

ZERO AND TOTAL FORCING DENSE GRAPHS

RANDY DAVILA^{1,2}, MICHAEL HENNING²

AND

RYAN PEPPER¹

¹*Department of Mathematics and Statistics
University of Houston-Downtown
Houston, Texas 77002*

²*Department of Mathematics and Applied Mathematics
University of Johannesburg
Auckland Park 2006, South Africa*

e-mail: DavilaR@uhd.edu
mahenning@uj.ac.za
PepperR@uhd.edu

Abstract

If S is a set of colored vertices in a simple graph G , then one may allow a colored vertex with exactly one non-colored neighbor to force its non-colored neighbor to become colored. If by iteratively applying this color change rule, all of the vertices in G become colored, then S is a zero forcing set of G . The minimum cardinality of a zero forcing set in G , written $Z(G)$, is the zero forcing number of G . If in addition, S induces a subgraph of G without isolated vertices, then S is a total forcing set of G . The total forcing number of G , written $F_t(G)$, is the minimum cardinality of a total forcing set in G . In this paper we introduce, and study, the notion of graphs for which all vertices are contained in some minimum zero forcing set, or some minimum total forcing set; we call such graphs ZF-dense and TF-dense, respectively. A graph is ZTF-dense if it is both ZF-dense and TF-dense. We determine various classes of ZTF-dense graphs, including among others, cycles, complete multipartite graphs of order at least three that are not stars, wheels, n -dimensional hypercubes, and diamond-necklaces. We show that no tree of order at least three is ZTF-dense. We show that if G and H are connected graphs of order at least two that are both ZF-dense, then the join $G + H$ of G and H is ZF-dense.

Keywords: zero forcing sets, zero forcing number, ZF-dense.

2020 Mathematics Subject Classification: 05C69.

1. INTRODUCTION

Coloring the vertices of a graph G and allowing this initial coloring to propagate throughout the vertex set of G is known as a *dynamic coloring* of G . In this paper, we will focus on the dynamic coloring due to the *forcing process*, which we recall the definition from [10] as follows. Let G be a finite and simple graph with vertex set $V(G)$, and let $S \subseteq V(G)$ be a set of initially “colored” vertices, all remaining vertices being “uncolored”. All vertices contained in S are said to be *S-colored*, while all vertices not in S are *S-uncolored*. At each discrete time step, if a colored vertex has exactly one uncolored neighbor, then this colored vertex *forces* its uncolored neighbor to become colored. If v is such a colored vertex, then we call v a *forcing vertex*, and say that v has been *played*. The initial set of vertices S is a *zero forcing set*, if by iteratively applying this forcing process all of $V(G)$ becomes colored. We call such a set, an *S-forcing set*. If S is a zero forcing set of G and v is an *S-colored* vertex which has been played, then v is called an *S-forcing vertex*. The *zero forcing number* of G , written $Z(G)$, is the cardinality of a minimum forcing set in G . If S is an *S-forcing set* of G which also induces a subgraph without isolated vertices, then S is a *total forcing set*, abbreviated *TF-set*, of G . The *total forcing number* of G , written $F_t(G)$, is the cardinality of a minimum TF-set in G .

Zero forcing in graphs was first introduced and studied in an AIM Special Work Group [3] in 2008, and has subsequently been extensively studied in the literature; MathSciNet lists over 140 papers to date on the topic. For a small sample of recent (2020) papers on zero forcing we refer the reader to [4, 12, 15, 16, 20]. The notion of total forcing in graphs was first introduced by Davila [7] in 2015 as a strengthening of zero forcing in graphs, and has been studied, for example, in [8–11].

Motivation. There is a close connection between power domination in a graph and zero forcing in graphs, as discussed, for example, by Benson *et al.* [5] in 2018. The power domination process in a graph G can be described as choosing a set S of vertices in G and applying the zero forcing process to the closed neighborhood $N[S]$ of S . Indeed, as first observed by Aazemi [1, 2], a set S is a power dominating set of a graph G if and only if $N[S]$ is a zero forcing set of G . Benson *et al.* [5] proved that the zero forcing number of a graph is at most its maximum degree times its power domination number.

There is also a close connection with domination and zero forcing, and with total domination and total forcing in graphs (see, for example, [11, 13]). A frequently studied problem with domination type parameters is to determine the set of vertices that belong to every or to some minimum dominating set, as this is important in obtaining algorithmic and complexity results, as well as bounds

on the parameters. We refer the reader to [6, 18, 21, 23] for a small sample of such papers.

In this paper, we study an analogous concept for zero forcing and total forcing in graph. We call a graph with the property that every vertex is contained in some minimum zero forcing set (respectively, minimum TF-set) a *ZF-dense graph* (respectively, a *TF-dense graph*). If a graph is both ZF-dense and TF-dense, then we say that the graph is *ZTF-dense*.

Definitions and notation. For notation and graph terminology, we will typically follow the monograph [19]. Specifically, this paper will only consider finite and simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* and *size* of G will be denoted by $n(G) = |V(G)|$ and $m(G) = |E(G)|$, respectively. A *nontrivial graph* is a graph of order at least 2. Two vertices $v, w \in V(G)$ are said to be *neighbors*, or *adjacent*, if $vw \in E(G)$. The *open neighborhood* of a vertex $v \in V(G)$, written $N_G(v)$, is the set of all neighbors of v , whereas the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$, written $d_G(v)$, is the number of neighbors of v in G ; and so, $d_G(v) = |N_G(v)|$. The complete graph, path, and cycle, on n vertices will be denoted by K_n , P_n , and C_n , respectively.

A graph G is connected if for all vertices v and w in G , there exists a (v, w) -path. The length of a shortest (v, w) -path in G is the *distance* between v and w , and is written $d_G(v, w)$ or simply $d(v, w)$ if G is clear from context.

