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THE NEIGHBOR-LOCATING-CHROMATIC NUMBER OF TREES AND UNICYCLIC GRAPHS

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Abstract

A k-coloring of a graph is *neighbor-locating* if any two vertices with the same color can be distinguished by the colors of their respective neighbors, that is, the sets of colors of their neighborhoods are different. The *neighbor-locating chromatic number* $\chi_{NL}(G)$ is the minimum k such that a neighbor-locating k-coloring of G exists. In this paper, we give upper and lower bounds on the neighbor-locating chromatic number in terms of the order and the degree of the vertices for unicyclic graphs and trees. We also obtain tight upper bounds on the order of trees and unicyclic graphs in terms of the neighbor-locating chromatic number. Further partial results for trees are also established.

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1. Introduction

We refer to location in graphs as a way to distinguish all their vertices subject to some restrictions. There are mainly two types of location, metric location and neighbor location. Roughly speaking, in metric location the vertices are distinguished by means of distances to other vertices, while in neighbor location only the neighbors of a vertex are taken into account to distinguish them. Metric-locating sets (also known as resolving sets) were introduced simultaneously in [15, 18], meanwhile neighbor-locating sets were introduced in [19].

In [11], the notion of metric location was extended to vertex partitions, introducing the so called *locating partitions*, also known as *resolving-partitions*, and in [8], there were first studied the so-called *locating colorings*, obtained by considering locating partitions formed by independent sets of vertices. Both resolving partitions and locating colorings have been extensively studied since then. See for example [7, 12–14, 16, 17] and [3, 5, 6, 9, 20], respectively.

In [1], neighbor-locating partitions formed by independent sets are studied, which are named *neighbor-locating colorings*. More specifically, we consider vertex colorings such that any two vertices with the same color can be distinguished from each other by the colors of their respective neighbors. This concept was already introduced and studied in [4] with the name of *adjacency locating colorings* as a tool to study the chromatic locating number of the join of graphs.

In this paper, we focus on neighbor-locating colorings of trees and unicyclic graphs. The remaining part of this paper is organized as follows. We finish this section by giving the terminology used in the paper. In Section 2, the definition of neighbor-locating colorings is given, together with some properties and bounds. Section 3 deals with unicyclic graphs, proving that the upper bound on the order given in Section 2 is tight for unicyclic graphs. In Section 4, neighbor-locating colorings of trees are studied. Among other results, a tree that attains the upper bound given in Section 2 restricted to trees is provided. Finally, in Section 5, we summarize our results and pose some open problems.

All the graphs considered in this paper are connected, undirected, simple and finite. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Let $n(G)$ denote the order of G, i.e., $n(G) = |V(G)|$. The *neighborhood* of a vertex $v \in V(G)$ is the set $N(v) = \{w \in V(G) : vw \in E(G)\}$. The *degree* of v, defined as the cardinality of $N(v)$, is denoted by $\deg(v)$. When $\deg(v) = 1$, v is called a *leaf*. The *maximum degree* $\Delta(G)$ of G is defined to be

$\Delta(G) = \max\{deg(v) : v \in V(G)\}$. The *distance* between two vertices $v, w \in V(G)$ is denoted by $d(v, w)$. The *diameter* of G is $\mathrm{diam}(G) = \max\{d(v, w) : v, w \in V(G)\}$. The path and the cycle of order n are denoted by P_n and C_n, respectively.

Let $[k]$ denote the set $\{1, \ldots, k\}$. A *proper coloring* of G is a map $f : V(G) \to [k]$, for some $k \geq 1$, such that $f(u) \neq f(v)$ whenever $uv \in E(G)$. The elements of $[k]$ are called *colors* and we say that vertex v is colored with $f(v)$. We refer to f as a proper k-*coloring*, if we want to emphasize the number of potential colors. If $W \subseteq V(G)$, then $f(W) = \{f(w) : w \in W\} \subseteq [k]$. Every proper k-coloring defines a k-partition $\Pi(f) = \{S_1(f), \ldots, S_k(f)\}$, where $S_i(f) = \{u : f(u) = i\}$, such that $S_i(f)$ is an independent set of vertices for every $i \in [k]$. Given a proper k-coloring f of a graph G and a vertex $v \in V(G)$, the *color-degree* of v is the number of different colors of its neighborhood, that is, $|\{f(x) : x \in N(v)\}|$. Note that the color-degree of a vertex v is at most $\deg(v)$.

2. NEIGHBOR-LOCATING COLORINGS

A proper coloring f of a graph G is called a *neighbor-locating coloring*, an NL-*coloring* for short, if for every pair of different vertices u, v with $f(u) = f(v)$, the sets of colors of their neighborhoods are different, that is, $f(N(u)) \neq f(N(v))$. We say that f is a k-NL-*coloring* if f is a proper k-coloring.

The *neighbor-locating chromatic number* $\chi_{NL}(G)$, NLC-*number* for short, is the minimum k such that there exists a k-NL-coloring of G. Note that the trivial graph and the path of order 2 are the only graphs with NLC-number equal to 1 and 2, respectively.

As a straightforward consequence of these definitions the following remark is derived.

Remark 1. Let f be a k-NL-coloring of a nontrivial graph G and let n_j denote the number of vertices of degree j, for $1 \leq j \leq \Delta(G)$. Then,

(1) For every $i, j \in [k]$, there are at most $\binom{k-1}{j}$ vertices in $S_i(f)$ with color-degree j.

(2) $|S_i(f)| \leq \displaystyle\sum_{j=1}^{\Delta(G)} \binom{k-1}{j}$, for every $i \in [k]$.

(3) For every $j \in [k]$, there are at most $k\binom{k-1}{j}$ vertices in G with color-degree j.

(4) $\displaystyle\sum_{j=1}^{i} n_j \leq k \sum_{j=1}^{i} \binom{k-1}{j}$, for every $i \in [\Delta(G)]$.

Neighbor-locating colorings and the neighbor-locating chromatic number of a graph have been studied in [1, 2, 4]. An upper bound on the order of a graph

in terms of the NLC-number is given in [1].

