\mathbf{a}^\top \mathbf{j} \neq 0$ , (2) shows that  $\mathbf{a} \in \text{span}\langle \mathbf{d}, \mathbf{j} \rangle$ . Hence, we may write  $\mathbf{a} = r\mathbf{d} + s\mathbf{j}$ , where  $r \neq 0$  as  $\dot{G}$  is not net-regular.

By (5), we have  $A^2\mathbf{j} = (\rho + \mu)A\mathbf{j} - \rho\mu\mathbf{j}$ , which together with (2), gives

$$(\rho - \lambda)\mathbf{d} - \mu(\rho - \lambda)\mathbf{j} = p\mathbf{a}(\mathbf{a}^\top \mathbf{j}) = p(\mathbf{a}^\top \mathbf{j})(r\mathbf{d} + s\mathbf{j}).$$

By equating the coefficients of  $\mathbf{d}$  and  $\mathbf{j}$ , we find that  $s = -\mu r$ , and so

$$(6) \quad \mathbf{a} = r(\mathbf{d} - \mu\mathbf{j}).$$

Using (3), we obtain  $d_i = pr^2(d_i^\pm - \mu)^2 - \mu\lambda$ , and by setting  $\alpha = pr^2$ , we arrive at (4).

Now, if  $\dot{G}$  is  $d$ -regular, then  $d = \alpha(d_i^\pm - \mu)^2 - \mu\lambda$  for every vertex  $i$ . Evidently, this equation has 2 solutions in  $d_i^\pm$  and since  $\dot{G}$  is not net-regular, both solutions appear as net-degrees; hence,  $\dot{G}$  is net-biregular. The sum of the corresponding net-degrees follows from the previous equation for  $d$ . Lastly, from (3) we see that the coordinates of  $\mathbf{a}$  are equal in absolute value. If  $D$  is the diagonal matrix of  $\pm 1$ s with 1 in the  $i$ th position precisely when  $a_i$  is positive, then  $D^{-1}AD$  is the adjacency matrix of a switching equivalent signed graph, say  $\dot{H}$ . Moreover,  $D\mathbf{a}$  is a constant eigenvector associated with  $\rho$  in  $\dot{H}$ , which means that  $\dot{H}$  is net-regular.

Finally, suppose that  $\dot{G}$  is biregular with degrees  $d_1$  and  $d_2$ , and assume that  $\dot{G}$  is not net-biregular. Then, for at least one  $j$  ( $j \in \{1, 2\}$ ), there are vertices of degree  $d_j$  which differ in net-degree. By (6), the corresponding coordinates of  $\mathbf{a}$  are different, while by (3), they are equal in absolute value. Using  $D$  formed exactly as before, we obtain a signed graph  $\dot{H}$  for which we have  $A_{\dot{H}}D\mathbf{a} = \rho D\mathbf{a}$  where  $D\mathbf{a}$  has 2 different coordinates. By (6), this means that  $\dot{H}$  has 2 net-degrees, and so it is net-biregular. ■

Of course, there is an analogous statement with  $\mu$  in the role of the unique non-main eigenvalue. Here is a closer description of  $\dot{G}$  being net-biregular.

**Corollary 9.** *If the signed graph  $\dot{G}$  of Theorem 8 is net-biregular, then  $\dot{G}$  is biregular and its net-degrees determine an equitable vertex bipartition.*

**Proof.** From (4) we see that  $\dot{G}$  must be biregular. In addition, since  $\dot{G}$  is non-regular, its vertices are equal in degree if and only if they are equal in net-degree. By (6), the eigenvector  $\mathbf{a}$  of (2) has 2 different coordinates, say  $a_u$  and  $a_w$ , which correspond to different net-degrees and determine the vertex set partition  $V = U \dot{\cup} W$ . It remains to show that this partition is equitable. For  $v \in V$ , let  $d_{vu}^\pm$  and  $d_{vw}^\pm$  denote its net-degree in  $U$  and  $W$ , respectively. Then we also have  $d_v^\pm = d_{vu}^\pm + d_{vw}^\pm$ . If  $v \in U$ , since  $\mathbf{a}$  is associated with  $\rho$ , we have  $d_{vu}^\pm a_u + d_{vw}^\pm a_w = \rho a_u$ , i.e.,  $d_{vu}^\pm a_u + (d_v^\pm - d_{vu}^\pm) a_w = \rho a_u$  and  $(d_v^\pm - d_{vu}^\pm) a_u + d_{vw}^\pm a_w = \rho a_u$ . The last two equalities lead to

$$d_{vu}^\pm = \frac{\rho a_u - d_v^\pm a_w}{a_u - a_w} \quad \text{and} \quad d_{vw}^\pm = a_u \frac{d_v^\pm - \rho}{a_u - a_w}.$$

In a very similar way, we obtain

$$d_{vu}^\pm = a_w \frac{d_v^\pm - \rho}{a_w - a_u} \quad \text{and} \quad d_{vw}^\pm = \frac{\rho a_w - d_v^\pm a_u}{a_w - a_u},$$

for  $v \in W$ . In other words, the net-degrees determine an equitable vertex bipartition. ■

It is not difficult to construct some examples. For instance, by making a switch with respect to 3 mutually adjacent vertices of the Paley graph with 9 vertices, we obtain a regular and net-biregular signed graph with spectrum  $[4, 1^4, (-2)^4]$ . Also, the cone over the complete bipartite signed graph  $\dot{K}_{4,4}$ , in which negative edges form a perfect matching, is biregular and net-biregular, while its spectrum is  $[4, 2^3, (-2)^5]$ .

