Discussiones Mathematicae
Graph Theory 43 (2023) 587–606
https://doi.org/10.7151/dmgt.2387

TOTAL 2-DOMINATION NUMBER IN DIGRAPHS AND ITS DUAL PARAMETER

Doost Ali Mojdeh[1]

Department of Mathematics
University of Mazandaran
Babolsar, Iran

e-mail: damojdeh@umz.ac.ir

AND

Babak Samadi

Babolsar, Iran

e-mail: samadibabak62@gmail.com

Abstract

A subset S of vertices of a digraph D is a total 2-dominating set if every vertex not in S is adjacent from at least two vertices in S, and the subdigraph induced by S has no isolated vertices. Let D^{-1} be a digraph obtained by reversing the direction of every arc of D.

In this work, we investigate this concept which can be considered as an extension of double domination in graphs G to digraphs D, along with total 2-limited packing ($L_2^t(D)$) of digraphs D which has close relationships with the above-mentioned concept. We prove that the problems of computing these parameters are NP-hard, even when the digraph is bipartite. We also give several lower and upper bounds on them. In dealing with these two parameters our main emphasis is on directed trees, by which we prove that $L_2^t(D) + L_2^t(D^{-1})$ can be bounded from above by $16n/9$ for any digraph D of order n. Also, we bound the total 2-domination number of a directed tree from below and characterize the directed trees attaining the bound.

Keywords: total 2-domination number, total 2-limited packing number, directed tree.

2020 Mathematics Subject Classification: 05C69, 05C20.

[1]Corresponding author.

1. Introduction and Preliminaries

Throughout this paper, we consider $D = (V(D), A(D))$ as a finite digraph with vertex set $V = V(D)$ and arc set $A = A(D)$ with neither loops nor multiple arcs (although pairs of opposite arcs are allowed). Also, $G = (V(G), E(G))$ stands for a simple finite graph with vertex set $V(G)$ and edge set $E(G)$. We use [3] and [26] as references for basic terminology and notation in digraphs and graphs, respectively, which are not defined here.

For any two vertices $u, v \in V(D)$, we write (u, v) as the *arc* with direction from u to v, and say u is *adjacent to* v, or v is *adjacent from* u. We also say u and v are *adjacent with* each other. Given a subset S of vertices of D and a vertex $v \in V(D)$, we write $N_S^-(v) = \{u \in S \mid (u, v) \in A(D)\}$ and $N_S^+(v) = \{u \in S \mid (v, u) \in A(D)\}$. The *in-degree* (*out-degree*) of v from (to) S is $\deg_S^-(v) = |N_S^-(v)|$ ($\deg_S^+(v) = |N_S^+(v)|$). Moreover, $N_S^-[v] = N_S^-(v) \cup \{v\}$ and $N_S^+[v] = N_S^+(v) \cup \{v\}$. In particular, if $S = V(D)$, then we simply write $N_D^-(v)$, $N_D^+(v)$, $N_D^-[v]$, $N_D^+[v]$, $\deg_D^-(v)$ and $\deg_D^+(v)$ instead of $N_{V(D)}^-(v)$, $N_{V(D)}^+(v)$, $N_{V(D)}^-[v]$, $N_{V(D)}^+[v]$, $\deg_{V(D)}^-(v)$ and $\deg_{V(D)}^+(v)$, respectively (we moreover remove the subscript D if there is no ambiguity with respect to the digraph D).

For a graph G, $\Delta = \Delta(G)$ and $\delta = \delta(G)$ represent the maximum and minimum degrees of G. In addition, for a digraph D, ($\Delta^+ = \Delta^+(D)$ and $\delta^+ = \delta^+(D)$) $\Delta^- = \Delta^-(D)$ and $\delta^- = \delta^-(D)$ represent the maximum and minimum (out-degrees) in-degrees of D. Given two subsets A and B of vertices of D, by $(A, B)_D$ we mean the sets of arcs of D going from A to B. Given a subset $S \subseteq V(D)$, by $D\langle S\rangle$ we mean the subdigraph of D induced by S. Finally, we let $N[v] = N^-[v] \cup N^+[v]$ for each vertex v of D.

We denote the *converse* of a digraph D by D^{-1}, obtained by reversing the direction of every arc of D. A vertex $v \in V(D)$ with $\deg^+(v) + \deg^-(v) = 1$ is called an *end-vertex*. A *penultimate vertex* is a vertex adjacent with an end-vertex. A digraph D is *connected* if its underlying graph is connected. A *directed tree* is a digraph in which its underlying graph is a tree. A digraph D is *bipartite* if it is obtained from a bipartite graph G by replacing each edge xy of G by either (x, y) or (y, x), or the pair (x, y) and (y, x).

A vertex $v \in V(D)$ ($v \in V(G)$) is said to dominate itself and its out-neighbors (neighbors). A subset $S \subseteq V(D)$ ($S \subseteq V(G)$) is a *dominating set* in D (G) if all vertices are dominated by the vertices in S. The *domination number* $\gamma(D)$ ($\gamma(G)$) is the minimum cardinality of a dominating set in the digraph D (graph G).

Domination and its related topics in graphs have received a lot of attention from a large number of researchers over the last few decades because of their important theoretical aspects a wealth of real-world applications. But the papers on domination in digraphs are much less common than in graphs despite the fact that the number of papers on digraphs has grown significantly over the last ten

years. One reason for this unbalanced situation that exists in the literature could be the fact that the theory of graphs is significantly more developed than the theory of digraphs. However, many domination parameters have been investigated in digraphs. For example, the reader can consult the papers [5, 15, 19], and [6, 7, 21] in recent years. This paper also contributes to decreasing the unbalanced situation.

The concept of domination in digraphs was introduced by Fu [10] while the very well-known same topic in graphs was introduced by Berge [4] and Ore [24]. The reader is referred to [18] for more details on this topic. Ouldrabah *et al.* [25] defined the concept of k-domination in digraphs, as a transformation of the same topic in graphs (see [9]), as follows. A subset S of vertices in a digraph D is a k-dominating set if $|N^-(v) \cap S| \geq k$ for every vertex v in $V(D) \setminus S$. The k-domination number $\gamma_k(D)$ of D is the minimum cardinality of a k-dominating set in D. Clearly, this concept is a generalization of the concept of domination in digraphs.