Theorem 2 [1]. *Let G be a nontrivial graph such that $\chi_{_{NL}}(G) = k$. Then,*

(1) $n(G) \leq k\,(2^{k-1} - 1)$, *and this bound is tight.*

(2) *If* $\Delta(G) \leq k-1$, *then* $n(G) \leq k \sum_{j=1}^{\Delta(G)} \binom{k-1}{j}$.

Paths and cycles are the only connected graphs with $\Delta(G) = 2$. In this case, we have

$$n(G) \leq k(k-1) + k\binom{k-1}{2} = \frac{k^3 - k^2}{2}. \tag{1}$$

Behtoei and Anbarloei [4] calculate the exact value of the NLC-number of paths and cycles, and from their results, it can be easily derived that the preceding upper bound is attained for every $k \geq 2$. Their proof is constructive, that is, they give specific NL-colorings for all paths and cycles. Another k-NL-coloring of paths and cycles that leads to the same results is given in [2]. Also in [4], the NLC-number for complete and bipartite complete graphs is given, and for fans and wheels is implicitly given, since the authors calculate the *locating chromatic number* of these graphs, that is equal to the NLC-number for graphs of diameter 2.

Next, we provide an upper bound on the order of a graph that turns out to be better than the bounds given in Theorem 2 for some families of graphs, including trees and unicyclic graphs, as we will see in the remaining sections. This bound involves the *cycle rank* of a graph G, denoted by $c(G)$, defined as $c(G) = |E(G)| - n(G) + 1$ (see [10]). Notice that trees and unicyclic graphs are those graphs with cycle rank equal to 0 and 1, respectively.

Theorem 3. *If G is a graph such that $\chi_{_{NL}}(G) = k \geq 3$, then*

$$n(G) \leq \frac{1}{2}\left(k^3 + k^2 - 2k\right) + 2(c(G) - 1).$$

Moreover, if the equality holds, then G has maximum degree 3, and it has exactly $k(k-1)$ leaves, $\frac{k(k-1)(k-2)}{2}$ vertices of degree 2, and $k(k-1)+2(c(G)-1)$ vertices of degree 3.

Proof. Let n_1, n_2, n_3 and $n_{\geq 3}$ be respectively the number of leaves, the number of vertices of degree 2, the number of vertices of degree 3 and the number of vertices of degree at least 3 of G. On the one hand, we know that

$$\begin{aligned} n_1 + 2\,n_2 + \sum_{\deg(u)\geq 3} \deg(u) &= \sum_{u\in V(G)} \deg(u) = 2|E(G)| = 2\,(n(G) + c(G) - 1) \\ &= 2\left(n_1 + n_2 + n_{\geq 3} + c(G) - 1\right). \end{aligned}$$

From here, we deduce that

$$n_1 = \sum_{\deg(u)\geq 3} (\deg(u) - 2) - 2(c(G) - 1) \geq n_{\geq 3} - 2(c(G) - 1). \tag{2}$$

If $\chi_{NL}(G) = k$, then by (2) and Remark 1,

$$\begin{aligned} n(G) = n_1 + n_2 + n_{\geq 3} &\leq k(k-1) + k\binom{k-1}{2} + n_1 + 2(c(G) - 1) \\ &\leq k(k-1) + k\binom{k-1}{2} + k(k-1) + 2(c(G) - 1) \\ &= \frac{1}{2}\left(k^3 + k^2 - 2k\right) + 2(c(G) - 1). \end{aligned}$$

Now, assume that there is a graph G attaining this bound. In such a case, the inequalities in the preceding expression must be equalities. Thus, $n_{\geq 3} = n_1 + 2(c(G) - 1) = k(k-1) + 2(c(G) - 1)$ and $n_1 + n_2 = k(k-1) + k\binom{k-1}{2}$. Hence, $n_1 = k(k-1)$ and $n_2 = k\binom{k-1}{2} = \frac{k(k-1)(k-2)}{2}$. From (2), we deduce that $n_{\geq 3} = n_1 + 2(c(G) - 1)$ if and only if there are no vertices of degree greater than 3. Hence, $n_3 = n_{\geq 3} = n_1 + 2(c(G) - 1) = k(k-1) + 2(c(G) - 1)$. ■

3. UNICYCLIC GRAPHS

A connected graph G is *unicyclic* if it contains precisely one cycle. As a direct consequence of Theorem 3, we have the following bound for unicyclic graphs taking into account that the cycle rank for unicyclic graphs is 1.

Corollary 4. *Let G be a unicyclic graph. If $\chi_{NL}(G) = k \geq 3$, then*

$$n(G) \leq \frac{1}{2}\left(k^3 + k^2 - 2k\right).$$

Moreover, if the equality holds, then G has maximum degree 3, and it contains exactly $k(k-1)$ leaves, $\frac{k(k-1)(k-2)}{2}$ vertices of degree 2, and $k(k-1)$ vertices of degree 3.

Hence, for $k \in \{3, 4\}$, the maximum order of a unicyclic graph with NLC-number k is given by the general bound of Theorem 2, whereas for $k \geq 5$, the bound given in the preceding corollary is more adjusted (see Table 1). In all cases, these bounds are tight. For $k = 3$, the cycle of order 9 is a unicyclic graph attaining the bound (see Figure 1(a)) and for $k = 4$, the graph in Figure 1(b) provides an example of unicyclic graph of order 28.

$\chi_{NL}(G)$	general bound	unicyclic graphs			
k	$k\,(2^{k-1}-1)$	$\frac{1}{2}(k^3+k^2-2k)$	n_1	n_2	n_3
3	9	**15**	**6**	**3**	**6**
4	28	**36**	**12**	**12**	**12**
5	75	70	20	30	20
6	186	120	30	60	30
7	441	189	42	105	42

Table 1. Upper bounds on the order of a graph and of a unicyclic graph for some values of $\chi_{NL}(G)$. In the last columns, n_i is the number of vertices of degree i, $i \in [3]$, of a unicyclic graph attaining the specific upper bound. Cases in boldface are not feasible.