#### 4. VERTEX-DELETED SUBGRAPHS WITH 3 EIGENVALUES

In this section we consider the question of whether a vertex-deleted subgraph of a connected signed graph  $\dot{G}$  with 3 eigenvalues also has 3 eigenvalues. We distinguish 3 cases depending on the number of simple eigenvalues of  $\dot{G}$ . First, if all of them are simple then all vertex-deleted subgraphs have fewer than 3 eigenvalues; this case is trivial. If  $\dot{G}$  has 2 simple eigenvalues, then  $\dot{G}$  is switching equivalent to a complete bipartite graph, by Lemma 5. If so, then every vertex-deleted subgraph is also switching equivalent to a complete bipartite graph; such a subgraph has 3 eigenvalues unless  $\dot{G}$  is the star  $\dot{K}_{1,n-1}$  and the degree of the deleted vertex is  $n - 1$ . The remaining case is more complicated and it is considered in the following two theorems.



**Theorem 10.** *Let  $\dot{G}$  be a connected signed graph with spectrum  $[\rho, \mu^m, \lambda^l]$ , with  $m, l \geq 2$  and  $\mu > \lambda$ . Let  $\dot{H} = \dot{G} - v$  and let  $\mathbf{r}$  be the characteristic  $(0, 1, -1)$ -vector that determines the neighbourhood of  $v$  in  $\dot{H}$ . If  $\dot{H}$  has 3 eigenvalues, then  $\mathbf{r}$  is an eigenvector associated with an eigenvalue of  $\dot{H}$  distinct from  $\mu$  and  $\lambda$ , and:*

- (i) *for  $\rho > \mu$ , the spectrum of  $\dot{H}$  is  $[\mu^m, \rho + \lambda, \lambda^{l-1}]$  with  $\|\mathbf{r}\|^2 = -\rho\lambda$ ;*
- (ii) *for  $\rho < \lambda$ , the spectrum of  $\dot{H}$  is  $[\mu^{m-1}, \rho + \mu, \lambda^l]$  with  $\|\mathbf{r}\|^2 = -\rho\mu$ ;*
- (iii) *for  $\rho \in (\lambda, \mu)$ , the spectrum of  $\dot{H}$  is  $[\mu^{m-1}, \rho^2, \lambda^{l-1}]$  with  $\rho = \mu + \lambda$ ,  $\|\mathbf{r}\|^2 = -\mu\lambda$  or  $[\mu^m, \rho + \lambda, \lambda^{l-1}]$  with  $\|\mathbf{r}\|^2 = -\rho\lambda$ , or  $[\mu^{m-1}, \rho + \mu, \lambda^l]$  with  $\|\mathbf{r}\|^2 = -\rho\mu$ .*

*Conversely, if  $\mathbf{r}$  is an eigenvector of  $\dot{H}$  associated with an eigenvalue distinct from  $\mu$  and  $\lambda$ , then  $\dot{H}$  has 3 eigenvalues when either  $\rho$  is an eigenvalue of multiplicity 2 in  $\dot{H}$  or  $\dot{H}$  does not have  $\rho$  as an eigenvalue.*

**Proof.** Computing  $\text{tr}(A_{\dot{G}})$  and  $\text{tr}(A_{\dot{G}}^2)$ , we obtain

$$(7) \quad \rho + m\mu + l\lambda = 0,$$

$$(8) \quad \rho^2 + m\mu^2 + l\lambda^2 = 2e,$$

where  $e$  denotes the number of edges of  $\dot{G}$ . Let  $g$  be the number of edges in  $\dot{G}$  but not in  $\dot{H}$ , so that  $g = \|\mathbf{r}\|^2$ .

Suppose first that  $\rho > \mu$ . Observe that  $\lambda < 0$ , since  $\dot{G}$  must have at least one negative eigenvalue (see Proposition 1), and then we also have  $\mu \geq 0$ . By eigenvalue interlacing, the eigenvalues of  $\dot{H}$  are  $\nu, \mu^{m-1}, \theta, \lambda^{l-1}$ , where  $\rho \geq \nu \geq \mu \geq \theta \geq \lambda$ . Now, since  $\dot{H}$  has 3 eigenvalues we have one of the following situations:

- (a)  $\nu = \rho$  and  $\theta \in \{\mu, \lambda\}$ ,
- (b)  $\nu = \mu$  and  $\theta \notin \{\mu, \lambda\}$ ,
- (c)  $\nu \notin \{\rho, \mu\}$  and  $\theta = \mu$ ,
- (d)  $\nu \notin \{\rho, \mu\}$  and  $\theta = \lambda$ .

For (a), when  $\theta = \mu$  we have  $\rho + m\mu + (l-1)\lambda = 0$ , which together with (7) leads to the contradiction  $\lambda = 0$ . When  $\theta = \lambda$  in a similar way we get  $\mu = 0$ , but then  $\dot{H}$  and  $\dot{G}$  have the same number of edges (by (8)), a contradiction since  $\dot{G}$  is connected.

For (b), we use the equalities (7) and (8) along with  $\text{tr}(A_{\dot{H}}) = 0$  and  $\text{tr}(A_{\dot{H}}^2) = 2(e-g)$  to obtain  $\theta = \rho + \lambda$  and  $g = -\rho\lambda$ . To complete the proof of (i) it remains to prove that  $\mathbf{r}$  is an eigenvector of  $\dot{H}$  associated with  $\rho + \lambda$ .