A *tree* is a connected graph which contains no cycle as a subgraph. A vertex of degree 1 in a tree is called a *leaf* and a vertex with a leaf neighbor is a *support vertex*. A *strong support vertex* is a vertex with at least two leaf neighbors. A *branch vertex* of a tree is a vertex of degree at least 3 in the tree. A *star* is a non-trivial tree with at most one vertex which is not a leaf, and if the star in question has n leaves, we denote the star by $K_{1,n}$.

For $k \geq 2$ a graph G is *k-partite* if its vertex set $V(G)$ can be partitioned into k subsets V_1, V_2, \dots, V_k (called partite sets) in such a way that no two vertices of V_i are adjacent for all $i \in [k]$. Further, if every vertex of V_i is adjacent to every vertex not in V_i for all $i \in [k]$, then G is a *complete k-partite graph*. A 2-partite graph is called a *bipartite graph*. A graph is a *multipartite graph* if it is a k -partite graph for some $k \geq 2$.

The *Cartesian product* $G \square H$ of two graphs G and H is the graph whose vertex set is $V(G) \times V(H)$. Two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G \square H$ if either $g_1 = g_2$ and $h_1 h_2$ is an edge in H , or $h_1 = h_2$ and $g_1 g_2$ is an edge in G .

The *join graph* of two graphs G and H , written $G + H$ (also written $G \vee H$ or $G \nabla H$ in the literature), is the graph obtained from the disjoint union of G and H by joining each vertex of G to every vertex of H .

We use the standard notation $[k] = \{1, \dots, k\}$.

2. KNOWN RESULTS AND PRELIMINARY OBSERVATIONS

In this section we provide preliminary observations and results on both ZF-dense and TF-dense graphs. First, we recall some known elementary result. As defined earlier, a *nontrivial graph* is a graph of order at least 2.

Observation 1. *If G is a nontrivial connected graph, then $Z(G) \leq F_t(G)$.*

Any initially forcing vertex in the forcing process must be colored along with all but one of its neighbors. Thus the forcing number of a graph is always bounded from below by the minimum degree, as first observed in [3].

Observation 2 [3]. *If G is a graph with minimum degree δ , then $Z(G) \geq \delta(G)$.*

The zero forcing number and total forcing number of paths, cycles, complete graphs and stars is easy to compute.

Observation 3 [10]. *The following holds.*

- (a) For $n \geq 2$, $Z(P_n) = 1$ and $F_t(P_n) = 2$.
- (b) For $n \geq 3$, $Z(C_n) = F_t(C_n) = 2$.
- (c) For $n \geq 3$, $Z(K_n) = F_t(K_n) = n - 1$.
- (d) For $n \geq 3$, $Z(K_{1,n-1}) = n - 2$ and $F_t(K_{1,n-1}) = n - 1$.

By Observation 3, for $n \geq 2$, $Z(P_n) = 1$ and $F_t(P_n) = 2$. Every minimum zero forcing set of a path consists of a leaf of the path. In particular for $n \geq 3$, no internal vertex of a path P_n belongs to a minimum zero forcing set of the path. Thus, P_n is ZF-dense if and only if $n \in \{1, 2\}$. For $n \geq 2$, coloring any two consecutive vertices on the path P_n produces a minimum TF-set in the path. Thus, P_n is TF-dense if and only if $n \geq 2$. By Observation 3 for $n \geq 3$, $Z(C_n) = F_t(C_n) = 2$. Coloring any two consecutive vertices of the cycle C_n results in both a minimum zero forcing set and a minimum TF-set. Thus, C_n is both ZF-dense and TF-dense.

By Observation 3 for $n \geq 3$, $Z(K_n) = F_t(K_n) = n - 1$. Coloring any arbitrary $(n - 1)$ -element subset of vertices in K_n results in both a minimum forcing set and a minimum TF-set. Thus, K_n is both ZF-dense and TF-dense.

By Observation 3 for $n \geq 3$, $Z(K_{1,n-1}) = n - 2$. Every minimum forcing set of the star $K_{1,n-1}$ consists of $n - 2$ leaves of the star. Since the central vertex of such a star does not belong to a minimum forcing set of the star, $K_{1,n-1}$ is not ZF-dense. By Observation 3 for $n \geq 3$, $F_t(K_{1,n-1}) = n - 1$. Coloring the central vertex of $K_{1,n-1}$ and any combination of $n - 2$ leaves of the star results in a minimum TF-set. Thus, $K_{1,n-1}$ is TF-dense. We state the above observations formally as follows.

Observation 4. *The following holds.*

- (a) *All paths of order at least 3 are TF-dense, but not ZF-dense.*
- (b) *All cycles are ZTF-dense.*
- (c) *All complete graphs on at least three vertices are ZTF-dense.*
- (d) *All stars of order at least 3 are TF-dense, but not ZF-dense.*

3. FAMILIES OF ZERO AND TOTAL FORCING DENSE GRAPHS

In this section, we present several families of ZTF-dense graphs, such as certain multipartite graphs, wheel graphs, and the n -dimensional hypercube. If we exclude stars, then all multipartite graphs of order at least 3 are ZTF-dense, as the following result shows.

Proposition 5. *If G is a complete multipartite graph of order $n \geq 3$ that is not a star, then G is ZTF-dense. Moreover, if $G \neq K_n$, then $Z(G) = F_t(G) = n - 2$.*

Proof. Let G be a complete multipartite graph of order $n \geq 3$ that is not a star. Thus, G is a complete k -partite graph for some $k \geq 2$. Let V_1, V_2, \dots, V_k be the partite sets of G , where $1 \leq |V_1| \leq |V_2| \leq \dots \leq |V_k|$. If $|V_k| = 1$, then $n = k$ and $G = K_k$, and by Observation 4, G is ZTF-dense. Further, by Observation 3, $Z(K_n) = F_t(K_n) = n - 1$. Hence, we may assume that $|V_k| \geq 2$. Since G is not a star, either $k = 2$ and $|V_1| \geq 2$ or $k \geq 3$.