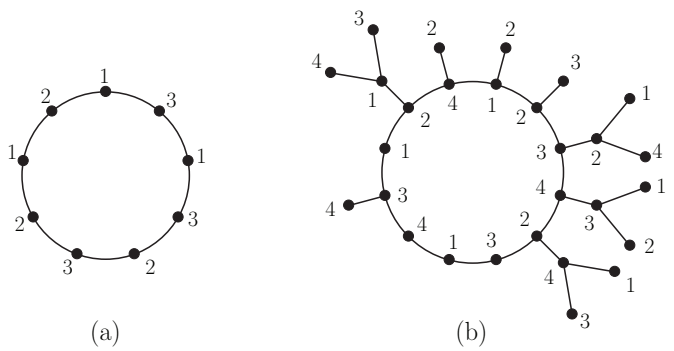

Figure 1. (a) The cycle of order 9 is a unicyclic graph of order 9 and NLC-number 3. (b) An example of unicyclic graph of order 28 and NLC-number 4.

For $k \geq 5$, we prove that the bound give in Corollary 4 is tight by means of some special k-NL-colorings of cycles and comb graphs of certain order. Concretely, cycles of order $k(k-1)(k-2)/2$ and comb graphs of order $2k(k-1)$. We denote

$$a(k) = \frac{k(k-1)(k-2)}{2}.$$

Note that $a(k)$ is the maximum number of vertices of color-degree 2 for a k-NL-coloring.

Proposition 5. *For every $k \geq 3$, there is a k-NL-coloring of the cycle of order $a(k)$ such that every vertex has color-degree 2.*

Proof. We proceed by induction on $k \geq 3$. If $k = 3$, then $a(3) = 3$ and any bijection from $V(C_3)$ to $[3]$ is a 3-NL-coloring with all vertices having color-degree 2. Let $k \geq 4$ and suppose by induction hypothesis that there is a $(k-1)$-NL-coloring f of $C_{a(k-1)}$ with all vertices having color-degree 2. We extend f to a k-NL-coloring f' of $C_{a(k)}$ with all vertices having color-degree 2.

Since $a(k-1)$ is the maximum number of vertices of color degree 2 for a $(k-1)$-NL-coloring, for every three distinct elements $i, j, h \in [k-1]$, there is a vertex with color i such that its neighbors are colored with j and h. Therefore, for every pair of colors $i, j \in [k-1]$, $i \neq j$, we can choose an edge $e(i,j)$ with endpoints colored with i and j. For every one of these $\binom{k-1}{2}$ edges, we proceed in the following way.

Let $e(i,j) = xy$ with $f(x) = i$ and $f(y) = j$. Insert three new vertices x', z' and y' in the edge xy, so that xx', $x'z'$, $z'y'$ and $y'y$ are edges of the new cycle, and define $f'(x') = j$, $f'(y') = i$, $f'(z') = k$ and $f'(u) = f(u)$, whenever $u \in V(C_{a(k-1)})$. Then, $f'(N(x')) = \{i, h\}$, $f'(N(z')) = \{i, j\}$, $f'(N(y')) = \{j, h\}$ and $f'(N(u)) = f(N(u))$, whenever $u \in V(C_{a(k-1)})$ (see Figure 2). Notice that the color, color-degree and the colors of the neighbors remain unchanged for the vertices belonging to $C_{a(k-1)}$ when extending f to f'.

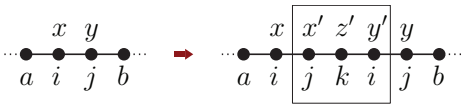

Figure 2. Inserting three new vertices in the edge $e(i,j) = xy$.

By construction, since $a(k-1) + 3\binom{k-1}{2} = \frac{(k-1)(k-2)(k-3)}{2} + 3\frac{(k-1)(k-2)}{2} = \frac{k(k-1)(k-2)}{2} = a(k)$, we obtain a proper k-coloring f' of a cycle of order $a(k)$ with all vertices having color-degree 2. We claim that f' is a k-NL-coloring of $C_{a(k)}$. Indeed, suppose that $f'(u) = f'(v)$, for $u, v \in V(C_{a(k)})$, $u \neq v$.

If $u, v \in V(C_{a(k-1)})$, then $f(u) = f'(u) = f'(v) = f(v)$. Hence, $f'(N(u)) = f(N(u)) \neq f(N(v)) = f'(N(v))$, because f is an NL-coloring of $C_{a(k-1)}$.

If $u \in V(C_{a(k-1)})$ and $v \notin V(C_{a(k-1)})$, then $f'(v) = f'(u) \neq k$. Hence, $f'(N(v)) \neq f'(N(u))$, because $k \in f'(N(v))$ and $k \notin f(N(u)) = f'(N(u))$.

Finally, suppose that $u, v \notin V(C_{a(k-1)})$. If $f'(u) = f'(v)$, then u and v are inserted in distinct edges, say $e(i,j)$ and $e(i',j')$, respectively, with $i \neq j$, $i' \neq j'$ and $\{i,j\} \neq \{i',j'\}$. If $f'(u) = f'(v) = k$, then $f'(N(u)) = \{i,j\} \neq \{i',j'\} = f'(N(v))$. If $f'(u) = f'(v) = h \neq k$, we may assume without loss of generality that $h = i = i'$. Then, $f'(N(u)) = \{j,k\} \neq \{j',k\} = f'(N(v))$. ■

An example of the procedure described in the proof of Proposition 5 is shown in Figure 3 for $k = 4, 5$.

For every integer $m \geq 3$, the *comb graph* B_m is a tree of order $2m$ obtained by attaching a leaf at every vertex of a path of order m.