Let  $Q_\xi$  denote the matrix of the orthogonal projection of  $\mathbb{R}^{n-1}$  onto the eigenspace of an eigenvalue  $\xi$  of  $\dot{H}$ . By Proposition 3, we have

$$P_{\dot{G}}(x) = P_{\dot{H}}(x) \left( x - \frac{\|Q_\mu \mathbf{r}\|^2}{x - \mu} - \frac{\|Q_\theta \mathbf{r}\|^2}{x - \theta} - \frac{\|Q_\lambda \mathbf{r}\|^2}{x - \lambda} \right).$$

Since the multiplicity of  $\mu$  and  $\lambda$  in  $\dot{G}$  is not less than the multiplicity of the same eigenvalue in  $\dot{H}$ , we have  $Q_\mu \mathbf{r} = Q_\lambda \mathbf{r} = \mathbf{0}$ , which means that  $\mathbf{r}$  is orthogonal to the eigenspaces of  $\mu$  and  $\lambda$ , equivalently  $\mathbf{r}$  belongs to the eigenspace of  $\theta = \rho + \lambda$ .

For (c), as in the previous case, we obtain  $\nu = \rho + \lambda$  and  $g = -\rho\lambda$ , along with the conclusion that  $\mathbf{r}$  belongs to the eigenspace of  $\nu$ .

For (d), we find that  $\nu = \rho + \mu$ , which is impossible as  $\mu \geq 0$  and  $\nu \neq \rho$ .

This completes the proof of (i), while (ii) follows analogously.

Now suppose that  $\rho \in (\lambda, \mu)$ . Here  $\mu > 0, \lambda < 0$  and the possible eigenvalues of  $\dot{H}$  are  $\mu^{m-1}, \nu, \theta, \lambda^{l-1}$ . The cases that arise are considered in the same way as before, and yield the results summarized in (iii).

Assume now that  $\mathbf{r}$  belongs to the eigenspace of an eigenvalue of  $\dot{H}$  distinct from  $\mu$  and  $\lambda$ . If  $\rho$  is an eigenvalue of  $\dot{H}$  with multiplicity 2, then (by eigenvalue interlacing)  $\dot{H}$  has 3 eigenvalues, and so it remains to consider the case in which  $\rho$  does not belong to the spectrum of  $\dot{H}$ . In this case  $\dot{H}$  has at most 4 eigenvalues. If  $\lambda, \mu$  are the only eigenvalues of  $\dot{H}$  then  $\rho = 0$  and we obtain the contradiction  $g = 0$ . Now suppose that  $\dot{H}$  has 4 eigenvalues,  $\mu, \lambda, \nu$  and  $\theta$ . By Proposition 3, we have

$$P_{\dot{G}}(x) = P_{\dot{H}}(x) \left( x - \frac{\|Q_\mu \mathbf{r}\|^2}{x - \mu} - \frac{\|Q_\lambda \mathbf{r}\|^2}{x - \lambda} - \frac{\|Q_\nu \mathbf{r}\|^2}{x - \nu} - \frac{\|Q_\theta \mathbf{r}\|^2}{x - \theta} \right).$$

If  $\mathbf{r}$  belongs to the eigenspace of (say)  $\nu$ , then we obtain

$$(9) \quad (x - \nu)(x - \rho)(x - \mu)^m(x - \lambda)^l = (x(x - \nu) - \|Q_\nu \mathbf{r}\|^2)P_{\dot{H}}(x).$$

Observe that the multiplicities of  $\mu$  and  $\lambda$  in  $\dot{H}$  are  $m - 1$  and  $l - 1$ , respectively, which implies  $(x - \mu)(x - \lambda) = (x(x - \nu) - \|Q_\nu \mathbf{r}\|^2)$ ; but then since  $(x - \rho)$  must appear on the right hand side of (9), we conclude that  $\rho$  is an eigenvalue of  $\dot{H}$ . This contradiction completes the proof. ■

We note a consequence of Theorem 10 in the case that  $\dot{G}$  is connected and switching equivalent to its underlying graph. Then  $\rho$  is the largest eigenvalue of  $\dot{G}$ , and so  $\rho$  is not an eigenvalue of  $\dot{H}$ . Hence, if  $\mathbf{r}$  is an eigenvector of  $\dot{H}$  associated with an eigenvalue other than  $\mu$  or  $\lambda$  then  $\dot{H}$  has 3 eigenvalues. Plenty of examples can be found among unsigned graphs; for instance  $\dot{H}$  can be the Petersen graph, with  $\dot{G}$  the cone over  $\dot{H}$ . We further consider cones by setting  $\dot{G} \cong K_1 \nabla \dot{H}$  in Theorem 10.

**Corollary 11.** *Suppose that  $K_1 \nabla \dot{H}$  has spectrum  $[\rho, \mu^m, \lambda^l]$ , where  $m, l \geq 2$  and  $\mu > \lambda$ . If  $\dot{H}$  is connected with 3 eigenvalues, then  $\dot{H}$  is net-regular and either:*

- (i)  $\dot{H}$  has spectrum  $[\rho + \lambda, \mu^m, \lambda^{l-1}]$ , with  $\rho > \rho + \lambda > \mu > \lambda$  or
- (ii)  $\dot{H}$  has spectrum  $[\mu^{m-1}, \lambda^l, \rho + \mu]$ , with  $\rho < \rho + \mu < \lambda < \mu$ .