We show firstly that $Z(G) \geq n - 2$. Suppose, to the contrary, that $Z(G) \leq n - 3$. Let S be a minimum forcing set of G , and so $|S| = Z(G) \leq n - 3$. If some partite set of G contains two or more S -uncolored vertices, then it would not be possible to color these vertices. Hence, every partite set of G contains at most one S -uncolored vertex. This implies that there are three distinct partite sets of G , each of which contains an S -uncolored vertex. But then every vertex of G has at least two S -uncolored neighbors, implying that S is not a forcing set of G , a contradiction. Therefore, $Z(G) \geq n - 2$.

We show next that $Z(G) = F_t(G) = n - 2$. If u is an arbitrary vertex in V_k and v is an arbitrary vertex in $V(G) \setminus V_k$, then the set $S = V(G) \setminus \{u, v\}$ is a TF-set of G , implying that $n - 2 \leq Z(G) \leq F_t(G) \leq |S| = n - 2$. Consequently, $Z(G) = F_t(G) = n - 2$. Moreover if w is an arbitrary vertex of G , then we can choose the vertices u and v distinct from w , and therefore we can choose the set S to contain the vertex w . Thus, every vertex of G belongs to some minimum forcing set of G and to some minimum TF-set of G . Hence, G is ZTF-dense. ■

The *wheel graph* W_n of order $n \geq 4$ is the graph obtained from a cycle C_{n-1} by adding a new vertex and joining it to every vertex on the cycle. We call the cycle C_{n-1} the *outer cycle* of the wheel, and we call the new added vertex

adjacent to every vertex on the outer cycle the *central vertex* of the wheel. We show next that all wheel graphs are ZTF-dense.

Proposition 6. *If G is a wheel graph of order $n \geq 4$, then G is ZTF-dense. Moreover, $Z(G) = F_t(G) = 3$.*

Proof. Let G be the wheel graph W_n of order $n \geq 4$ obtained from a cycle $v_1v_2 \cdots v_{n-1}v_1$ by adding a new vertex v and joining it to every vertex on the cycle. Thus, v has degree $n-1$ in G , while every vertex of the wheel different from v has degree 3 in G . By Observations 1 and 2, we have $F_t(G) \geq Z(G) \geq \delta(G) = 3$.

We show next that $F_t(G) \leq 3$. Let S consist of the central vertex v of the wheel together with any two consecutive vertices on the outer cycle. Renaming vertices if necessary, we may assume that $S = \{v, v_{n-1}, v_1\}$. We note that $G[S] \cong K_3$, and so S induces a graph without isolated vertices. Further, the set S is a forcing set of G since if $x_i = v_i$ for $i \in [n-3]$, then the sequence x_1, \dots, x_{n-3} of played vertices in the forcing process results in all vertices of G colored, where x_i denotes the forcing vertex played in the i th step of the forcing process. More precisely, when the vertex x_i is played in the forcing process, it forces the vertex v_{i+1} to be colored for $i \in [n-3]$. Therefore, S is a TF-set of G , and so $F_t(G) \leq |S| = 3$. Consequently, $F_t(G) = Z(G) = 3$. Further, since we can choose the set S to contain the central vertex v of the wheel together with any two consecutive vertices on the outer cycle, every vertex of G belongs to a minimum forcing set and a minimum TF-set of G . Thus, the wheel graph G of order $n \geq 4$ is ZTF-dense. ■

We next consider the n -dimensional hypercube Q_n . From our perspective it is important that Q_n can be represented as the n^{th} power of K_2 with respect to the Cartesian product operation \square , that is, $Q_1 = K_2$ and $Q_n = Q_{n-1} \square K_2$ for $n \geq 2$. The 4-dimensional hypercube Q_4 is illustrated in Figure 1, where the darkened vertices form a TF-set of Q_4 . Peters [22] determined the forcing number of the hypercube Q_n , and showed that $Z(Q_n) = 2^{n-1}$. We show next that all hypercubes Q_n are ZTF-dense.

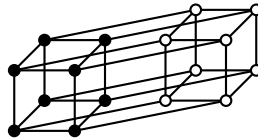


Figure 1. Total forcing (and zero forcing) in the hypercube Q_4 .

Proposition 7. *For $n \geq 3$, if G is the n -dimensional hypercube Q_n , then G is ZTF-dense. Moreover, $Z(G) = F_t(G) = 2^{n-1}$.*

Proof. For $n \geq 3$, let G be the n -dimensional hypercube Q_n . Thus, $G = Q_{n-1} \square K_2$. By Peters's result (see [22]), $Z(Q_n) = 2^{n-1}$. Let S be the set of initially colored vertices obtained by coloring one of the copies of Q_{n-1} in the product $Q_{n-1} \square K_2$. We note that $G[S] \cong Q_{n-1}$, and so since $n \geq 3$, the set S induces a subgraph without isolated vertices. Each vertex in S is adjacent with exactly one vertex outside of S , and thus, each S -colored vertex is S -forcing. Allowing these vertices to force results in all of $V(Q_n)$ becoming colored. Thus, S is a TF-set with $|S| = 2^{n-1}$. Hence, $Z(G) \leq F_t(G) \leq 2^{n-1} = Z(Q_n)$. Consequently, we must have equality throughout this inequality chain. In particular, $Z(G) = F_t(G) = 2^{n-1}$. Further, since we can choose the set S to contain the vertices from either copy of Q_{n-1} in G , every vertex of G belongs to a minimum forcing set and a minimum TF-set of G . Thus, the hypercube Q_n is ZTF-dense for all $n \geq 3$. ■