As digraphs are extensions of graphs (note that a graph can be considered as a symmetric digraph), we can expect that a well-known concept in graph theory can be extended to digraph theory in different ways. For example, Arumugam *et al.* [2] investigated two extensions of the *total domination* (in graphs) to digraphs in two different ways, namely, *open domination* and *total domination* in digraphs.

The *k-tuple domination number* $\gamma_{\times k}(G)$ of a graph G with $\delta(G) \geq k-1$ is the minimum cardinality of a subset $S \subseteq V(G)$ such that $|N[v] \cap S| \geq k$, for each vertex $v \in V(G)$. In particular, the 2-tuple domination number is called a *double domination number*. This concept was first introduced by Harary and Haynes in [17]. Gallant *et al.* [12] introduced the concept of limited packing in graphs as follows. The *k-limited packing number* $L_k(G)$ of a graph G is the maximum cardinality of a subset $B \subseteq V(G)$ such that $|N[v] \cap B| \leq k$, for each vertex $v \in V(G)$. Note that $L_1(G) = \rho(G)$ is the well-known *packing number* in the graph G.

The concept of double domination in graphs can be extended to digraphs in two different ways. One can say a subset $S \subseteq V(D)$ is a *double dominating set* in a digraph D with $\delta^-(D) \geq 1$ if every vertex is dominated by at least two vertices in S. The *double domination number* $\gamma_{\times 2}(D)$ is the minimum cardinality of a double dominating set in D. But such a parameter cannot be defined for some important families of digraphs like acyclic digraphs (the digraphs with no directed cycle), especially directed trees. Indeed, we would rather consider the following definition.

Definition 1. Let D be a digraph with no isolated vertices. A subset $S \subseteq V(D)$ is a *total 2-dominating set* in D if $D\langle S\rangle$ has no isolated vertices and every vertex in $V(D) \setminus S$ is dominated by at least two vertices in S. The *total 2-domination number* $\gamma_2^t(D)$ is the minimum cardinality of a total 2-dominating set in D.

We remark that the definition of total 2-dominating sets is more general than that of double dominating sets, as every double dominating set is a total 2-dominating set (so $\gamma(D) \leq \gamma_2^t(D) \leq \gamma_{\times 2}(D)$ for all digraphs D with $\delta^-(D) \geq 1$).

Regarding the 2-limited packing in graphs, we can extend this concept to digraphs in two different ways. A subset $B \subseteq V(D)$ is a 2-*limited packing* in the digraph D if $|N^+[v] \cap B| \leq 2$, for every vertex $v \in V(D)$. The 2-*limited packing number* $L_2(D)$ is the maximum cardinality of a 2-limited packing in D. In this work, we consider the following definition which can be interpreted as the dual version of Definition 1.

Definition 2. A subset $B \subseteq V(D)$ is a *total* 2-*limited packing* in the digraph D if every vertex in B is adjacent with at most one vertex in B and every vertex in $V(D) \setminus B$ is adjacent to at most two vertices in B. The *total* 2-*limited packing number* $L_2^t(D)$ is the maximum cardinality of a total 2-limited packing in D.

Comparing the last two definitions we can readily observe that $\rho(D) \leq L_2^t(D) \leq L_2(D)$ for all digraphs D, in which $\rho(D)$ is the usual packing number of D (see [21]). Recall that $\rho(D)$ is the maximum cardinality of subset $B \subseteq V(D)$ for which $|N^+[v] \cap B| \leq 1$, for all $v \in V(D)$.

Note that, for various reasons, the small values of k (especially $k \in \{1, 2\}$) regarding the above-mentioned parameters have attracted more attention from the experts in domination theory rather than the large ones. One reason is that for the large values of k, we lose some important families of graphs (for example, the k-tuple domination number cannot be studied for trees when $k \geq 3$), or we deal with a trivial problem (for example, for every graph G with $k > \Delta(G)$, we have $L_k(G) = |V(G)|$). Another reason is that many results for the case $k \in \{1, 2\}$ can be easily generalized to the general case k. Moreover, one may obtain stronger results for the small values of k rather than the large ones. For more evidences on these pieces of information the reader can be referred to [8, 17] and [20].

This paper is organized as follows. We initiate the investigation of the parameters given in Definition 1 and Definition 2. We derive their computational complexity and give some bounds on these parameters in Section 2 and Section 3. We show that the problems given in Definition 1 and Definition 2 are dual problems (on the instances of directed trees) in Section 4. Also, we bound the total 2-domination number of a directed tree from below and characterize the directed trees attaining the bound. In Section 5, with emphasis on directed trees, we prove that $L_2^t(D) + L_2^t(D^{-1})$ can be bounded from above by $16n/9$ for any digraph D of order n.

Given $\eta \in \{\gamma_2^t, L_2^t, \gamma, \rho\}$, by a $\eta(G)$-set in any graph/digraph G, we mean a total 2-dominating set, total 2-limited packing, dominating set or packing in G of cardinality $\eta(G)$, respectively.

2. COMPUTATIONAL COMPLEXITY

We use the well-known NP-complete EXACT 3-COVER problem (see [13]) in the proof of the main theorem of this section.

(1)

EXACT 3-COVER (X3C)
INSTANCE: A finite set X of cardinality $3q$ and a collection C of 3-element subsets of X.
QUESTION: Is there a subcollection C' of C such that every element of X appears in exactly one element of C'?

We now consider the following two well-known decision problems in domination theory.

(2)

DOMINATING SET PROBLEM
INSTANCE: A graph G and a positive integer k.
QUESTION: Is $\gamma(G) \leq k$?

(3)

PACKING PROBLEM
INSTANCE: A graph G and a positive integer k'.
QUESTION: Is $\rho(G) \geq k'$?

We make use of these two problems which are known to be NP-complete from [18] and [14], respectively, in order to study the complexity of the problems introduced in this paper. Note that DOMINATING SET PROBLEM is NP-complete even when restricted to bipartite graphs (see [22]).

The directed counterparts of the problems (2) and (3) can be naturally stated as follows.

(4)

DOMINATING SET PROBLEM for digraphs
INSTANCE: A digraph D and a positive integer k.
QUESTION: Is $\gamma(D) \leq k$?

(5)

PACKING PROBLEM for digraphs
INSTANCE: A digraph D and a positive integer k'.
QUESTION: Is $\rho(D) \geq k'$?

We deal with the following decision problems.

(6)

TOTAL 2-DOMINATING SET PROBLEM
INSTANCE: A digraph D with no isolated vertices and a positive integer j.
QUESTION: Is $\gamma_2^t(D) \leq j$?