**Proof.** By setting  $\mathbf{r} = \mathbf{j}$  in Theorem 10 we see that  $\mathbf{j}$  is an eigenvector of  $\dot{H}$ , and so  $\dot{H}$  is net-regular. Let  $A_{\dot{H}}\mathbf{j} = \nu\mathbf{j}$ ,  $\dot{G} \cong K_1 \nabla \dot{H}$  and  $n = 1 + l + m$ . From Theorem 10 we see that  $\nu \notin \{\mu, \lambda\}$  and there are five possible scenarios:

- (a)  $\rho > \mu$ ,  $\nu = \rho + \lambda$ ,  $n - 1 = -\rho\lambda$  and  $\dot{H}$  has spectrum  $[\mu^m, \rho + \lambda, \lambda^{l-1}]$ ;
- (b)  $\rho < \lambda$ ,  $\nu = \rho + \mu$ ,  $n - 1 = -\rho\mu$  and  $\dot{H}$  has spectrum  $[\mu^{m-1}, \rho + \mu, \lambda^l]$ ;
- (c)  $\lambda < \rho < \mu$ ,  $\nu = \rho + \lambda$ ,  $n - 1 = -\rho\lambda$  and  $\dot{H}$  has spectrum  $[\mu^m, \rho + \lambda, \lambda^{l-1}]$ ;
- (d)  $\lambda < \rho < \mu$ ,  $\nu = \rho + \mu$ ,  $n - 1 = -\rho\mu$  and  $\dot{H}$  has spectrum  $[\mu^{m-1}, \rho + \mu, \lambda^l]$ ;
- (e)  $\lambda < \rho < \mu$ ,  $\nu = \rho$ ,  $n - 1 = -\lambda\mu$ ,  $\rho = \mu + \lambda$  and  $\dot{H}$  has spectrum  $[\mu^{m-1}, \rho^2, \lambda^{l-1}]$ .

Note first that if  $\lambda$  has multiplicity  $l - 1$  in  $\dot{H}$  then  $l > 2$ , for otherwise  $\dot{H}$  has 2 simple eigenvalues and, by Lemma 5, is switching equivalent to a complete bipartite graph, say  $K_{r,s}$ . Then  $\mu = 0$  and  $\{\nu, \lambda\} = \{-\sqrt{rs}, \sqrt{rs}\}$ . Moreover,  $r + s = n - 1 = -\rho\lambda = (\lambda - \nu)\lambda = 2rs$ , whence  $\dot{H} \cong K_2$ , a contradiction. Similarly, if  $\mu$  has multiplicity  $m - 1$  in  $\dot{H}$ , then  $m > 2$ . Now we may apply [14, Theorem 5.5] to  $\dot{H}$ , because  $\dot{H}$  is net-regular,  $\nu$  is a simple eigenvalue of  $\dot{H}$  and each of  $\lambda, \mu$  has multiplicity  $\geq 2$  in  $\dot{H}$ .

For (a), by [14, Theorem 5.5] either  $\mu(\mu - \nu) = n - 1$  and  $\mu > 0 > \lambda > \nu$  or  $\lambda(\lambda - \nu) = n - 1$  and  $\nu > \mu > 0 > \lambda$ . In the former case, we have  $\nu > \mu + \lambda$  and so  $\lambda < \nu - \mu < \nu$ , a contradiction. In the latter case, we have part (i) of this corollary.

For (b), we again get either  $\mu(\mu - \nu) = n - 1$  and  $\mu > 0 > \lambda > \nu$  or  $\lambda(\lambda - \nu) = n - 1$  and  $\nu > \mu > 0 > \lambda$ . In the former case, we have part (ii) of this corollary. In the latter case,  $\mu$  is the largest eigenvalue of  $\dot{G}$  because  $\rho < \lambda$ . Hence  $\nu \leq \mu$ , a contradiction.

For (c), we have  $\lambda < 0$  and  $\mu > 0$ , because  $\lambda$  and  $\mu$  are respectively the least and largest eigenvalues of  $\dot{G}$ . By [14, Theorem 5.5], either  $\mu(\mu - \lambda) = n - 1$  or  $\lambda(\lambda - \mu) = n - 1$ . Since  $\mu > \rho$ , we have  $-\lambda\mu > -\lambda\rho$ , and so in the former case  $\mu(\mu - \lambda) < -\lambda\mu$ . Since  $\mu > 0$ , we have  $\mu - \lambda < -\lambda$  and the contradiction  $\mu < 0$ . In the latter case, we have  $\lambda(\lambda - \mu) = n - 1 = -\rho\lambda$ , whence  $\rho = \mu - \lambda > \mu$ , a contradiction.

For (d), we have  $\lambda < 0$ ,  $\mu > 0$  and either  $\mu(\mu - \lambda) = n - 1$  or  $\lambda(\lambda - \mu) = n - 1$ . In the former case,  $\mu(\mu - \lambda) = n - 1 = -\rho\mu$ , whence  $\rho = \lambda - \mu < \lambda$ , a contradiction. In the latter case,  $\rho > \lambda$  and so  $\lambda(\lambda - \mu) = -\rho\mu < -\lambda\mu$ , whence  $\lambda - \mu > -\mu$  and the contradiction  $\lambda > 0$ .

For (e), we have  $\nu = \lambda + \mu$ , and so by [14, Theorem 5.5],  $\dot{G}$  has just 2 eigenvalues, a contradiction. ■

It remains to consider signed graphs with 3 eigenvalues such that none of them is simple.

**Theorem 12.** Let  $\dot{G}$  be a connected signed graph with spectrum  $[\rho^r, \mu^m, \lambda^l]$ , with  $r, m, l \geq 2$ . Let  $\dot{H} = \dot{G} - v$  and let  $\mathbf{r}$  denote the characteristic  $(0, 1, -1)$ -vector that determines the neighbourhood of  $v$  in  $\dot{H}$ .

If  $\dot{H}$  has 3 eigenvalues then, to within a permutation of the eigenvalues, its spectrum is  $[\rho^{r+1}, \mu^{m-1}, \lambda^{l-1}]$ ; moreover,  $\rho = \mu + \lambda$ ,  $\|\mathbf{r}\|^2 = -\mu\lambda$ , and  $\mathbf{r}$  belongs to the eigenspace of  $\rho$  in  $\dot{H}$ .