We remark that the ZTF-dense graphs described in the statements of Propositions 5, 6, and 7 all have equal forcing number and total forcing numbers. However in general, if G is a ZTF-dense graph, then it is possible that $Z(G) < F_t(G)$. We illustrate this with the family of cubic graphs known as diamond-necklaces. Following the notation in [17], for $k \geq 2$ an integer, let N_k be the connected cubic graph constructed as follows. Take k disjoint copies D_1, D_2, \dots, D_k of a diamond, where $V(D_i) = \{a_i, b_i, c_i, d_i\}$ and where $a_i b_i$ is the missing edge in D_i . Let N_k be obtained from the disjoint union of these k diamonds by adding the edges $\{b_i a_{i+1} \mid i \in [k-1]\}$ and adding the edge $b_k a_1$. Let $A = \{a_1, a_2, \dots, a_k\}$, $B = \{b_1, b_2, \dots, b_k\}$, $C = \{c_1, c_2, \dots, c_k\}$ and $D = \{d_1, d_2, \dots, d_k\}$. We call N_k a *diamond-necklace with k diamonds*. Let $\mathcal{N}_{\text{cubic}} = \{N_k \mid k \geq 2\}$. A diamond-necklace, N_6 , with six diamonds is illustrated in Figure 2, where the darkened vertices form a TF-set in the graph.

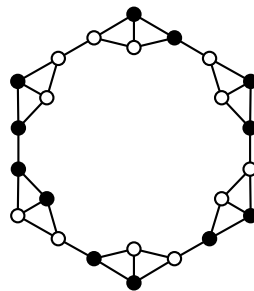


Figure 2. A diamond-necklace N_6 .

The authors in [9] determined the forcing and total forcing numbers of a diamond-necklace

Proposition 8. If $G \in \mathcal{N}_{\text{cubic}}$ has order n , then $F_t(G) = \frac{1}{2}n$ and $Z(G) = \frac{1}{4}n + 2$.

We show next that all diamond-necklaces are ZTF-dense. In the proof of the following result, we adopt our earlier notation for a diamond-necklace. This result shows that there exist ZTF-dense graphs G with the difference $F_t(G) - Z(G)$ arbitrarily large.

Proposition 9. *If $G \in \mathcal{N}_{cubic}$, then G is ZTF-dense.*

Proof. Let $G \in \mathcal{N}_{cubic}$ have order n . Thus, $G = N_k$ is a diamond-necklace with k diamonds for some $k \geq 2$, where $n = 4k$. By Proposition 8, $F_t(G) = 2k$. As observed in [9], the set $S_1 = (A \setminus \{a_k\}) \cup C \cup \{b_k\}$ is a TF-set of G since the sequence x_1, x_2, \dots, x_{2k} of played vertices in the forcing process result in all vertices of G colored, where x_i denotes the forcing vertex played in the i th step of the process and where $x_{2i-1} = a_i$ and $x_{2i} = d_i$ for $i \in [k-1]$ and where $x_{2k-1} = b_{k-1}$ and $x_{2k} = a_k$; that is, the sequence of played vertices is given by $a_1, d_1, a_2, d_2, \dots, a_{k-1}, d_{k-1}, b_{k-1}, a_k$. Since $|S_1| = 2k = F_t(G)$, the set S_1 is a minimum TF-set of G . Analogously, for all $i \in [k]$ the set $S_i = (A \setminus \{a_i\}) \cup C \cup \{b_i\}$ is a minimum TF-set of G , as is the set $S'_i = (A \setminus \{a_i\}) \cup D \cup \{b_i\}$. Thus, every vertex of G belongs to some minimum TF-set of G . Hence, G is TF-dense.

Let $G \in \mathcal{N}_{cubic}$ have order n . Thus, $G = N_k$ is a diamond-necklace with k diamonds for some $k \geq 2$, where $n = 4k$. By Proposition 8, $F_t(G) = 2k$. As observed in [9], the set $S_1 = (A \setminus \{a_1\}) \cup C \cup \{b_1\}$ is a TF-set of G since the sequence x_1, x_2, \dots, x_{2k} of played vertices in the forcing process result in all vertices of G colored, where x_i denotes the forcing vertex played in the i th step of the process and where $x_1 = b_1$, $x_2 = d_1$, and $x_{2i+1} = a_{i+1}$ and $x_{2i+2} = d_{i+1}$ for $i \in [k-1]$; that is, the sequence of played vertices is given by $b_1, d_1, a_2, d_2, \dots, a_k, b_k$. Since $|S_1| = 2k = F_t(G)$, the set S_1 is a minimum TF-set of G . Analogously, for all $i \in [k]$ the set $S_i = (A \setminus \{a_i\}) \cup C \cup \{b_i\}$ is a minimum TF-set of G , as is the set $S'_i = (A \setminus \{a_i\}) \cup D \cup \{b_i\}$. Thus, every vertex of G belongs to some minimum TF-set of G . Hence, G is TF-dense.

We show next that G is ZF-dense. By Proposition 8, $Z(G) = k + 2$. As observed in [9], the set $D_1 = C \cup \{b_1, a_2\}$ is a forcing set of G since the sequence $x_1, x_2, \dots, x_{3k-2}$ of played vertices in the forcing process result in all vertices of G colored, where x_i denotes the forcing vertex played in the i th step of the process and where $x_{3i-2} = a_{i+1}$, $x_{3i-1} = d_{i+1}$, and $x_{3i} = b_{i+1}$ for $i \in [k-1]$, and where $x_{3k-2} = a_1$; that is, the sequence of played vertices is given by $a_2, d_2, b_2, a_3, d_3, b_3, \dots, a_k, d_k, b_k, a_1$. Since $|D_1| = k+2 = Z(G)$, the set D_1 is a minimum forcing set of G . Analogously, for all $i \in [k]$ the set $D_i = C \cup \{b_i, a_{i+1}\}$ is a minimum forcing set of G , as is the set $D'_i = D \cup \{b_i, a_{i+1}\}$, where $a_{k+1} = a_1$. Thus, every vertex of G belongs to some minimum forcing set of G . Hence, G is ZF-dense. As observed earlier, G is TF-dense. Therefore, G is ZTF-dense. ■

4. ZTF-DENSE TREES

Recall that by Observation 4, no path of order at least 3 and no star of order at least 3 is ZF-dense. In this section, we show that the only ZF-dense trees are K_1 and K_2 . For this purpose, we first prove the following lemma.