(7)

TOTAL 2-LIMITED PACKING PROBLEM
INSTANCE: A digraph D and a positive integer j'.
QUESTION: Is $L_2^t(D) \geq j'$?

We next present NP-completeness results for the digraph problems listed above. Recall first that for a graph G, the *complete biorientation* $cb(G)$ of G is a digraph D obtained from G by replacing each edge $xy \in E(G)$ by the pair of arcs (x, y) and (y, x).

Theorem 3. *The problems given in the rectangles* (6) *and* (7) *are NP-complete, even when restricted to bipartite digraphs.*

Proof. The problem (6) is clearly in NP since checking that a given set is indeed a total 2-dominating set of cardinality at most j can be done in polynomial time.

We reduce the problem (2) to the problem (4). Let G be a graph. We then consider the complete biorientation $cb(G)$ of G. It is easy to check that a set $S \subseteq V(G)$ is a dominating set in G if and only if $S \subseteq V(cb(G))$ is a dominating set in $cb(G)$. This shows that $\gamma(G) = \gamma(cb(G))$. We now deduce from the problem (2) and fact that it is NP-complete for bipartite graphs that the corresponding problem (4) is NP-complete for bipartite digraphs, as well.

We now reduce the problem (4) to the problem (6) for bipartite digraphs. We begin with a bipartite digraph D with $V(D) = \{v_1, \ldots, v_n\}$. For each $1 \leq i \leq n$, we add new vertices w_i and u_i, and arcs (w_i, u_i), (u_i, w_i) and (u_i, v_i). We denote the resulting bipartite digraph by D'. Note that every total 2-dominating set S' in D' contains both w_i and u_i, for each $1 \leq i \leq n$. Moreover, $|S' \cap V(D)|$ must be at least as large as $\gamma(D)$ so as to the vertices of D can be total 2-dominated by S'. On the other hand, for each $\gamma(D)$-set S, $S \cup \left(\bigcup_{i=1}^{n} \{w_i, u_i\}\right)$ is a total 2-dominating set in D'. The above argument shows that $\gamma_2^t(D') = 2n + \gamma(D)$. Now by taking $j = k + 2n$, we have $\gamma_2^t(D') \leq j$ if and only if $\gamma(D) \leq k$. So, the problem given in (6) is NP-complete for bipartite digraphs.

From now on, we discuss the NP-completeness of TOTAL 2-LIMITED PACKING PROBLEM. The problem belongs to NP as it can be readily checked in polynomial time that a given set is indeed a total 2-limited packing of cardinality at least j'. We first check the NP-completeness of PACKING PROBLEM for bipartite graphs. In order to complete the proof, we need the following claim.

Claim A. *PACKING PROBLEM is NP-complete even for bipartite graphs.*

Proof. It is clear that PACKING PROBLEM belongs to NP as we can check, in polynomial time, a subset B of vertices is a packing of cardinality at least k'. We make use of a reduction of X3C to PACKING PROBLEM. Let $X = \{x_1, \ldots, x_{3q}\}$ and $C = \{C_1, \ldots, C_m\}$ be an arbitrary instance of X3C. For any element x_i, we consider a 3-path $x_i y_i z_i$. Corresponding to each 3-set C_j, we associate a 4-path $c_j d_j e_j f_j$. Now to obtain the graph G, we add edges $x_i c_j$ if $x_i \in C_j$. It is easily checked that G is bipartite and that its construction can be accomplished in polynomial time. We set $k' = 4q + m$.

Let B be a $\rho(G)$-set of cardinality at least k'. Since B is a maximum packing in G, it follows that B contains precisely one vertex in $\{x_i, y_i, z_i\}$ for any $1 \leq i \leq 3q$. We may assume, without loss of generality, that $B \cap \{x_i, y_i, z_i\} = \{z_i\}$ for each $1 \leq i \leq 3q$. Since every vertex in $B \cap \{c_1, \ldots, c_m\}$ is adjacent to three vertices in X, and no two vertices of $B \cap \{c_1, \ldots, c_m\}$ have a common neighbor, we deduce that $|B \cap \{c_1, \ldots, c_m\}| \leq q$. If $|B \cap \{c_1, \ldots, c_m\}| \leq q - 1$, then at least $m + 1$ vertices in $\bigcup_{j=1}^{m} \{d_j, e_j, f_j\}$ belong to B. Therefore, $|B \cap \{d_j, e_j, f_j\}| \geq 2$ for some $1 \leq j \leq m$, which contradicts the definition of the packing B. Hence, $|B \cap \{c_1, \ldots, c_m\}| = q$. Consequently, $C' = \{C_j \mid c_j \in B\}$ is a solution for the problem X3C.

Conversely, suppose that the problem X3C has a solution C' with $|C'| = q$. It is then straightforward to see that $B = \{z_i\}_{i=1}^{3q} \cup \{c_j, f_j \mid c_j \in C'\} \cup \{f_j \mid c_j \notin C'\}$ is a packing of cardinality $4q + m$. Therefore, $\rho(G) \geq 4q + m = k'$. □

Similarly to the equality $\gamma(G) = \gamma(cb(G))$ we can see that $\rho(G) = \rho(cb(G))$ for any graph G. Therefore, Claim A implies that the problem (5) is NP-complete for bipartite digraphs.

We now construct a bipartite digraph D'' from any bipartite digraph D with $V(D) = \{v_1, \ldots, v_n\}$ by adding a new vertex x_i and a new arc (v_i, x_i), for each $1 \leq i \leq n$. It is easy to check that $B \cup \{x_i\}_{i=1}^{n}$ is a total 2-limited packing in D'', in which B is a $\rho(D)$-set. Therefore, $L_2^t(D'') \geq n + \rho(D)$.

Finally, we reduce the problem (5) to TOTAL 2-LIMITED PACKING PROBLEM for bipartite digraphs. Let B'' be an $L_2^t(D'')$-set. Let $x_i \notin B''$, for some $1 \leq i \leq n$. If $v_i \notin B''$, then it must be adjacent to precisely two vertices in B'', for otherwise $B'' \cup \{x_i\}$ would be a total 2-limited packing in D'' which contradicts the maximality of B''. Then $(B'' \setminus \{w_i\}) \cup \{x_i\}$ is an $L_2^t(D'')$-set containing x_i, in which $w_i \in N^+(v_i) \cap B''$. Now if $v_i \in B''$, we easily observe that $(B'' \setminus \{v_i\}) \cup \{x_i\}$ is also an $L_2^t(D'')$-set. Therefore, we may assume that $x_i \in B''$ for all $1 \leq i \leq n$.