Conversely, if  $\mathbf{r}$  belongs to the eigenspace of an eigenvalue of  $\dot{H}$ , then  $\dot{H}$  has at most 4 eigenvalues, with 3 eigenvalues precisely when the eigenvalue associated with  $\mathbf{r}$  also belongs to the spectrum of  $\dot{G}$ .

**Proof.** Assume that  $\dot{H}$  has 3 eigenvalues and suppose first that the multiplicities of two of them (say  $\mu$  and  $\lambda$ ) are transferred from  $\dot{G}$ . Since  $\text{tr}(A_{\dot{G}}) = \text{tr}(A_{\dot{H}})$  we have  $\rho = 0$ , and since  $\text{tr}(A_{\dot{G}}^2) = \text{tr}(A_{\dot{H}}^2)$  we see that  $\dot{G}$  and  $\dot{H}$  have the same number of edges, a contradiction. By eigenvalue interlacing, the remaining case is the one in which every eigenvalue changes its multiplicity: one increases and two decrease. If the multiplicity of  $\rho$  increases, then as before we have  $\rho = \mu + \lambda$ . Using Proposition 3 and following the proof of Theorem 10, we find that  $\rho$  is afforded by  $\mathbf{r}$  in  $\dot{H}$  and that  $\|\mathbf{r}\|^2 = -\mu\lambda$ .

Conversely, suppose that  $\mathbf{r}$  belongs to the eigenspace of the eigenvalue  $\xi$  of  $\dot{H}$ . By Proposition 3, we have

$$(10) \quad (x - \xi)P_{\dot{G}}(x) = P_{\dot{H}}(x)(x(x - \xi) - \|Q_{\xi}\mathbf{r}\|^2).$$

If  $\dot{H}$  has 5 eigenvalues (more than this is impossible), those transferred from  $\dot{G}$  along with (say)  $\nu$  and  $\theta$ , then at least one of the factors  $(x - \nu)$  and  $(x - \theta)$  occurs only on the right hand side of (10), which is impossible. Therefore, the number of eigenvalues of  $\dot{H}$  is 3 or 4. If their number is 3, then we see immediately that  $\xi$  is an eigenvalue of  $\dot{G}$ . Conversely, if  $\xi$  is an eigenvalue of  $\dot{G}$ , and  $\nu$  is the fourth eigenvalue of  $\dot{H}$  (the one distinct from  $\rho, \mu, \lambda$ ), then as before  $(x - \nu)$  occurs only on the right hand side of (10), which is impossible, and we are done. ■

**Example 13.** To obtain an example for Theorem 12 it is convenient to start from  $\dot{H}$  as a signed graph with spectrum  $[\rho^{r+1}, \mu^{m-1}, \lambda^{l-1}]$ , and set  $\mathbf{r} = \mathbf{j}$ . In this case,  $\dot{H}$  is net-regular, and  $\dot{G}$  is a cone over  $\dot{H}$ . Moreover,  $\rho = \mu + \lambda$  and  $\dot{H}$  has  $-\mu\lambda$  vertices. By inspecting some known net-regular signed graphs with 3 eigenvalues, we arrive at a signed graph which can be found in [1] and satisfies all the numerical constraints. This is the signed graph obtained by reversing the sign of every edge belonging to a fixed Hamiltonian cycle of the Paley graph with 9 vertices. Its spectrum is  $[3^2, 0^5, (-3)^2]$  (with  $\rho = 0$ ). The corresponding cone has the spectrum  $[3^3, 0^4, (-3)^3]$ .

Motivated by the previous example, in which  $\dot{G}$  is a cone over  $\dot{H}$ , we give a closer description of both signed graphs in this particular case. The first statement of the next theorem is a general one.

**Theorem 14.** *The following statements hold.*

- (i) *Let  $\dot{G}$  be a signed graph with  $\rho$  as an eigenvalue of multiplicity  $r \geq 2$ , let  $\dot{H} = G - v$  and let  $\mathbf{r}$  be the characteristic vector that determines the neighbourhood of  $v$  in  $\dot{H}$ . If the eigenspace of  $\rho$  in  $\dot{H}$  has orthogonal basis  $\mathbf{r}, \mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$  and  $v$  is vertex 1 in  $\dot{G}$ , then  $(0, \mathbf{x}_1^\top)^\top, (0, \mathbf{x}_2^\top)^\top, \dots, (0, \mathbf{x}_k^\top)^\top$  are linearly independent eigenvectors associated with  $\rho$  in  $\dot{G}$ .*
- (ii) *In particular, if  $\dot{G} \cong K_1 \nabla \dot{H}$  with spectrum  $[\rho^r, \mu^m, \lambda^l]$  ( $r, m, l \geq 2$ ), if  $\mathbf{j}$  belongs to the eigenspace of  $\rho$  in  $\dot{H}$  and if  $\dot{H}$  has spectrum  $[\rho^{r+1}, \mu^{m-1}, \lambda^{l-1}]$ , then  $\rho$  is the unique main eigenvalue of  $\dot{H}$  and the unique non-main eigenvalue of  $\dot{G}$ .*

**Proof.** We have

$$A_{\dot{G}} \begin{pmatrix} 0 \\ \mathbf{x}_i \end{pmatrix} = \begin{pmatrix} \mathbf{r}^\top \mathbf{x}_i \\ \rho \mathbf{x}_i \end{pmatrix} = \rho \begin{pmatrix} 0 \\ \mathbf{x}_i \end{pmatrix},$$

for  $1 \leq i \leq k$ . Linear independence follows directly, and we have (i).