Lemma 10. *If T is a ZF-dense tree, then the following holds.*

- (a) T has no strong support vertex.
- (b) There is no path $P: v_1v_2v_3v_4$ in T where $d_T(v_i) = d_P(v_i)$ for $i \in \{1, 2, 4\}$.
- (c) There is no path $P: v_1v_2v_3v_4v_5$ in T where $d_T(v_i) = d_P(v_i)$ for $i \in \{1, 2, 4, 5\}$.
- (d) If $v_1v_2v_3$ is a path in T where $d_T(v_1) = 1$ and $d_T(v_2) = 2$, then $d_T(v_3) \geq 3$.

Proof. Let T be a ZF-dense tree. For an arbitrary vertex v of T , let S_v be a minimum zero forcing set of T that contains the vertex v . Thus, $|S_v| = Z(T)$ and $v \in S_v$. Further, let $S'_v = S_v \setminus \{v\}$.

(a) Suppose, to the contrary, that T has a strong support vertex v . Let v_1 and v_2 be two distinct leaf neighbors of v . Since S_v is a zero forcing set, at least one of v_1 and v_2 belong to the set S_v . Renaming v_1 and v_2 , if necessary, we may assume that $v_1 \in S_v$. But then the set S'_v is a zero forcing set, noting that as the first vertex played in the forcing process starting with the set S'_v we play the vertex v_1 , which results in the set S_v of colored vertices after the first step. Thereafter, we continue with the exact same sequence of vertices in the forcing process starting with the zero forcing set S_v to color the remaining uncolored vertices of T . Thus contradicts the minimality of the zero forcing set S_v .

(b) Suppose, to the contrary, that $P: v_1v_2v_3v_4$ is a path in T where $d_T(v_1) = d_T(v_4) = 1$ and $d_T(v_2) = 2$. We note that $d_T(v_3) \geq 2 = d_P(v_3)$. Let $v = v_3$ and consider the minimum zero forcing set S_v of T that contains the vertex v . Since S_v is a zero forcing set, at least one of v_1 , v_2 and v_4 belong to the set S_v . If $v_4 \in S_v$, then analogously as in the proof of part (a), the set S'_v is a zero forcing set, noting that as the first vertex played in the forcing process starting with the set S'_v we play the vertex v_4 . If $v_2 \in S_v$, then the set $S''_v = (S'_v \setminus \{v_2\}) \cup \{v_1\}$ is a zero forcing set, noting that as the first vertex played in the forcing process starting with the set S''_v we play the vertex v_1 . If $v_1 \in S_v$, then the set S'_v is a zero forcing set, noting that as the first vertex played in the forcing process starting with the set S'_v we play the vertex v_1 . In all three cases, we produce a zero forcing set of T of cardinality strictly less than $|S_v|$, contradicting the minimality of the zero forcing set S_v .

(c) Suppose, to the contrary, that $P: v_1v_2v_3v_4v_5$ is a path in T where $d_T(v_1) = d_T(v_5) = 1$ and $d_T(v_2) = d_T(v_4) = 2$. We note that $d_T(v_3) \geq 2 = d_P(v_3)$. Let $v = v_3$ and consider the minimum zero forcing set S_v of T . Since S_v is a zero forcing set, at least one of v_1 , v_2 , v_4 and v_5 belong to the set S_v . If $v_1 \in S_v$, then the set S'_v is a zero forcing set, noting that as the first vertex played in the forcing

process starting with the set S'_v we play the vertex v_1 . Analogously, if $v_5 \in S_v$, then the set S'_v is a zero forcing set. If $v_2 \in S_v$, then the set $S''_v = (S'_v \setminus \{v_2\}) \cup \{v_1\}$ is a zero forcing set, noting that as the first vertex played in the forcing process starting with the set S''_v we play the vertex v_1 . Analogously, if $v_4 \in S_v$, then the set $(S'_v \setminus \{v_4\}) \cup \{v_5\}$ is a zero forcing set. In all four cases, we produce a zero forcing set of T of cardinality strictly less than $|S_v|$, contradicting the minimality of the zero forcing set S_v .

(d) Let $v_1v_2v_3$ be a path in T where $d_T(v_1) = 1$ and $d_T(v_2) = 2$, and suppose, to the contrary, that $d_T(v_3) \leq 2$. By Observation 4, the tree T is not a path, since no path of order at least 3 is ZF-dense. This implies that $d_T(v_3) = 2$ and there exists a vertex of degree at least 3 in T . Let w be the vertex of degree at least 3 in T that is at minimum distance from v_1 in T . Further, let $d_T(v_1, w) = k$, and so $k \geq 3$ and let $Q: v_1v_2v_3 \cdots v_{k+1}$ be the path from v_1 to w in T , where $w = v_{k+1}$. We now consider the vertex $v = v_2$. If $v_i \in S_v$ for some $i \in [k+1] \setminus \{2\}$, then the set $S''_v = (S'_v \setminus \{v_i\}) \cup \{v_1\}$ is a zero forcing set, noting that as the first vertex played in the forcing process starting with the set S''_v we play the vertex v_1 . Thus, $v = v_2$ is the only vertex of Q that belongs to the set S_v . This implies that the set $S'_v \subseteq V(T) \setminus V(Q)$ is necessarily a zero forcing set of T , contradicting the minimality of the set S_v . This completes the proof of Lemma 10. ■

We are now in a position to prove that the only ZF-dense trees are K_1 and K_2 .