We then note that $|B'' \cap V(D)|$ must be less than or equal to $\rho(D)$. If this is not true, then it is not hard to see that B'' is not a total 2-limited packing in D'', a contradiction. Therefore, $L_2^t(D'') \leq n + \rho(D)$. It now follows that $L_2^t(D'') = n + \rho(D)$. Now by taking $j' = k' + n$, we have $L_2^t(D'') \geq j'$ if and only

if $\rho(D) \geq k'$. So, the problem given in (7) is NP-complete for bipartite digraphs. This completes the proof. ■

As a consequence of the result above, we conclude that the problems of computing the parameters given in Definition 1 and Definition 2 are NP-hard even when restricted to bipartite digraphs. Taking into account this fact, it is desirable to bound their values with respect to several different invariants of the digraph. In the next two sections we exhibit such results.

3. Bounding $\gamma_2^t(D)$ and $L_2^t(D)$ for General Digraphs

In this section, we discuss some results about the digraph parameters $\gamma_2^t(D)$ and $L_2^t(D)$. We first bound the total 2-limited packing number of a digraph from above just in terms of its order and minimum in-degree. We introduce a family of digraphs in order to characterize all digraphs attaining the upper bound. Let D' be a complete biorientation of a 1-factor with $V(D') = \{v_1, \ldots, v_{n'}\}$. Let $p = (r-1)n'$ where $r \geq 1$ is an integer. Add a set of new vertices $U = \{u_1, \ldots, u_{p/2}\}$ and new arcs (u_i, v_j) such that

(i) every vertex u_i is incident with precisely two such arcs, and

(ii) $\deg^-(v_j) = r$ for all $1 \leq j \leq n'$.

Now add some arcs among the vertices u_i and some arcs from $V(D')$ to U, such that r is the minimum in-degree of the constructed digraph. Let Ω be the family of digraphs D constructed as above.

Theorem 4. *Let D be a digraph of order n with $\delta^- \geq 1$. Then $L_2^t(D) \leq \frac{2n}{\delta^-+1}$ with equality if and only if $D \in \Omega$.*

Proof. Let B be an $L_2^t(D)$-set. By the definition, every vertex in $V(D) \setminus B$ has at most two out-neighbors in B. Thus,

$$|(V(D) \setminus B, B)_D| \leq 2(n - |B|). \tag{8}$$

On the other hand, since every vertex in B is adjacent with at most one vertex in B, we have

$$\begin{aligned} |B|(\delta^- - 1) &\leq \sum_{v \in B} \deg^-(v) - \sum_{v \in B} \deg_B^-(v) = \sum_{v \in B} (\deg^-(v) - \deg_B^-(v)) \\ &= \sum_{v \in B} \deg_{V(D)\setminus B}^-(v) = |(V(D) \setminus B, B)_D|. \end{aligned} \tag{9}$$

Together inequalities (8) and (9) imply the desired upper bound.

Suppose that $D \in \Omega$. It is easily seen that $V(D')$ is a total 2-limited packing in D. Moreover, $\delta^-(D) = r$ and $n = n' + p/2$. Therefore, $|V(D')| = n' = 2n/(\delta^- + 1)$. We now have $L_2(D) \geq 2n/(\delta^- + 1)$, implying the desired equality.

Let the upper bound hold with equality. Then both (8) and (9) hold with equality, necessarily. The equality in (9) shows that $\sum_{v \in B} \deg_B^-(v) = |B|$. Moreover, since every vertex in B is adjacent with at most one vertex in B, $D\langle B\rangle$ is a complete biorientation of a 1-factor. Also, all vertices in B have in-degree δ^- since $\sum_{v \in B} \deg^-(v) = \delta^-|B|$. The equality in (8) shows that every vertex in $V(D) \setminus B$ is adjacent to precisely two vertives in B. Thus, the membership D in Ω easily follows by choosing $D\langle B\rangle$, δ^- and $V(D) \setminus B$ for D', r and U, respectively, in the description of Ω. ■

We conclude this section by bounding the total 2-domination number of a digraph from below in terms of its order and maximum out-degree. Indeed, the following theorem for total 2-domination can be considered as a result analogous to Theorem 4 for total 2-limited packing. Similarly to that for Theorem 4, we introduce a family of digraphs so as to characterize all digraphs attaining the lower bound given in the next theorem. We begin with a directed 1-factor D' with $V(D') = \{v_1, \ldots, v_{n'}\}$. Choose $r \geq 1$ such that $q = (r - 1/2)n' \equiv 0 \pmod 2$. Add a set of new vertices $U = \{u_1, \ldots, u_{q/2}\}$ and new arcs (v_i, u_j) such that

(i) every vertex u_j is incident with precisely two such arcs, and

(ii) $\deg^+(v_i) = r$ for all $1 \leq i \leq n'$.

Now add some arcs among the vertices u_j and some arcs from U to $V(D')$, such that r is the maximum out-degree of the constructed digraph. Let Θ be the family of digraphs D constructed as above.

Theorem 5. *For any digraph D of order n with no isolated vertices of maximum out-degree Δ^+, $\gamma_2^t(D) \geq \frac{2n}{\Delta^+ + 3/2}$. Furthermore, the equality holds if and only if $D \in \Theta$.*

Proof. Let S be a $\gamma_2^t(D)$-set. Every vertex in $V(D) \setminus S$ is adjacent from at least two vertices in S, by the definition. Hence,

$$2(n - |S|) \leq |(S, V(D) \setminus S)_D|. \tag{10}$$

On the other hand, every vertex in S is adjacent with at least one vertex in S. Therefore,

$$\begin{aligned} |(S, V(D) \setminus S)_D| &= \sum_{v \in S} \deg^+_{V(D)\setminus S}(v) = \sum_{v \in S} (\deg^+(v) - \deg^+_S(v)) \\ &= \sum_{v \in S} \deg^+(v) - \sum_{v \in S} \deg^+_S(v) \leq (\Delta^+ - 1/2)|S|. \end{aligned} \tag{11}$$

The desired lower bound now follows by the inequalities (10) and (11).