For (ii), first note that  $\rho$  is the unique main eigenvalue of  $\dot{H}$  because  $\mathbf{j}$  is an eigenvector of  $\rho$  in  $\dot{H}$ . Taking  $k = r$  in (i) we obtain a basis for the eigenspace of  $\rho$  in  $\dot{G}$  consisting of vectors orthogonal to  $\mathbf{j}$ . Thus  $\rho$  is non-main in  $\dot{G}$ . If  $\dot{G}$  has another non-main eigenvalue, then the remaining one is main, but this means that the cone  $\dot{G}$  is net-regular and hence the complete graph with just 2 eigenvalues, a contradiction. ■

## 5. CONSTRUCTIONS OF SIGNED GRAPHS WITH AT MOST 3 EIGENVALUES

Here we give some examples of signed graphs with 3 eigenvalues, along with applications of some results from Section 3. The first construction is based on weighing matrices, while the second one is based on symmetric 3-class association schemes.

### 5.1. Weighing matrices

Let  $W = W(n, \alpha)$  be a weighing matrix of order  $n$  with weight  $\alpha$ , i.e., an  $n \times n$   $(0, 1, -1)$ -matrix such that  $W^\top W = \alpha I$ . For  $1 \leq m \leq n$ , we call the submatrix of  $W$  indexed by rows  $1, 2, \dots, m$  and columns  $1, 2, \dots, n$  a *partial weighing matrix* with *weighing extension*  $W$ .

**Theorem 15.** *Let  $W'_1$  and  $W'_2$  be two partial weighing matrices of size  $m \times n$  with weighing extensions  $W_1, W_2$  of weight  $\alpha$ . The following block matrix has spectrum  $[-\sqrt{\alpha^{n+m}}, \sqrt{\alpha^{n-m}}, 2\sqrt{\alpha^m}]$ :*

$$(11) \quad \mathcal{A}_m(W_1, W_2) = \begin{pmatrix} O_m & W_1' & W_2' \\ W_1'^\top & O_n & \frac{1}{\sqrt{\alpha}} W_1^\top W_2 \\ W_2'^\top & \frac{1}{\sqrt{\alpha}} W_2^\top W_1 & O_n \end{pmatrix}.$$

**Proof.** Let  $B$  be the matrix  $(W_1' \mid W_2')$ , and let

$$C = \begin{pmatrix} O_n & \frac{1}{\sqrt{\alpha}} W_1^\top W_2 \\ \frac{1}{\sqrt{\alpha}} W_2^\top W_1 & O_n \end{pmatrix}.$$

By a straightforward computation we arrive at the following equality (see, for example, [10]):

$$(2\sqrt{\alpha}I_{2n} - C)^{-1} = \begin{pmatrix} \frac{2}{3\sqrt{\alpha}}I_n & \frac{1}{3\alpha\sqrt{\alpha}}W_1^\top W_2 \\ \frac{1}{3\alpha\sqrt{\alpha}}W_2^\top W_1 & \frac{2}{3\sqrt{\alpha}}I_n \end{pmatrix}.$$

Therefore  $B(2\sqrt{\alpha}I_{2n} - C)^{-1}B^\top = 2\sqrt{\alpha}I_m$ , and it follows that the vectors

$$\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1}B^\top \mathbf{x} \end{pmatrix},$$

for  $\mathbf{x} \in \mathbb{R}^m$ , lie in the eigenspace for the eigenvalue  $2\sqrt{\alpha}$  of  $\mathcal{A}_m(W_1, W_2)$ . Thus  $2\sqrt{\alpha}$  is an eigenvalue of  $\mathcal{A}_m(W_1, W_2)$  with multiplicity at least  $m$ . A similar argument applied for  $C = O_n$  shows that  $-\sqrt{\alpha}$  is an eigenvalue with multiplicity at least  $n + m$ . In addition, by eigenvalue interlacing,  $\sqrt{\alpha}$  is an eigenvalue of  $\mathcal{A}_m(W_1, W_2)$  with multiplicity at least  $n - m$  since the submatrix

$$\begin{pmatrix} O_n & \frac{1}{\sqrt{\alpha}} W_1^\top W_2 \\ \frac{1}{\sqrt{\alpha}} W_2^\top W_1 & O_n \end{pmatrix}$$

has the spectrum  $[-\sqrt{\alpha}^n, \sqrt{\alpha}^n]$ . ■

This theorem enables us to construct an infinite family of signed graphs with spectrum  $[-\sqrt{\alpha}^{n+m}, \sqrt{\alpha}^{n-m}, 2\sqrt{\alpha}^m]$  for some appropriate  $\alpha$ . We remark that the matrix (11) does not always correspond to a signed graph. For a signed graph, the inner product of any row of  $W_1$  and any row of  $W_2$  has to be 0 or  $\pm\sqrt{\alpha}$ , and in [11] one can find a method for constructing a family of weighing matrices of weight 4 with this property. The method can be used to construct signed graphs with spectrum  $[-2^{n+m}, 2^{n-m}, 4^m]$ , as in the following example.

**Example 16.** Let  $W_1$  and  $W_2$  be as follows:

$$W_1 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 & -1 & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}.$$

Considering the first two rows of  $W_1$  and  $W_2$  as the matrices  $W'_1$  and  $W'_2$ , we see from Theorem 15 that the signed graph  $\dot{G}$  with adjacency matrix  $\mathcal{A}_2(W_1, W_2)$  has spectrum  $[-2^8, 2^4, 4^2]$ . Since  $\dot{G}$  has no vertices of degree 4, Theorem 4.3 shows that if a vertex-deleted subgraph of  $\dot{G}$  has 3 eigenvalues, then it is obtained by deleting a vertex of degree 8. There are exactly two such vertices, labelled by 1 and 2 in (11), and the deletion of either leads to a subgraph with spectrum  $[-2^7, 2^5, 4]$ . This is because, in the notation of Theorem 4.3,  $\mathbf{r}$  is an eigenvector of the vertex-deleted subgraph corresponding to the eigenvalue 2.