Theorem 11. *The only ZF-dense trees are K_1 and K_2 .*

Proof. Let T be a ZF-dense tree. We show firstly that T is a path. Suppose, to the contrary, that T is not a path. Let $P: v_1v_2v_3 \cdots v_d$ be a longest path in T , where $d \geq 3$. Necessarily, v_1 is a leaf in T . If $d = 3$, then T is a star, contradicting Observation 4. Hence, $d \geq 4$. If $d_T(v_2) \geq 3$, then by the maximality of the path P every neighbor of v_2 different from v_3 is a leaf, implying that v_2 is a strong support vertex, contradicting Lemma 10(a). Thus, $d_T(v_2) = 2$. By Lemma 10(d), $d_T(v_3) \geq 3$. Let u_2 be a neighbor of v_3 not on P . By Lemma 10(b), u_2 is not a leaf, and so $d_T(u_2) \geq 2$. By the maximality of the path P , every neighbor of u_2 different from v_3 is a leaf. Thus if $d_T(u_2) \geq 3$, then u_2 is a strong support vertex of T , contradicting Lemma 10(a). Hence, $d_T(u_2) = 2$. Let u_1 be the neighbor of u_2 different from v_3 . Thus, u_1 is a leaf of T . Hence, $u_1u_2v_3v_2v_1$ is a path in T , where u_1 and v_1 are leaves of T and u_2 and v_2 are support vertices of T of degree 2, contradicting Lemma 10(c). Therefore, T is a path. As observed earlier, a path P_n is ZF-dense if and only if $n \in \{1, 2\}$. Hence, $T \in \{K_1, K_2\}$. ■

Recall that a *nontrivial graph* is a graph of order at least 2. As an immediate consequence of Observation 4(a) and Theorem 11, the only non-trivial tree that is ZTF-dense is the tree K_2 .

5. THE JOIN OF TWO GRAPHS

In this section we study ZF-dense and ZTF-dense graphs under the join operation. We remark that Theorem 12(a) appears in the Ph.D. Dissertation of Taklimi [24]. Acknowledging this result, we also present a proof of this result as it helps clarify the proof of Theorem 12(b).

Theorem 12. *If G and H are non-trivial connected graphs, then the following holds.*

- (a) $Z(G + H) = \min\{n(G) + Z(H), n(H) + Z(G)\}$.
- (b) *If both G and H are ZF-dense, then $G + H$ is ZF-dense.*

Proof. Let S be an arbitrary minimum zero forcing set of $G + H$. Starting with the colored set S , let x_1, \dots, x_t be the sequence of played vertices in the forcing process that results in all vertices of $G + H$ colored, where x_i denotes the forcing vertex played in the i th step of the process. Let X be the resulting of played vertices, and so $X = \{x_1, \dots, x_t\}$. We proceed further by establishing some properties of the set X .

Claim 13. *We can choose the set X so that $X \subseteq V(G)$ or $X \subseteq V(H)$.*

Proof. Suppose that X contains vertices from both G and H . Renaming G and H if necessary, we may assume that $x_1 \in V(G)$. Let x_i be the vertex in X with smallest subscript that belongs to H ; that is, $x_i \in V(H)$ and $i \in [t] \setminus \{1\}$. By definition, the first vertex x_1 played in the sequence of played vertices has exactly one uncolored neighbor immediately before it is played. This implies that all vertices of H , except possibly for one vertex, are S -colored (and therefore belong to the set S). After the vertex x_1 is played, all vertices of H are colored. When the vertex $x_i \in V(H)$ is played in the i th step of the forcing process, all vertices of G , except for exactly one vertex, say v , are colored. This implies that after the vertex x_i is played, all vertices of $G + H$ are colored. Thus, x_i is the final vertex played, and so $x_i = x_t$. By our choice of the index i , we note therefore that x_t is the only vertex in X that belongs to H . However, replacing x_t in X with an arbitrary neighbor, v' say, of v in G produces a new sequence x_1, \dots, x_{t-1}, v' of played vertices in the forcing process that results in all vertices of $G + H$ colored. The resulting set $(X \setminus \{x_t\}) \cup \{v'\}$ of played vertices in this sequence belong entirely to the graph G . \square

Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$.

Claim 14. $Z(G + H) \geq \min\{n(G) + Z(H), n(H) + Z(G)\}$.

Proof. By Claim 13, we can choose the set X so that $X \subseteq V(G)$ or $X \subseteq V(H)$. Renaming the graphs G and H if necessary, we may assume that $X \subseteq V(G)$. As

observed in the proof of Claim 13, immediately before the first vertex $x_1 \in X$ is played, all vertices of H , except possibly for one vertex, belong to the set S . Thus, $S_H = V(H)$ or $S_H = V(H) \setminus \{v\}$ for some vertex $v \in V(H)$.

Suppose that all $S_H = V(H)$. In this case, the set S_G is a zero forcing set of G and x_1, \dots, x_t is a sequence of played vertices in the forcing process in the graph G that results in all vertices of G colored. Thus, $Z(G) \leq |S_G|$, and so $Z(G+H) = |S| = |S_G| + |S_H| \geq Z(G) + n(H) \geq \min\{n(G) + Z(H), n(H) + Z(G)\}$. Hence, we may assume that $S_H = V(H) \setminus \{v\}$ for some vertex $v \in V(H)$, for otherwise the desired result hold.

With this assumption, immediately before the first vertex x_1 is played in the forcing sequence in $G + H$, all neighbors of x_1 in $G + H$ are colored, except for the vertex $v \in V(H)$ which becomes colored when the vertex x_1 is played. In particular, we note that in this case all neighbors of x_1 in G belong to the set S ; that is, $N_G(x_1) \subseteq S_G$.