Suppose that $D \in \Theta$. Clearly, $r = \Delta^+(D)$ and $n = n' + q/2$. Moreover, $V(D')$ is a double dominating set in D. Since $|V(D')| = n' = 2n/(\Delta^+ + 3/2)$, it follows that $\gamma_2^t(D) \leq 2n/(\Delta^+ + 3/2)$, implying the equality in the lower bound.

Conversely, let the equality hold in the lower bound. Then both (10) and (11) hold with equality, necessarily. Therefore, every vertex in S is adjacent with exactly one vertex in S. This shows that $D\langle S\rangle$ is a directed 1-factor. Also, the equality $\sum_{v\in S} \deg^+(v) = \Delta^+|S|$ implies that every vertex of $D\langle S\rangle$ has the out-degree Δ^+. On the other hand, the equality in (10) shows that every vertex in $V(D) \setminus S$ is adjacent from exactly two vertices in S. That the digraph D is in Ω can be easily seen by choosing $D\langle S\rangle$, Δ^+ and $V(D) \setminus S$ for D', r and U, respectively, in the description of Θ. ■

4. Directed Trees

We first recall that a maximization problem **M** and a minimization problem **N**, defined on the same instances (such as graphs or digraphs), are *dual problems* if the value of every candidate solution M to **M** is less than or equal to the value of every candidate solution N to **N**. Often the "value" is cardinality. Analogously to many well known pairs of dual (graph or digraph) problems like *matching* and *vertex covering*, *packing* and *domination*, etc. the following theorem shows that the problems "total 2-domination" and "total 2-limited packing", on the instances of directed trees, are dual problems.

Recall that in a tree a support vertex is called a *weak* (*strong*) *support vertex* if it is adjacent to (more than) one leaf. Also, a *double star* $S_{a,b}$ is a tree with exactly two non-leaf vertices in which one support vertex is adjacent to a leaves and the other to b leaves.

Theorem 6. *For any directed tree T of order $n \geq 2$, $L_2^t(T) \leq \gamma_2^t(T)$.*

Proof. We proceed by induction on the order n. The result is obvious for $n = 2$. Let $\tilde{T}$ be the underlying tree of T. It is easy to check that the result is true when $diam(\tilde{T}) \leq 3$. In such a case, we have $L_2^t(T) \leq \gamma_2^t(T) = n$. So, we may assume that $diam(\tilde{T}) \geq 4$. This implies that $n \geq 5$. Assume that the inequality holds for all directed trees T' of order $3 \leq n' < n$. Let T be a directed tree of order $n \geq 5$. Suppose that r and x are two leaves of $\tilde{T}$ with $d(r, x) = diam(\tilde{T})$. We root $\tilde{T}$ at r. Let x be adjacent with y. Note that the choice of x shows that all children of y in $\tilde{T}$ are leaves. Let B and S be an $L_2^t(T)$-set and a $\gamma_2^t(T)$-set in T, respectively. Note that all end-vertices and penultimate vertices belong to every total 2-dominating set in T. Suppose that y is a strong support vertex in $\tilde{T}$. Then $S \setminus \{x\}$ is a total 2-dominating set in $T' = T - x$. Moreover, it is easy

to see that $B \setminus \{x\}$ is a total 2-limited packing in T'. Using now the induction hypothesis we have

$$L_2^t(T) - 1 \leq |B \setminus \{x\}| \leq L_2^t(T') \leq \gamma_2^t(T') \leq |S| - 1 = \gamma_2^t(T) - 1.$$

From now on, we assume that y is a weak support vertex of $\tilde{T}$. Hence, x is a sink or source (that is, $\deg^+(v) = 0$ or $\deg^-(v) = 0$ respectively) of T and y is the unique vertex adjacent with x. We consider two cases depending on the behavior of x.

Case 1. Suppose that $S \setminus \{x\}$ is a total 2-dominating set in the directed tree $T' = T - x$. Moreover, $B \setminus \{x\}$ is a total 2-limited packing in T'. Then, by using the induction hypothesis, we have again

$$L_2^t(T) - 1 \leq |B \setminus \{x\}| \leq L_2^t(T') \leq \gamma_2^t(T') \leq |S| - 1 = \gamma_2^t(T) - 1.$$

Case 2. Suppose that $S \setminus \{x\}$ is not a total 2-dominating set in T'. This shows that y is not adjacent with any vertex in $S \setminus \{x\}$. Here we need to consider two more possibilities.

Subcase 2.1. Suppose that $S'' = S \setminus \{x, y\}$ is a total 2-dominating set in $T'' = T - x - y$. On the other hand, it is clear that $B'' = B \setminus \{x, y\}$ is a total 2-limited packing in T''. We then have

$$L_2^t(T) - 2 \leq |B''| \leq L_2^t(T'') \leq \gamma_2^t(T'') \leq |S''| = \gamma_2^t(T) - 2.$$

Subcase 2.2. Suppose now that $S'' = S \setminus \{x, y\}$ is not a total 2-dominating set in T''. This assumption along with the fact that y is not adjacent with any vertex in $S \setminus \{x\}$ imply that there exists a vertex $z \in V(T'') \setminus S''$ such that $|N^-_{T''}(z) \cap S''| \leq 1$. This shows that $|N^-_T(z) \cap S| = 2$ and $y \in N^-(z)$, necessarily. Note that by our choice of x, all children of z in $\tilde{T}$ are leaves or support vertices. If z is adjacent with an end-vertex, then we have contradiction to the fact that y is not adjacent with any vertex in $S \setminus \{x\}$. Therefore, all children of z in $\tilde{T}$ are support vertices. Let $\tilde{T}_z$ be the subtree of $\tilde{T}$ rooted at z consisting of z and its descendants in $\tilde{T}$. Now consider the directed tree $T''' = T - V(\tilde{T}_z)$ (our choice of x and $diam(\tilde{T}) \geq 4$ imply that $|V(T''')| \geq 2$). Let z have k children in $\tilde{T}$. It is easy to see that $S''' = S \setminus V(\tilde{T}_z)$ is a total 2-dominating set in T''' with $|S'''| = \gamma_2^t(T) - 2k$. On the other hand, $B''' = B \setminus V(\tilde{T}_z)$ is a total 2-limited packing in T''' with $|B'''| \geq L_2^t(T) - 2k$. Therefore,

$$L_2^t(T) - 2k \leq |B'''| \leq L_2^t(T''') \leq \gamma_2^t(T''') \leq |S'''| = \gamma_2^t(T) - 2k.$$

This completes the proof. ∎

In what follows we construct a family of directed trees in order to characterize those ones attaining the lower bound in the next theorem.