## 5.2. Symmetric 3-class association schemes

A symmetric 3-class association scheme  $\mathcal{R}$  consists of a set  $X$  and a partition of  $X \times X$  into 4 non-empty binary relations  $R_0, R_1, R_2, R_3$  satisfying the following constraints:

- $R_0 = \{(x, x) \mid x \in X\}$ ;
- If  $(x, y) \in R_i$ , then  $(y, x) \in R_i$  and if  $(x, y) \in R_k$ , then the number of  $z \in X$  such that  $(x, z) \in R_i$  and  $(z, y) \in R_j$  is a constant  $p_{ij}^k$  depending on  $i, j, k$ , but not on a particular choice of  $x, y$ .

For  $0 \leq i \leq 3$ , we define the  $(0, 1)$ -matrix  $A_i$  with rows and columns indexed by the elements of  $X$ , and  $(x, y)$ -entry 1 if and only if  $(x, y) \in R_i$ . It follows that  $A_0 = I$  and  $A_i A_j = \sum_{k=0}^3 p_{ij}^k A_k$ . For  $i \in \{1, 2, 3\}$  let  $G_i$  be the graph with adjacency matrix  $A_i$ , and for distinct  $i, j \in \{1, 2, 3\}$  let  $\dot{G}_{i,j}$  be the signed graph with adjacency matrix  $A_i - A_j$ . The matrices  $A_i$  span a 4-dimensional commutative  $\mathbb{R}$ -algebra (called the Bose-Mesner algebra, cf. [3, Chapter 17]). It follows that the signed graphs  $\dot{G}_{i,j}$  have at most 4 eigenvalues.

**Theorem 17.** *Let  $\dot{G}_{i,j}$  be a signed graph arising from a 3-class association scheme. Then  $\dot{G}_{i,j}$  is strongly regular with parameters*

$$r = p_{ii}^0 + p_{jj}^0, \quad a = p_{ii}^i + p_{jj}^i - 2p_{ij}^i, \quad b = -2p_{ij}^j + p_{ii}^j + p_{jj}^j, \quad c = p_{ii}^k + p_{jj}^k - 2p_{ij}^k$$

where  $\{i, j, k\} = \{1, 2, 3\}$ . Moreover,  $\dot{G}_{i,j}$  has 3 eigenvalues if and only if  $a + b = 2c \neq 0$ , and 2 eigenvalues if and only if  $a + b = 2c = 0$ .

**Proof.** The first assertion follows from [12, Theorem 2.2], and the second from Lemma 7. Lastly,  $\dot{G}_{ij}$  is not a complete graph since  $A_k \neq 0$ , and so the third assertion follows from [18, Theorem 4.2]. ■

We note that  $b$  is replaced by  $-b$  in [12, Definition 1.4]. Now we are ready to provide some examples of signed graphs with at most 3 distinct eigenvalues.

**Example 18.** The 3-class *Johnson scheme*  $J(n, 3)$  ( $n \geq 6$ ), also known as the tetrahedral scheme, is defined on the 3-subsets of an  $n$ -set, with two subsets in the relation  $R_i$  if they intersect in  $3 - i$  elements. The following scheme provides the numbers  $p_{ij}^k$  relevant to  $\dot{G}_{1,3}$ ; they are obtained by a simple computation.

$$\begin{array}{cccccccccc} p_{11}^1 & p_{33}^1 & p_{13}^1 & p_{11}^3 & p_{13}^3 & p_{33}^3 & p_{11}^2 & p_{33}^2 & p_{13}^2 \\ n-2 & \binom{n-4}{3} & 0 & 0 & 3(n-6) & \binom{n-6}{3} & 4 & \binom{n-5}{3} & n-5 \end{array}$$

Therefore,

$$p_{11}^1 + p_{33}^1 - 2p_{13}^1 - 2p_{13}^3 + p_{11}^3 + p_{33}^3 = n - 2 + \binom{n-4}{3} - 6(n-6) + \binom{n-6}{3},$$

$$2(p_{11}^2 + p_{33}^2 - 2p_{13}^2) = 8 + 2\binom{n-5}{3} - 4n + 20.$$

Both of the above expressions are equal to  $\frac{1}{3}(n-2)(n-7)(n-9)$ . By Theorem 17,  $\dot{G}_{1,3}$  has 2 eigenvalues when  $n = 7$  or  $n = 9$ , and 3 eigenvalues otherwise.

Alternatively we can find the spectrum of  $\dot{G}_{i,j}$  by using the following information from [2, 9]:

$$\text{Spec}(A_1) = \left[ 3(n-3), (2n-9)^{n-1}, (n-7)\binom{n}{2}^{-n}, -3\binom{n}{3}^{-\binom{n}{2}} \right],$$

$$\text{Spec}(A_3) = \left[ \binom{n-3}{3}, (-n^2 + 9n)/2 - 10^{n-1}, (n-5)\binom{n}{2}^{-n}, -1\binom{n}{3}^{-\binom{n}{2}} \right].$$

Here the eigenvalues are ordered by common eigenvectors and so the eigenvalues of  $A_1 - A_3$  are  $\rho$ ,  $\mu^{n-1}$  and  $\lambda\binom{n}{3}^{-n}$ , where  $\rho, \mu, \lambda$  are not necessarily distinct, and

$$\rho = 3(n-3) - \binom{n-3}{3}, \quad \mu = \frac{1}{2}(n^2 - 5n + 2), \quad \lambda = -2.$$

Note that  $G_1$  is regular of degree  $r_1 = 3(n-3)$ ,  $G_3$  is regular of degree  $r_2 = \binom{n-3}{3}$  and the graph underlying  $\dot{G}_{1,3}$  is regular of degree  $r_1 + r_2$ .