Proposition 6. *For every $k \geq 5$, there is a k-NL-coloring of the comb graph $B_{k(k-1)}$.*

Proof. Let $k \geq 5$. Consider the comb $B_{k(k-1)}$ obtained by hanging a leaf to each vertex of a path P of order $k(k-1)$. We assign the color 1 to the leaves hanging from the first $k-1$ vertices of the path P; color 2 to the leaves hanging from the following $k-1$ vertices of P; and so on. For every $r \in [k]$, consider the set M_r containing the $k-1$ vertices of P adjacent to the leaves with color r. We define a bijection between the vertices of M_r and the $k-1$ colors of the set $L_r = [k] \setminus \{r\}$. Set $M_r = \{x_1^r, \ldots, x_{k-1}^r\}$ so that $x_i^r x_{i+1}^r \in E(P)$ for every $i \in [k-2]$, and $x_{k-1}^r x_1^{r+1} \in E(P)$, whenever $r < k$.

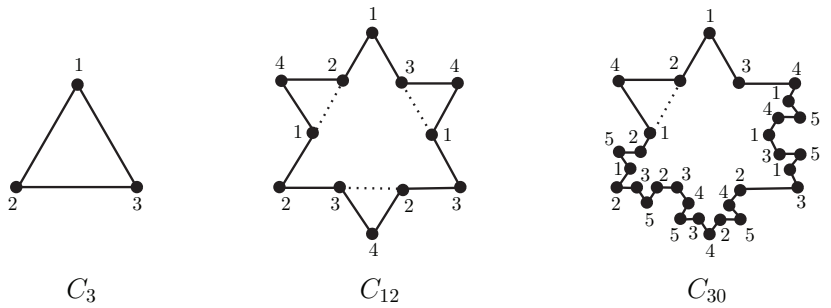

Figure 3. Obtaining a k-NL-coloring of $C_{a(k)}$, $k \in \{4, 5\}$ with all vertices of color degree 2 from a $(k-1)$-NL-coloring of $C_{a(k-1)}$ satisfying the same condition. In this case, $a(3) = 3$, $a(4) = 12$ and $a(5) = 30$. For $k = 4, 5$ we insert 3 new vertices to the selected edges with endpoints colored with i, j, for every $i, j \in [k-1]$, $i \neq j$.

We assign the colors of L_r in cyclically decreasing order to the vertices $x_1^r, \ldots, x_{k-1}^r$ beginning with a concrete color according to the following rules:

(1) If r is even, then we begin with $r+1$ modulo k. Therefore, x_1^r and x_2^r are colored respectively with $r+1$ and $r-1$ modulo k.

(2) If r is odd and $r < k$, then we begin with $r-2$ modulo k. Therefore, x_{k-2}^r and x_{k-1}^r are colored respectively with $r+1$ and $r-1$ modulo k.

(3) If r is odd and $r = k$, then we proceed as in the case r odd and $r < k$, but we switch the colors of the last three vertices so that x_{k-3}^k, x_{k-2}^k and x_{k-1}^k have color $k-1$, 1 and 2, respectively.

See the defined k-NL-coloring of the comb $B_{k(k-1)}$ for $k \in \{5, 6, 7\}$ in Figure 4.

Notice that the colors of two consecutive vertices of M_r differ by one unit modulo k, except for the first two vertices, when r is even, and for the last two vertices, when r is odd. Besides, the first vertex of M_r is always colored with an odd number and the last vertex of M_r is colored with an even number, whenever k is even or when k is odd and $r \notin \{1, k-1, k\}$. We claim that this procedure gives a k-NL-coloring of the comb $B_{k(k-1)}$.

By construction, only the leaves have color-degree 1. Hence, it only remains to prove that for every pair of non-leaves with the same color, the sets of colors of their neighborhoods are different.

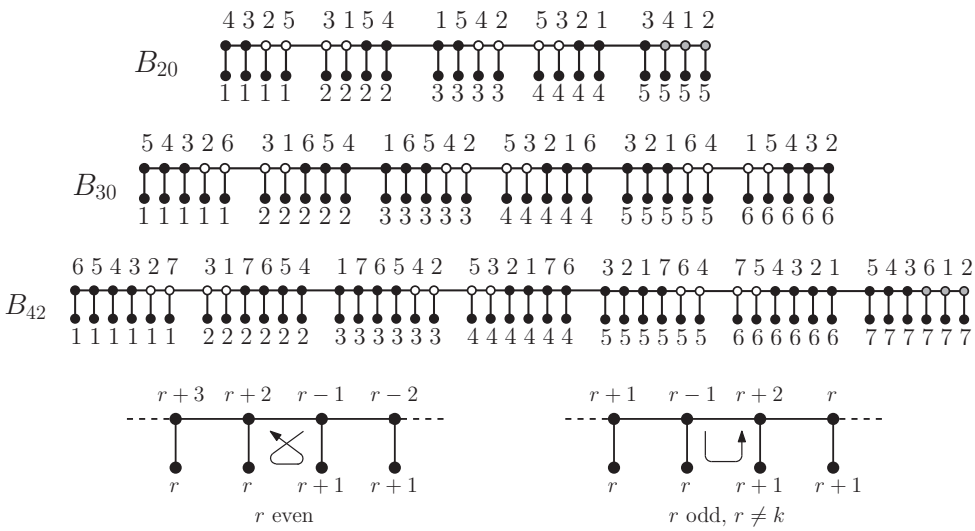

Figure 4. A 5-NL-coloring of the comb B_{20}, a 6-NL-coloring of the comb B_{30} and a 7-NL-coloring of the comb B_{42}. In white, adjacent vertices of M_r with no consecutive colors modulo k. In B_{20} and in B_{42}, we have shifted the colors of the vertices in gray with respect to the general rule used to the vertices of M_r, when r is odd. Below, the general rule for coloring adjacent vertices of consecutive groups M_r and M_{r+1} and the leaves hanging from them. In all cases, the colors involved are $r-1$, r, $r+1$ and $r+2$.

Let $l \in [k]$. There are exactly $k-1$ non-leaves colored with l, and exactly one of them belongs to M_r, for every $r \in [k] \setminus \{l\}$. Let v_l^r denote the vertex in M_r colored with l. Notice that the colors of the neighbors of v_l^r are $\{r, l-1, l+1\}$, except when v_l^r occupies the first or the last position in M_r. Concretely, this happens for $r \in \{l-2, l-1, l+1\}$, if l is even, and for $r \in \{l-1, l+1, l+2\}$, if l is odd, whenever $l \neq \{1, 2, 3, k-1, k\}$. Those last cases are analysed separately.