We now consider the set $S' = S_G \setminus \{x_1\}$. If S' is a zero forcing set in G , then $Z(G) \leq |S'| = |S_G| - 1$. Thus, $Z(G + H) = |S| = |S_G| + |S_H| = (|S'| + 1) + (n(H) - 1) = |S'| + n(H) \geq Z(G) + n(H) \geq \min\{n(G) + Z(H), n(H) + Z(G)\}$. Hence we may assume that S' is not a zero forcing set in G . This implies that at least one vertex of X is a neighbor of x_1 in G . Let x_j be a vertex of X of smallest subscript such that x_j is a neighbor of x_1 in G ; that is, $x_j \in N_G(x_1)$ and $j \in [t] \setminus \{1\}$.

We now consider the set $S^* = S_G \setminus \{x_j\}$. We show that S^* is a zero forcing set in G by showing that the sequence of vertices x_1, \dots, x_t in the forcing process colors all vertices of G . When we play the vertex x_1 , its unique S^* -uncolored neighbor, namely the vertex x_j , becomes colored. After the vertex x_1 is played, the set of colored vertices in G with respect to the coloring S^* is exactly the same as the set of colored vertices in G with respect to the coloring S_G . We note that after the vertex x_1 is played in $G + H$ all vertices in H are colored, implying that each vertex x_i where $i \in [t] \setminus \{1\}$ forces a unique vertex in G to be colored. These observations imply that once the vertex x_1 is played in the set S^* , we may proceed exactly as before by playing the vertices x_2, \dots, x_t in turn in the forcing process to color all vertices of G . Thus, S^* is a zero forcing set in G , and so $Z(G) \leq |S^*| = |S_G| - 1$. Therefore, $Z(G + H) = |S| = |S_G| + |S_H| = (|S^*| + 1) + (n(H) - 1) = |S^*| + n(H) \geq Z(G) + n(H) \geq \min\{n(G) + Z(H), n(H) + Z(G)\}$. This completes the proof of Claim 14. \square

Claim 15. $Z(G + H) \leq \min\{n(G) + Z(H), n(H) + Z(G)\}$.

Proof. Coloring all vertices in G and coloring the vertices in a minimum zero forcing set in H we form a zero forcing set in $G + H$. Analogously, coloring all vertices in H and coloring the vertices in a minimum zero forcing set in G we form a zero forcing set in $G + H$. Thus, $Z(G + H) \leq \min\{n(G) + Z(H), n(H) + Z(G)\}$. \square

As an immediate consequence of Claims 14 and 15, the zero forcing number of the join $G + H$ of G and H is determined, and is given by

$$Z(G + H) = \min\{n(G) + Z(H), n(H) + Z(G)\}.$$

This completes the proof of part (a). To prove part (b), suppose that both G and H are ZF-dense. Renaming G and H if necessary, we may assume that $n(H) + Z(G) = \min\{n(G) + Z(H), n(H) + Z(G)\}$. Coloring all vertices in H and coloring the vertices in a minimum zero forcing set in G we form a minimum zero forcing set in $G + H$ of cardinality $n(H) + Z(G)$. Since G is ZF-dense, we can choose the minimum zero forcing set in G to contain any specified vertex in G . Therefore, there exists a minimum zero forcing set in $G + H$ that contains any specified vertex in $G + H$, implying that $G + H$ is ZF-dense. This completes the proof of part (b), and completes the proof of Theorem 12. ■

We close with the following result.

Theorem 16. *If G and H are non-trivial connected graphs, then the following holds.*

- (a) $F_t(G + H) = \min\{n(G) + Z(H), n(H) + Z(G)\}$.
- (b) *If both G and H are ZF-dense, then $G + H$ is TF-dense.*

Proof. In view of Theorem 12, we may assume, renaming the graphs G and H if necessary, that $Z(G + H) = n(G) + Z(H)$. Notice that $V(G) \cup S_H$ is a zero forcing set of $G + H$, where S_H denotes a minimum zero forcing set of H . By definition of the join operation, the set $V(G) \cup S_H$ induces a subgraph of $G + H$ without isolated vertices, and is therefore a TF-set of $G + H$. Hence, $F_t(G + H) \leq Z(G + H)$. By Observation 1, $F_t(G + H) \leq Z(G + H)$. Consequently, $F_t(G + H) = Z(G + H) = \min\{n(G) + Z(H), n(H) + Z(G)\}$. This proves part (a). Part (b) follows readily. ■

As an immediate consequence of Theorems 12 and 16, we have the following result.

Corollary 17. *Let G and H be non-trivial connected graphs. If both G and H are ZF-dense, then $G + H$ is ZTF-dense.*

6. CONCLUDING REMARKS

In this paper, several classes of ZTF-dense graphs are given, such as complete multipartite graphs, wheels, n -dimensional hypercubes, diamond-necklaces, as well as constructions to build ZF-dense graphs. It would be interesting to obtain other classes of ZTF-dense or ZF-dense graphs.

As mentioned in the introductory section, vertices that belong to every or to some minimum dominating set of a graph are well studied, and such sets are often useful in obtaining algorithmic and complexity results, as well as bounds on domination type parameters. Since there is a connection between domination type parameters and zero forcing, it would be interesting to obtain algorithmic and complexity results, and to determine upper bounds on the zero forcing number in classes of ZTF-dense or ZF-dense or TF-dense graphs.