Let $F = F_0$ be a directed forest containing r copies of the directed path P_2 with arcs $(v_{11}, v_{12}), \ldots, (v_{r1}, v_{r2})$ and r' copies of directed stars $H_1, \ldots, H_{r'}$ of order at least 3 with central vertices $u_1, \ldots, u_{r'}$, respectively. Let $q = r+r'-1$. We construct the sequence $F_0, F_1, \ldots, F_q$ of digraphs as follows. Let F_1 be obtained from F_0 by adding a vertex w_1 and two arcs (x_1, w_1) and (y_1, w_1) for some $x_1, y_1 \in A = \{v_{i1}, v_{i2}\}_{i=1}^{r} \cup \{u_i\}_{i=1}^{r'}$ such that x_1 and y_1 are not vertices of a same P_2-copy. We now obtain F_2 from F_1 by adding a vertex w_2 and two arcs (x_2, w_2) and (y_2, w_2) such that $x_2, y_2 \in A$, $x_2 \in N_{F_0}[\{x_1, y_1\}]$ and $y_2 \in V(F_0) \setminus N_{F_0}[\{x_1, y_1\}]$ (note that $\{x_1, y_1\} = N^-_{F_1}(w_1)$). We now suppose that F_j is obtained from F_{j-1} by adding a vertex w_j with two arcs (x_j, w_j) and (y_j, w_j) such that (i) $x_j, y_j \in A$, and (ii) precisely one of them belongs to $B_{j-1} = N_{F_0}[N^-_{F_{j-1}}(\{w_1, \ldots, w_{j-1}\})]$. We define Γ as the family of all digraphs F_q constructed as above. In what follows, we first need to show that the above construction is well-defined. Moreover, Γ is a family of directed trees (Figure 1 depicts a representative member of Γ).

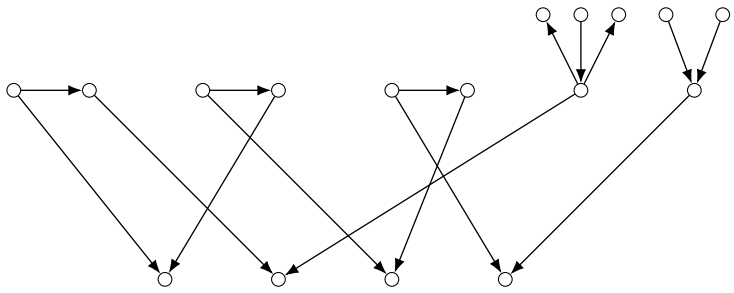

Figure 1. A member of Γ.

Proposition 7. *The following statements hold.*

(a) *If F_0 is neither the directed path P_2 nor a directed star on at least three vertices, then there always exist two arcs (x_j, w_j) and (y_j, w_j) with the given properties* (i) *and* (ii), *for all $1 \leq j \leq q$.*

(b) *F_q is a directed tree.*

Proof. (a) Let F_0 be neither the directed path P_2 nor a directed star on at least three vertices. Therefore, $r + r' \geq 2$. Clearly, there are such arcs for the vertex w_1. Let $2 \leq i \leq q$ be the smallest index for which there is not a pair of arcs (x_i, w_i) and (y_i, w_i) with the properties (i) and (ii). Now, consider the digraph F_{i-1}. We then add the vertex w_i. Since $i \leq q = r + r' - 1$, it follows that there exists a vertex $y_i \in A \setminus B_{i-1}$. Thus, we have the arcs (x_i, w_i) and (y_i, w_i) in which x_i is a vertex in B_{i-1}. This contradicts our choice of i.

(b) It is easy to see that F_1 is a directed forest. Assume now that F_i, $i \geq 1$, is a directed forest. It follows from the way we construct F_{i+1} from F_i that the underlying graph of F_{i+1} has no cycle as a subgraph. So, F_{i+1} is a directed forest

as well. In particular, F_q is a directed forest. Now let $H_1, \ldots, H_{r'}$ be those r' directed stars in the definition of $F = F_0$ of order $t_1, \ldots, t_{r'} \geq 3$, respectively. Then, $|A(F_q)| = 3r + r' + t_1 + \cdots + t_{r'} - 2 = |V(F_q)| - 1$. This implies that F_q is a directed tree. ■

We are now in a position to present the main theorem of this section.

Theorem 8. *Let T be a directed tree of order n with e end-vertices and p penultimate vertices. Then*

$$\gamma_2^t(T) \geq \frac{2n + e - p + 2}{3}.$$

The equality holds if and only if $T \in \Gamma$.

Proof. Let $S = \{v_1, \ldots, v_{|S|}\}$ be a $\gamma_2^t(T)$-set. Note that all end-vertices and penultimate vertices belong to S, necessarily. Therefore, all pendant arcs belong to $A(T\langle S\rangle)$. Since every vertex in $V(T) \setminus S$ is adjacent from at least two vertices in S and $T\langle S\rangle$ has no isolated vertices, it follows that

$$\begin{aligned} \deg^+(v_1) + \cdots + \deg^+(v_{|S|}) &= |(S, V(T) \setminus S)_T| + |A(T\langle S\rangle)| \\ &\geq 2(n - |S|) + e + \tfrac{|S|-e-p}{2}. \end{aligned} \tag{12}$$

On the other hand,

$$\deg^+(v_1) + \cdots + \deg^+(v_{|S|}) \leq n - 1. \tag{13}$$

The desired lower bound now follows from (12) and (13).

Let $T \in \Gamma$. Let S' be the set of vertices of the copies P_2 and H_i, $1 \leq i \leq r'$. It is easy to see that S' is a total 2-dominating set in T of cardinality $2r + t_1 + \cdots + t_{r'}$, in which $t_i = |V(H_i)|$ for $1 \leq i \leq r'$. Moreover, $n = 2r + t_1 + \cdots + t_{r'} + r + r' - 1$ and $e - p = t_1 + \cdots + t_{r'} - 2r'$. So, $\gamma_2^t(T) \leq 2r + t_1 + \cdots + t_{r'} = (2n + e - p + 2)/3$ which implies the equality in the lower bound.