In Table 2, the colors of the neighbors of v_l^r, $r \neq l$, are summarized. Observe that, for a fixed value of l, the sets of colors of the neighbors of the vertices v_l^r, $r \neq l$, are pairwise different in all cases, so that we have a k-NL-coloring for each case. ■

Theorem 7. *For every $k \geq 5$, there is a unicyclic graph U_k of order $\frac{1}{2}\big(k^3 + k^2 - 2k\big)$ such that $\chi_{NL}(U_k) = k$.*

Proof. Consider the k-NL-coloring of the cycle $C_{a(k)}$ with all vertices having color-degree 2 constructed in the proof of Proposition 5. There is an edge xy with its endpoints x and y colored respectively with 2 and $k-1$. Consider the k-NL-coloring of the comb $B_{k(k-1)}$ given in the proof of Proposition 6. Let x' and y' be the vertices of degree 2 of the comb $B_{k(k-1)}$. These vertices are colored with

l even, $l \neq 2, k-1, k$	
r	colors of $N(v_l^r)$
$r \notin \{l-2, l-1, l+1\}$	$\{r, l-1, l+1\}$
$l-1$	$\{l-2, l-1, l+1\}$
$l+1$	$\{l+1, l+2, l+3\}$
$l-2$	$\{l-3, l-2, l+1\}$

l odd, $l \neq 1, 3, k-1, k$	
r	colors of $N(v_l^r)$
$r \notin \{l-1, l+1, l+2\}$	$\{r, l-1, l+1\}$
$l-1$	$\{l-3, l-2, l-1\}$
$l+1$	$\{l-1, l+1, l+2\}$
$l+2$	$\{l-1, l+2, l+3\}$

$l = 1$	
r	colors of $N(v_1^r)$
$r \notin \{2, 3, 4\}$	$\{r, 2, k\}$
2	$\{2, 3, k\}$
3	$\{3, 4, k\}$
k even	$\{k-2, k-1, k\}$
k odd	$\{2, k-2, k-1\}$

$l = 2$	
r	colors of $N(v_2^r)$
$r \notin \{1, 3, k\}$	$\{r, 1, 3\}$
1	$\{1, 3, k\}$
3	$\{3, 4, 5\}$
k even	$\{3, k\}$
k odd	$\{1, k\}$

$l = 3$, $k \geq 6$ even	
r	colors of $N(v_3^r)$
$r \notin \{2, 4, 5, k-1, k\}$	$\{r, 2, 4\}$
2	$\{1, 2, k\}$
4	$\{2, 4, 5\}$
5	$\{2, 5, 6\}$
$k-1$	$\{2, 4, k-1\}$
k	$\{2, 4, k\}$

$l = 3$, $k \geq 7$ odd	
r	colors of $N(v_3^r)$
$r \notin \{2, 4, 5, k-1, k\}$	$\{r, 2, 4\}$
2	$\{1, 2, k\}$
4	$\{2, 4, 5\}$
5	$\{2, 5, 6\}$
$k-1$	$\{2, k-1, k\}$
k	$\{4, k-1, k\}$

$l = k-1$, k even	
r	colors of $N(v_{k-1}^r)$
$r \notin \{1, k-2, k\}$	$\{r, k-2, k\}$
1	$\{1, k-2\}$
$k-2$	$\{k-4, k-3, k-2\}$
k	$\{1, k-2, k\}$

$l = k-1$, k odd	
r	colors of $N(v_{k-1}^r)$
$r \notin \{1, k-3, k-2, k\}$	$\{r, k-2, k\}$
1	$\{1, k-2\}$
$k-3$	$\{k-4, k-3, k\}$
$k-2$	$\{k-3, k-2, k\}$
k	$\{1, 3, k\}$

$l = k$, even	
r	colors of $N(v_k^r)$
$r \notin \{1, k-2, k-1\}$	$\{r, 1, k-1\}$
1	$\{1, 2, 3\}$
$k-2$	$\{1, k-3, k-2\}$
$k-1$	$\{1, k-2, k-1\}$

$l = k$, odd	
r	colors of $N(v_k^r)$
$r \notin \{1, k-1\}$	$\{r, 1, k-1\}$
1	$\{1, 2, 3\}$
$k-1$	$\{k-3, k-2, k-1\}$

Table 2. Colors of the neighborhoods of non-leaves of the comb $B_{k(k-1)}$.

$k-1$ and 2, respectively. Consider the unicyclic graph U_k obtained from the union of the cycle and the comb, by deleting the edge xy from the cycle $C_{a(k)}$ and adding the edges xx' and yy' (see in Figure 5 the case $k = 6$). Notice that the order of U_k is $n(U_k) = n\big(C_{a(k)}\big) + n\big(B_{k(k-1)}\big) = \frac{k(k-1)(k-2)}{2} + 2k(k-1) = \frac{1}{2}\big(k^3 + k^2 - 2k\big)$.

We claim that the considered k-NL-colorings of the cycle and the comb induce a k-NL-coloring in U_k. We have only changed the colors of the neighborhoods of x' and y'. On the one hand x' has color $k-1$ and the colors of its neighbors are $\{1, 2, k-2\}$. On the other hand, y has color 2 and the colors of its neighbors are $\{3, k-1, k\}$ if k is even, and $\{1, k-1, k\}$, if k is odd. We can check in the tables given in the proof of Proposition 6 that any other vertex of the comb $B_{k(k-1)}$ has different color or different set of colors in their neighborhoods from those of x' and y'. Hence, we have a k-NL-coloring of U_k. ■

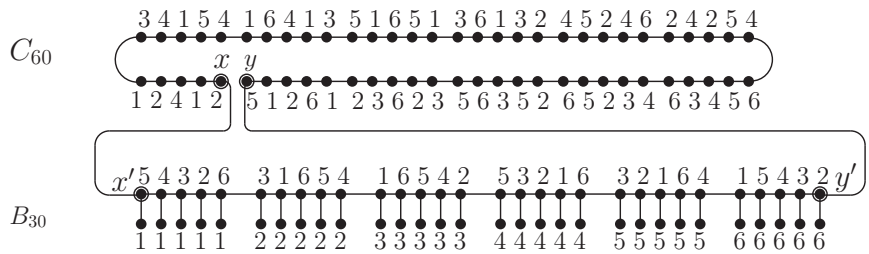

Figure 5. A 6-NL-coloring of the unicyclic graph U_6 of order 120.