For $r \geq 3$, let \mathcal{G}_r be the class of connected r -regular graphs. If $G \in \mathcal{G}_r$ is a ZF-dense graph of sufficiently large order, determine or estimate the best possible constant a_r such that $Z(G) \leq a_r \times n(G)$, and if $G \in \mathcal{G}_r$ is a TF-dense graph of sufficiently large order, determine or estimate the best possible constant b_r such that $F_t(G) \leq b_r \times n(G)$. These constants are given by

$$a_r = \sup_{G \in \mathcal{G}_r} \frac{Z(G)}{n(G)} \quad \text{and} \quad b_r = \sup_{G \in \mathcal{G}_r} \frac{F_t(G)}{n(G)}.$$

As shown in Propositions 8 and 9 we know that $a_3 \geq \frac{1}{4}$ and $b_3 \geq \frac{1}{2}$. It would be interesting to determine the exact values of a_3 and b_3 .

REFERENCES

- [1] A. Aazami, Hardness Results and Approximation Algorithms for Some Problems in Graphs, Ph.D Thesis (University of Waterloo, 2008).
- [2] A. Aazami, *Domination in graphs with bounded propagation: algorithms, formulations and hardness results*, J. Comb. Optim. **19** (2010) 429–456.
<https://doi.org/10.1007/s10878-008-9176-7>
- [3] AIM Minimum Rank-Special Graphs Work Group, *Zero forcing sets and the minimum rank of graphs*, Linear Algebra Appl. **428** (2008) 1628–1648.
<https://doi.org/10.1016/j.laa.2007.10.009>
- [4] M. Alishahi, E. Rezaei-Sani and E. Sharifi, *Maximum nullity and zero forcing number on graphs with maximum degree at most three*, Discrete Appl. Math. **284** (2020) 179–194.
<https://doi.org/10.1016/j.dam.2020.03.027>
- [5] K.F. Benson, D. Ferrero, M. Flagg, V. Furst, L. Hogben, V. Vasilevska and B. Wissman, *Zero forcing and power domination for graph products*, Australas. J. Combin. **70** (2018) 221–235.
- [6] E.J. Cockayne, M.A. Henning and C.M. Mynhardt, *Vertices contained in all or in no minimum total dominating set of a tree*, Discrete Math. **260** (2003) 37–44.
[https://doi.org/10.1016/S0012-365X\(02\)00447-8](https://doi.org/10.1016/S0012-365X(02)00447-8)
- [7] R. Davila, Bounding the Forcing Number of a Graph, Masters Thesis (Rice University, 2015).

- [8] R. Davila, Total and Zero Forcing in Graphs, Ph.D Thesis (University of Johannesburg, 2019).
- [9] R. Davila and M.A. Henning, *Total forcing and zero forcing in claw-free cubic graphs*, Graphs Combin. **34** (2018) 1371–1384.
<https://doi.org/10.1007/s00373-018-1934-4>
- [10] R. Davila and M.A. Henning, *On the total forcing number of a graph*, Discrete Appl. Math. **257** (2019) 115–127.
<https://doi.org/10.1016/j.dam.2018.09.001>
- [11] R. Davila and M.A. Henning, *Total forcing versus total domination in cubic graphs*, Appl. Math. Comput. **354** (2019) 385–395.
<https://doi.org/10.1016/j.amc.2019.02.060>
- [12] R. Davila and M.A. Henning, *Zero forcing in claw-free cubic graphs*, Bull. Malays. Math. Sci. Soc. **43** (2020) 673–688.
<https://doi.org/10.1007/s40840-018-00705-5>
- [13] R. Davila and M.A. Henning, *Zero forcing versus domination in cubic graphs*, manuscript.
- [14] R. Davila, M.A. Henning, C. Magnant and R. Pepper, *Bounds on the connected forcing number of a graph*, Graphs Combin. **34** (2018) 1159–1174.
<https://doi.org/10.1007/s00373-018-1957-x>
- [15] S.M. Fallat, L. Hogben, J. C.-H. Lin and B. Shader, *The inverse eigenvalue problem of a graph, zero forcing, and related parameters*, Notices Amer. Math. Soc. **67** (2020) 257–261.
- [16] M. Fürst and D. Rautenbach, *A short proof for a lower bound on the zero forcing number*, Discuss. Math. Graph Theory **40** (2020) 355–360.
<https://doi.org/10.7151/dmgt.2117>
- [17] M.A. Henning and C. Löwenstein, *Locating-total domination in claw-free cubic graphs*, Discrete Math. **312** (2012) 3107–3116.
<https://doi.org/10.1016/j.disc.2012.06.024>
- [18] M.A. Henning and M.D. Plummer, *Vertices contained in all or in no minimum paired-dominating set of a tree*, J. Comb. Optim. **10** (2005) 283–294.
<https://doi.org/10.1007/s10878-005-4107-3>
- [19] M.A. Henning and A. Yeo, Total Domination in Graphs (Springer Monographs in Mathematics, 2013).
<https://doi.org/10.1007/978-1-4614-6525-6>
- [20] S. Li and W. Sun, *On the zero forcing number of a graph involving some classical parameters*, J. Comb. Optim. **39** (2020) 365–384.
<https://doi.org/10.1007/s10878-019-00475-1>
- [21] C.M. Mynhardt, *Vertices contained in every minimum dominating set of a tree*, J. Graph Theory **31** (1999) 163–177.
[https://doi.org/10.1002/\(SICI\)1097-0118\(199907\)31:3<163::AID-JGT2>3.0.CO;2-T](https://doi.org/10.1002/(SICI)1097-0118(199907)31:3<163::AID-JGT2>3.0.CO;2-T)

- [22] T. Peters, *Positive semidefinite maximum nullity and zero forcing number*, Electron. J. Linear Algebra **23** (2012) 815–830.
<https://doi.org/10.13001/1081-3810.1559>
- [23] A. Samodivkin, *Roman domination excellent graphs: trees*, Commun. Comb. Optim. **3** (2018) 1–24.
- [24] F.A. Taklimi, *Zero Forcing Sets for Graphs*, Ph.D Thesis (University of Regina, 2013).

Received 20 March 2020
Revised 9 December 2020
Accepted 12 December 2020