We find that  $\rho \geq \mu$  if and only if  $n(n-2)(n-7) \leq 0$ . When  $n = 7$ ,  $\dot{G}_{1,3}$  has spectrum  $[8^7, -2^{28}]$  and by [14, Corollary 4.4] each vertex-deleted subgraph of  $\dot{G}_{1,3}$  has 3 eigenvalues. Such a subgraph necessarily has spectrum  $[6, 8^6, -2^{27}]$ . When  $n = 6$  the spectrum of  $\dot{G}_{1,3}$  is  $[8, 4^5, -2^{14}]$  and each vertex of  $\dot{G}_{1,3}$  has degree 10. Since  $-\rho\lambda \neq 10$ , Theorem 10(i) shows that no vertex-deleted subgraph of  $\dot{G}_{1,3}$  has 3 eigenvalues.



Secondly, we find that  $\rho \leq \lambda$  if and only if  $(n-1)(n-2)(n-9) \geq 0$ . When  $n = 9$ , the spectrum of  $\dot{G}_{1,3}$  is  $[19^8, -2^{76}]$  and by [14, Corollary 4.4] each vertex-deleted subgraph has spectrum  $[17, 19^7, -2^{75}]$ . When  $n > 9$ , no vertex-deleted subgraph of  $\dot{G}_{1,3}$  has 3 eigenvalues for otherwise, in the notation of Theorem 10(ii), we have  $\|\mathbf{r}\|^2 = -\rho\mu$ ; but  $r_1 + r_2 = -\rho\mu$  if and only if  $(n-1)(n-2)(n-9) = 0$ , equivalently  $n = 9$ .

The remaining case is  $n = 8$ , when Theorem 10(iii) shows in similar fashion that a vertex-deleted subgraph of  $\dot{G}_{1,3}$  does not have 3 distinct eigenvalues. In summary, a vertex-deleted subgraph of the signed graph  $\dot{G}_{1,3}$  derived from  $J(n, 3)$  has 3 eigenvalues if and only if  $n$  is 7 or 9.

The definition of  $J(n, 3)$  may be extended to  $J(5, 3)$ , but this scheme is degenerate in our context because then  $A_3 = 0$ . In fact,  $J(5, 3)$  is a 2-class association scheme, with  $G_1 \cong L(K_5)$  and  $G_2 \cong \overline{G}_1$  (the Petersen graph). Here  $\dot{G}_{1,2}$  has spectrum  $[-3^5, 3^5]$ , while any vertex-deleted subgraph has spectrum  $[-3^4, 0, 3^4]$ .

**Example 19.** The 3-class *Hamming scheme*  $H(3, q)$  is defined on the triples of  $q$  symbols (words of length 3 over an alphabet with  $q$  letters), where two triples are in the relation  $R_i$  if they differ in  $i$  coordinates ( $i = 0, 1, 2, 3$ ). In this case we have the following.

$$\begin{array}{cccccccccc} p_{11}^1 & p_{33}^1 & p_{13}^1 & p_{11}^3 & p_{13}^3 & p_{33}^3 & p_{11}^2 & p_{33}^2 & p_{13}^2 \\ q-2 & (q-2)(q-1)^2 & 0 & 0 & 3(q-2) & (q-2)^3 & 2 & (q-2)^2(q-1) & q-1 \end{array}$$

Therefore,

$$\begin{aligned} p_{11}^1 + p_{33}^1 - 2p_{13}^1 - 2p_{11}^3 + p_{13}^3 + p_{33}^3 &= q-2 + (q-2)(q-1)^2 - 6(q-2) + (q-2)^3, \\ 2(p_{11}^2 + p_{33}^2 - 2p_{13}^2) &= 4 + 2(q-2)^2(q-1) - 4q + 4. \end{aligned}$$

Note that both of the above expressions are equal to  $2q(q-2)(q-3)$ . By Theorem 17, the signed graph  $\dot{G}_{1,3}$  has at most 3 distinct eigenvalues. Moreover,  $c = 0$  if and only if  $q$  is 2 or 3, and these are the cases in which  $\dot{G}_{1,3}$  has only 2 distinct eigenvalues. Thus, by [14, Corollary 4.4] any vertex deleted subgraph of  $\dot{G}_{1,3}$  has 3 distinct eigenvalues when  $q \in \{2, 3\}$ . On the other hand, by the proof of [8, Theorem 4.2] the eigenvalues of  $\dot{G}_{1,3}$  other than its net-degree are the roots of the quadratic

$$x^2 + \frac{b-a}{2}x + \frac{a+b}{2} - r.$$

Now, by Theorem 17, we conclude that the eigenvalues of  $\dot{G}_{1,3}$  are

$$\rho = 3(q-1) - (q-1)^3, \quad \mu = q^2 - 2, \quad \lambda = -2.$$

For  $q > 3$ , we find  $\rho < \lambda$ , and so by Theorem 10(ii), if a vertex-deleted subgraph of  $\dot{G}_{1,3}$  has only 3 distinct eigenvalues then  $\|\mathbf{r}\|^2 = -\rho\mu$ . This equality holds if

and only if

$$q(q-1)^2(q+2)(q-3) = 0.$$

Accordingly, no vertex-deleted subgraph of  $\dot{G}_{1,3}$  has 3 eigenvalues when  $q > 3$ .

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