Corollary 8. *For every $k \geq 5$, the bound given in Theorem 4 is tight.*

4. TREES

In this section, some relations between the NLC-number and other parameters for trees are approached. An upper bound on the order of a tree in terms of its NLC-number follows from Theorem 3 taking into account that the cycle rank of trees is equal to 0.

Corollary 9. *Let T be a nontrivial tree. If $\chi_{NL}(T) = k \geq 3$, then*

$$n(T) \leq \frac{1}{2}\big(k^3 + k^2 - 2k\big) - 2.$$

Moreover, if the equality holds, then T has maximum degree 3 and it contains exactly $k(k-1)$ leaves, $\frac{k(k-1)(k-2)}{2}$ vertices of degree 2, and $k(k-1)-2$ vertices of degree 3.

$\chi_{NL}(G)$	general bound	trees			
k	$k\,(2^{k-1}-1)$	$\frac{1}{2}(k^3+k^2-2k)-2$	n_1	n_2	n_3
3	9	**13**	**6**	**3**	**4**
4	28	**34**	**12**	**12**	**10**
5	75	68	20	30	18
6	186	118	30	60	28
7	441	187	42	105	40

Table 3. Upper bounds on the order of a graph and of a tree for some values of $\chi_{NL}(G)$. In the last columns, n_i is the number of vertices of degree i, $i \in [3]$, of a tree attaining the specific upper bound for trees. Cases in boldface are not feasible.

Table 3 summarizes the different upper bounds on the order of a graph in terms of the NLC-number, k, for general graphs and trees. The cases in bold are not feasible because the bound for general graphs (see Theorem 2) is smaller than the specific bound for trees given in Corollary 9. The last column shows the number of vertices of degree 1 (n_1), of degree 2 (n_2) and of degree 3 (n_3) that has a tree attaining the upper bound, as shown in Corollary 9.

For $k = 3$, the path P_9 is an example of a tree attaining the general upper bound. For $k = 4$, a tree attaining the general upper bound is displayed in Figure 6 (left). For $k = 5$, the upper bound given for trees is 68. Figure 6 (right) shows a tree on 66 vertices and NLC-number 5. Determining if there exists a tree with NLC-number 5 and 67 or 68 vertices remains as an open problem. Next proposition shows that there is a tree attaining the specific upper bound for trees whenever $k \geq 6$.

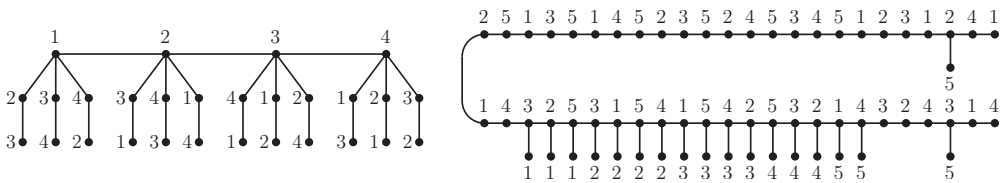

Figure 6. A tree T_1 of order 28 with $\chi_{NL}(T_1) = 4$ (left) and a tree T_2 of order 66 with $\chi_{NL}(T_2) = 5$ (right).

Theorem 10. *For every $k \geq 6$, there is a tree T_k of order $\frac{1}{2}(k^3+k^2-2k)-2$ such that $\chi_{NL}(T_k) = k$.*

Proof. Consider the k-NL-coloring of the unicyclic graph U_k of order $\frac{1}{2}(k^3+k^2-2k)$ described in Theorem 7. Consider the leaf of color 2 hanging from the vertex x colored with $k-1$. Delete both vertices and add the edge joining the remaining neighbors of x. Do the same with the leaf colored with $k-1$ hanging

from the vertex y of color 2. Remove the edge joining the vertex u of degree 2 and color 2 with the vertex v of degree 3 and color $k-1$. Attach a leaf colored with $k-1$ to vertex u and a leaf colored with 2 to vertex v. We obtain a tree T_k of order $\frac{1}{2}(k^3+k^2-2k)-2$ (see an example in Figure 7).

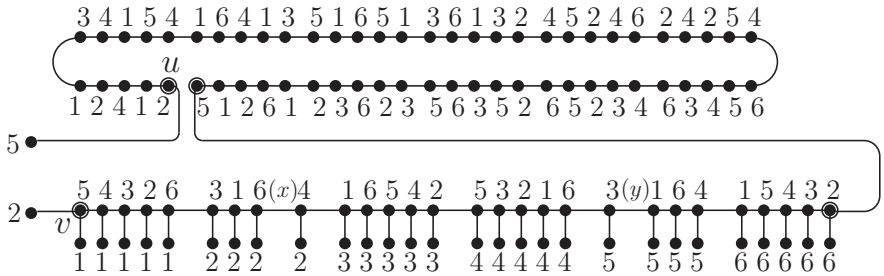

Figure 7. A 6-NL-coloring of a tree of order 118 constructed from a 6-NL-coloring of a unicyclic graph of order 120.