Suppose now that we have the equality in the lower bound of theorem. Then both the inequalities in (12) and (13) hold with equality, necessarily. In particular,

$$\deg^+(v_1) + \cdots + \deg^+(v_{|S|}) = n - 1 = \sum_{v \in V(T)} \deg^+(v)$$

shows that $V(T) \setminus S$ is independent. Moreover, the equalities $|(S, V(T) \setminus S)_T| = 2(n - |S|)$ and $|A(T\langle S\rangle)| = e + (|S| - e - p)/2$ imply that every vertex in $V(T) \setminus S$ is adjacent from precisely two vertices in S, and $T\langle S\rangle$ is a disjoint union of digraphs which are isomorphic to the directed paths P_2 and the directed stars H_t of orders at least three whose end-vertices are not adjacent with the vertices in $V(T) \setminus S$. Choose a vertex $w_1 \in V(T) \setminus S$. Then w_1 is adjacent from two vertices in S which do not belong to a same component of $T' = T\langle S\rangle$, for otherwise the underlying

graph of T contains a cycle. Since T is connected and its underlying graph has no cycle, there exists a vertex $w_2 \in V(T) \setminus S$ which is adjacent from exactly one vertex in $N_{T'}[N^-(w_1)]$. In general, by choosing the vertex w_{j-1} in such a way, we find the vertex w_j for which exactly one of its two in-neighbors belongs to $N_{T'}[N^-(\{w_i\}_{i<j})]$. The above argument shows that $T \in \Gamma$. ■

5. Sum and Product of $\Psi(D)$ and $\Psi(D^{-1})$ when $\Psi \in \{\gamma_2^t, L_2^t\}$

For the rest of the paper, we study the sum and product of $\Psi \in \{\gamma_2^t, L_2^t\}$ of a digraph and its converse. In order to obtain such inequalities concerning the total 2-limited packing number, we make use of the structures of directed trees. Note that the study of these kinds of inequalities was first presented by Chartrand *et al.* [5] for the domination number. Since then bounds on $\Psi(D) + \Psi(D^{-1})$ or $\Psi(D)\Psi(D^{-1})$ appeared in literature, in which Ψ is a digraph parameter. For example, the reader can be referred to the papers [11, 15] and [16].

Nordhaus and Gaddum [23] in 1956, gave lower and upper bounds on the sum and product of the chromatic numbers of a graph G and its complement $\overline{G}$ in terms of the order of G. Since then, bounds on $\Theta(G) + \Theta(\overline{G})$ or $\Theta(G)\Theta(\overline{G})$ are called *Nordhaus-Gaddum inequalities*, where Θ is any graph parameter. The search of the Nordhaus-Gaddum type inequalities has centered the attention of a large number of investigations, and in domination theory, this has probably been even more emphasized. For more information about this subject the reader can consult [1].

Indeed, the above-mentioned inequalities concerning digraphs and their converse can be interpreted as modified Nordhaus-Gaddum theorems for digraphs, where the converse of a digraph replaces the complement of a graph.

Proposition 9. *Let D be a connected digraph of order $n \geq 2$. Then $\gamma_2^t(D) = n$ if and only if $\deg^-(v) \leq 1$ for each vertex v which is neither a penultimate vertex nor an end vertex.*

Proof. The sufficiency of the condition is clear. Now let $\gamma_2^t(D) = n$. Suppose that there exists a vertex v with $\deg^-(v) \geq 2$ which is neither a penultimate vertex nor an end vertex. We can deduce that $V(D) \setminus \{v\}$ is a total 2-dominating set in D, which is impossible. ■

As an immediate consequence of Proposition 9, we have the following result.

Corollary 10. *For any connected digraph D of order $n \geq 2$, $\gamma_2^t(D) + \gamma_2^t(D^{-1}) = 2n$ $\left(\gamma_2^t(D)\gamma_2^t(D^{-1}) = n^2\right)$ if and only if $\deg^-(v), \deg^+(v) \leq 1$ for each vertex v which is neither a penultimate vertex nor an end vertex.*

We now turn our attention to the total 2-limited packing number. Let D be a connected digraph of order n. Then $L_2^t(D) + L_2^t(D^{-1}) = 2n$ when $n = 1, 2$. So, in what follows we may assume that $n \geq 3$.

Theorem 11. *For any connected digraph D of order $n \geq 3$,*

$$L_2^t(D) + L_2^t(D^{-1}) \leq \frac{16n}{9}.$$

This bound is sharp.

Proof. We first prove that the inequality holds for all directed stars on $n \geq 3$ vertices.

Claim B. *For any directed star S of order $n \geq 3$, $L_2^t(S) + L_2^t(S^{-1}) \leq 16n/9$.*

Proof. Let u be the central vertex of S, $\deg^+(u) = a$ and $\deg^-(u) = b$. We have,

(14)
$$L_2^t(S) + L_2^t(S^{-1}) = \begin{cases} a + b + 2 = n + 1, & \text{if } a = 0 \text{ or } b = 0, \\ a + b + 2 = n + 1, & \text{if } a = b = 1, \\ a + b + 3 = n + 2, & \text{if } \min\{a, b\} = 1 \text{ and } \max\{a, b\} \geq 2, \\ a + b + 4 = n + 3, & \text{if } a, b \geq 2. \end{cases}$$

We now have $L_2^t(S) + L_2^t(S^{-1}) \leq 16n/9$ in all four possible values for $L_2^t(S) + L_2^t(S^{-1})$. □

We are now able to extend the inequality in Claim B to directed trees on at least three vertices as follows. Indeed, we prove that

$$L_2^t(T) + L_2^t(T^{-1}) \leq 16n/9, \tag{15}$$

by induction on the order $n \geq 3$ of directed tree T. If $n = 3$, then the result follows from Claim B. Suppose now that the result is true for all directed trees T' of order $3 \leq n' < n$. Let T be a directed tree of order n. If T is a directed star, then the result again follows by Claim B. So, we assume that T is not a directed star. Therefore, T has an arc (x, y) such that $T - (x, y)$ is the union of two non-trivial directed trees T_1 and T_2 of order $n_1 < n$ and $n_2 < n$, respectively. Moreover, by the symmetry between T and T^{-1}, we may assume that $x \in V(T_1)$ and $y \in V(T_2)$. If $n_1 = n_2 = 2$, then T is obtained from orienting the edges of a path on four vertices. It follows that $L_2^t(T) + L_2^t(T^{-1}) \leq 6 \leq 16 \cdot 4/9$. So, in what follows we may assume that $n_1 \geq 3$. We now distinguish two cases depending on n_2.