We claim that in such a way we have a k-NL-coloring of T_k. Indeed, we have only changed the colors of the neighborhoods of the vertices adjacent to x and to y in T_k. Following the notations of the proof of Proposition 6, we have $x = v_{k-1}^2$ and $y = v_2^{k-1}$, and, if $k \geq 6$, the vertices adjacent to them in M_2 and M_{k-1} are respectively v_{k-2}^2, v_k^2 and v_1^{k-1}, v_3^{k-1}. After deleting the vertices x and y from the comb, the colors of their neighborhoods are given in Table 4.

z	v_1^{k-1}	v_3^{k-1}	v_{k-2}^2	v_k^2
color of z	1	3	$k-2$	k
colors of $N(z)$ in T_k	$\{3, k-1, k\}$	$\{1, 4, k-1\}$	$\{2, k-3, k\}$, if $k \geq 7$ $\{1, 2, 6\}$, if $k = 6$	$\{1, 2, k-2\}$

Table 4. Set of colors of the neighborhoods of $v_1^{k-1}, v_3^{k-1}, v_{k-2}^2$ and v_k^2.

Using Tables 2 and 4, we check that vertices in T_k with the same color, have different sets of colors in their neighborhoods. Therefore, we have a k-NL-coloring of T_k. ■

Finally, some other results involving the NLC-number of trees are shown.

Proposition 11. *Let T be a tree of order at least 5. If T is a star, then $\chi_{NL}(T) = n(T)$; otherwise $\chi_{NL}(T) \leq n(T) - 2$.*

Proof. If $\mathrm{diam}(T) = 2$, then T is a star, and $\chi_{NL}(T) = n(T)$ (see [1]).

If $\mathrm{diam}(T) = 3$, then T is a double star, that is, T has exactly two adjacent vertices u and v which are not leaves; u is adjacent to r leaves and v is adjacent to s leaves, with $1 \leq r \leq s \leq n(T) - 3$. Consider a coloring f such that

$f(u) = 1$, $\{f(x) : xu \in E(T)\} = \{2, 3, \ldots, r+1\}$, $f(v) = s+1$ and $\{f(x) : xv \in E(T)\} = \{1, 2, \ldots, s\}$. It is easy to check that f is an NL-coloring of T. Hence, $\chi_{NL}(T) \leq s+1 \leq n(T) - 2$.

Otherwise, $\mathrm{diam}(T) \geq 4$. Then, let x, y be two vertices at distance 4, and let a, b, c be three vertices such that xa, ab, bc, cy are edges of T. Consider a coloring f such that $\{f(u) : u \in V(T) \setminus \{x, b, y\}\} = \{1, 2, \ldots, n(T) - 3\}$, and $f(x) = f(b) = f(y) = n(T) - 2$. It is easy to check that f is an NL-coloring of T. Thus, $\chi_{NL}(T) \leq n(T) - 2$. ■

Proposition 12. *Let T be a tree. If $\chi_{NL}(T) = k$, then $\Delta(T) \leq (k-1)^2 + \frac{k-1}{2}$.*

Proof. Suppose to the contrary that $\chi_{NL}(T) = k$ and $\Delta(T) > (k-1)^2 + \frac{k-1}{2}$.

Consider a k-NL-coloring of T. We claim that if u is a vertex of degree $\Delta(T)$, then u has at most $(k-1)^2$ neighbors of degree at most 2. Indeed, assume that u has color k, then the neighbors of u are colored with some $i \in [k-1]$. Besides, the neighbors of u with degree at most 2 have color-degree at most 2, but they all have u as a neighbor. Hence, there are at most $k-1$ neighbors of u with degree at most 2 and colored with i, for each $i \in [k-1]$, implying that there are at most $(k-1)^2$ neighbors of u with degree at most 2.

Thus, u must have more than $\frac{k-1}{2}$ neighbors of degree at least 3. Therefore, the number n_1 of leaves satisfies $n_1 > \Delta(T) + \frac{k-1}{2} > (k-1)^2 + 2\frac{k-1}{2} = k(k-1)$. But, by Remark 1, T has at most $k(k-1)$ leaves, a contradiction. ■

5. Conclusions and Open Problems

In [1], the order of a graph is bounded from above by a function of its NLC-number. In the present paper, we provide an upper bound that is better for both unicyclic graphs (see Corollary 4) and trees (see Corollary 9). Moreover, we show that these new bounds are tight for NLC-number $k \geq 5$ in the case of unicyclic graphs (see Theorem 7) and for NLC-number $k \geq 6$ in the case of trees (see Theorem 10). In this last case, the given tree attaining the bound is a caterpillar. For trees with NLC-number 5, the maximum order according to our bound is 68, but we have obtained trees with order at most 66. The existence of trees with NLC-number 5 and order 67 or 68 is an open problem.

According to Theorem 2, an upper bound on the order of a graph G with fixed maximum degree Δ and NLC-number k satisfying $\Delta \leq k-1$ is

$$n(G) \leq k \sum_{j=1}^{\Delta} \binom{k-1}{j}. \tag{3}$$

As we have already pointed out in Section 2, the family of cycles provides examples of graphs attaining the bound for $\Delta = 2$ and $k \geq 3$ (see [4]). Nevertheless, it

is an open problem to determine if this bound is tight whenever $3 \leq \Delta(G) \leq k-1$. At the moment, if G is either a tree or a unicyclic graph then, by Theorem 3,

$$n(G) \leq \frac{1}{2}\big(k^3 + k^2 - 2k\big)$$

and it can be easily checked that this value is smaller than the considered bound when either $\Delta = 3$ and $k \geq 6$ or $\Delta \geq 4$ and $k \geq 5$. Hence, the bound can be achieved neither by trees nor by unicyclic graphs in these cases. For $\Delta = 3$ and $k = 5$, the unicyclic graph U_5 of order 70 described in Section 3 attains the bound. For $\Delta = 3$ and $k = 4$, the unicyclic graph of order 28 depicted in Figure 1(b) attains the bound. For $\Delta \geq k$, and taking into account that $\binom{n}{m} = 0$ whenever $m > n$, the bound given in equation (3) turns out to be

$$n(G) \leq k \sum_{j=1}^{k-1} \binom{k-1}{j} = k(2^{k-1} - 1)$$

and it is attained for a graph with $\Delta = (k-1)2^{k-2}$ (see [1]). It remains an open problem to determine if it is tight for other values of Δ such that $\Delta \geq k$.

In Proposition 12 we give an upper bound on the maximum degree in terms of the NLC-number. However, this bound is not tight. Indeed, for a tree T with NLC-number 3, this proposition states that $\Delta(T) \leq 5$. However, by Theorem 2(1), we have $n(T) \leq 9$ in this case, and it is easy to verify that $\chi_{NL}(T) = 3$ and $n(T) \leq 9$ imply $\Delta(T) \leq 4$. In general, we think that the maximum degree of a tree is bounded in terms of the XNL-number as follows.

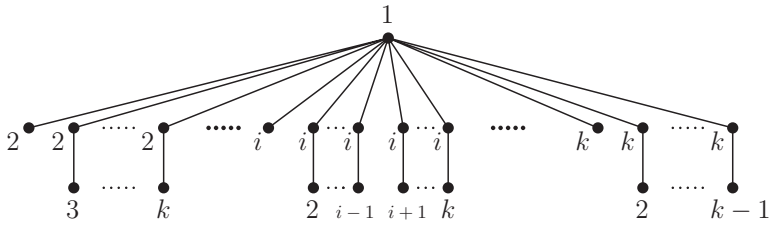

Figure 8. Tree T with $\Delta(T) = (k-1)^2$ and $\chi_{NL}(T) = k$.

Conjecture 13. *Let $k \geq 2$. If T is a tree with $\chi_{NL}(T) = k$, then $\Delta(T) \leq (k-1)^2$, and this bound is tight for every integer $k \geq 2$.*

An example of a tree T with $\chi_{NL}(T) = k$ and $\Delta(T) = (k-1)^2$ is given in Figure 8.

Finally, if G is a connected graph such that $\mathrm{diam}(G) = d \leq 23$, then it is possible verify that $\chi_{NL}(G) \geq \chi_{NL}(P_{d+1})$. We propose the following conjecture.

Conjecture 14. *Let G be a graph of diameter d. Then, $\chi_{NL}(G) \geq \chi_{NL}(P_{d+1})$.*

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References

[1] L. Alcon, M. Gutierrez, C. Hernando, M. Mora and I.M. Pelayo, *Neighbor-locating colorings in graphs*, Theoret. Comput. Sci. **806** (2020) 144–155. https://doi.org/10.1016/j.tcs.2019.01.039

[2] L. Alcon, M. Gutierrez, C. Hernando, M. Mora and I.M. Pelayo, *The neighbor-locating-chromatic number of pseudotrees* (2019). arXiv:1903.11937v2

[3] E.T. Baskoro and A. Asmiati, *Characterizing all trees with locating-chromatic number* 3, Electron. J. Graph Theory Appl. **1** (2013) 109–117. https://doi.org/10.5614/ejgta.2013.1.2.4

[4] A. Behtoei and M. Anbarloei, *The locating chromatic number of the join of graphs*, Bull. Iranian Math. Soc. **40** (2014) 1491–1504.

[5] A. Behtoei and M. Anbarloei, *A bound for the locating chromatic numbers of trees*, Trans. Comb. **4** (2015) 31–41.

[6] A. Behtoei and B. Omoomi, *On the locating chromatic number of Kneser graphs*, Discrete Appl. Math. **159** (2011) 2214–2221. https://doi.org/10.1016/j.dam.2011.07.015

[7] G.G. Chappell, J. Gimbel and C. Hartman, *Bounds on the metric and partition dimensions of a graph*, Ars Combin. **88** (2008) 349–366.

[8] G. Chartrand, D. Erwin, M.A. Henning, P.J. Slater and P. Zhang, *The locating-chromatic number of a graph*, Bull. Inst. Combin. Appl. **36** (2002) 89–101.

[9] G. Chartrand, D. Erwin, M.A. Henning, P.J. Slater and P. Zhang, *Graphs of order n with locating-chromatic number $n-1$*, Discrete Math. **269** (2003) 65–79. https://doi.org/10.1016/S0012-365X(02)00829-4

[10] G. Chartrand, L. Lesniak and P. Zhang, Graphs and digraphs, Fifth Edition (Chapman and Hall/CRC Press, Taylor and Francis Group, 2011).

[11] G. Chartrand, E. Salehi and P. Zhang, *The partition dimension of a graph*, Aequationes Math. **59** (2000) 45–54. https://doi.org/10.1007/PL00000127

[12] M. Fehr, S. Gosselin and O.R. Oellermann, *The partition dimension of Cayley digraphs*, Aequationes Math. **71** (2006) 1–18. https://doi.org/10.1007/s00010-005-2800-z

[13] H. Fernau, J.A. Rodríguez-Velázquez and I. González-Yero, *On the partition dimension of unicyclic graphs*, Bull. Math. Soc. Sci. Math. Roumanie **57(105)** (2014) 381–391.

[14] C. Grigorious, S. Stephen, R. Rajan and M. Miller, *On the partition dimension of circulant graphs*, Comput. J. **60** (2017) 180–184. https://doi.org/10.1093/comjnl/bxw079

[15] F. Harary and R. Melter, *On the metric dimension of a graph*, Ars Combin. **2** (1976) 191–195.

[16] C. Hernando, M. Mora and I.M. Pelayo, *Resolving dominating partitions in graphs*, Discrete Appl. Math. **266** (2019) 237–251. https://doi.org/10.1016/j.dam.2018.12.001

[17] J.A. Rodríguez-Velázquez, I. González Yero and M. Lemańska, *On the partition dimension of trees*, Discrete Appl. Math. **166** (2014) 204–209. https://doi.org/10.1016/j.dam.2013.09.026

[18] P.J. Slater, *Leaves of trees*, in: Proc. 6th Southeastern Conf. on Combinatorics, Graph Theory, and Computing, Congr. Numer. **14** (1975) 549–559.

[19] P.J. Slater, *Dominating and reference sets in a graph*, J. Math. Phys. Sci. **22** (1988) 445–455.

[20] D. Welyyanti, E.T. Baskoro, Darmaji, R. Simanjuntak and S. Uttunggadewa, *On locating-chromatic number of complete n-ary tree*, AKCE Int. J. Graphs Comb. **10** (2013) 309–315.

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