Case 1. $n_2 \geq 3$. By the induction hypothesis, we have $L_2^t(T_1) + L_2^t(T_1^{-1}) \leq 16n_1/9$ and $L_2^t(T_2) + L_2^t(T_2^{-1}) \leq 16n_2/9$. The inequality (15) now follows from the fact that $L_2^t(T) \leq L_2^t(T_1) + L_2^t(T_2)$ and $L_2^t(T^{-1}) \leq L_2^t(T_1^{-1}) + L_2^t(T_2^{-1})$.

Case 2. $n_2 = 2$. Let $V(T_2) = \{y, z\}$. We now consider two subcases.

Subcase 2.1. Suppose that there exist an $L_2^t(T)$-set B and an $L_2^t(T^{-1})$-set B^{-1} such that $\{y, z\} \nsubseteq B \cap B^{-1}$. By the induction hypothesis, we have $L_2^t(T_1) + L_2^t(T_1^{-1}) \leq 16n_1/9 = 16(n-2)/9$. Therefore,

$$\begin{aligned} L_2^t(T) + L_2^t(T^{-1}) &\leq L_2^t(T_1) + |B \cap \{y, z\}| + L_2^t(T_1^{-1}) + \left|B^{-1} \cap \{y, z\}\right| \\ &\leq \frac{16(n-2)}{9} + 3 < \frac{16n}{9}. \end{aligned} \tag{16}$$

Subcase 2.2. Suppose now that $\{y, z\} \subseteq B \cap B^{-1}$ for all $L_2^t(T)$-sets B and $L_2^t(T^{-1})$-sets B^{-1}. This implies that $x \notin B \cup B^{-1}$. We have, $T_1 = T - \{y, z\}$. Suppose that $T_{11}, \ldots, T_{1p}$ are the components of $T_1 - x$. If $|V(T_{11})|, \ldots, |V(T_{1p})| \geq 3$, we have $L_2^t(T_{1i}) + L_2^t(T_{1i}^{-1}) \leq 16|V(T_{1i})|/9$ for all $1 \leq i \leq p$, by the induction hypothesis. Therefore,

$$\begin{aligned} L_2^t(T) + L_2^t(T^{-1}) &\leq \sum_{i=1}^{p} (L_2^t(T_{1i}) + L_2^t(T_{1i}^{-1})) + |B \cap \{x, y, z\}| \\ &+ |B^{-1} \cap \{x, y, z\}| \leq \frac{16}{9} \sum_{i=1}^{p} |V(T_{1i})| + 4 \\ &= \frac{16}{9}(n-3) + 4 < \frac{16n}{9}. \end{aligned} \tag{17}$$

So, we assume that $|V(T_{1i})| \leq 2$ for some $1 \leq i \leq p$. Without loss of generality, we assume that $|V(T_{11})|, \ldots, |V(T_{1q})| \leq 2$ for $1 \leq q \leq p$. By the induction hypothesis, we have

$$L_2^t(T_{1i}) + L_2^t(T_{1i}^{-1}) \leq 16|V(T_{1i})|/9, \tag{18}$$

for all $q + 1 \leq i \leq p$ (if there is such an index i).

Let $x_1, \ldots, x_q$ be the unique vertices in $V(T_{11}), \ldots, V(T_{1q})$, respectively, which are adjacent with x. We now present the following claim.

Claim C. *If $q \geq 4$, then at least one of the vertices $x_1, \ldots, x_q$ does not belong to $B \cap B^{-1}$.*

Proof. Suppose that $q \geq 4$. Suppose to the contrary that $x_1, \ldots, x_q \in B \cap B^{-1}$. Since T is a directed tree, the subdigraph H induced by $\{y, x, x_1, \ldots, x_4\}$ is a directed star on six vertices. In fact, H is isomorphic to one of the directed stars depicted in Figure 2. In the first three directed stars (from left to right) we have contradiction with the fact that B is a 2-limited packing in D, and in the last two directed stars we have contradiction with the fact that B^{-1} is a 2-limited packing in D^{-1}. Thus, at least one of the vertices $x_1, \ldots, x_q$ does not belong to $B \cap B^{-1}$. □

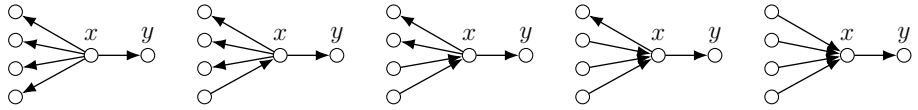

Figure 2. All the possible directed stars on six vertices with the center x and fixed arc (x, y).

Let $q \geq 4$. We now assume, without loss of generality, that $x_1 \notin B \cap B^{-1}$. If $V(T_{11}) = \{x_1\}$, then

$$\begin{aligned} L_2^t(T) + L_2^t(T^{-1}) &\leq |B \cap \{y, z\}| + |B^{-1} \cap \{y, z\}| + |B \cap \{x_1\}| + |B^{-1} \cap \{x_1\}| \\ &+ |B \cap V(T - \{y, z, x_1\})| + |B^{-1} \cap V(T^{-1} - \{y, z, x_1\})| \\ &\leq 4 + 1 + L_2^t(T - \{y, z, x_1\}) + L_2^t(T^{-1} - \{y, z, x_1\}) \\ &\leq 5 + 16(n-3)/9 < 16n/9. \end{aligned} \tag{19}$$

If $V(T_{11}) = \{x_1, x_1'\}$ for some vertex x_1', then

$$\begin{aligned} L_2^t(T) + L_2^t(T^{-1}) &\leq 4 + |B \cap \{x_1, x_1'\}| + |B^{-1} \cap \{x_1, x_1'\}| \\ &+ |B \cap V(T - \{y, z, x_1, x_1'\})| \\ &+ |B^{-1} \cap V(T^{-1} - \{y, z, x_1, x_1'\})| \\ &\leq 4 + 3 + L_2^t(T - \{y, z, x_1, x_1'\}) + L_2^t(T^{-1} - \{y, z, x_1, x_1'\}) \\ &\leq 7 + 16(n-4)/9 < 16n/9. \end{aligned} \tag{20}$$

It remains for us to prove the desired inequality when $q \leq 3$. We consider two subcases depending on p and q.

Subcase 2.2.1. Let $p = q$. If $q = 1$, then we deal with a directed tree whose underlying graph is a path on four or five vertices. Clearly, in such cases the desired inequality holds. If $q = 2$ or 3, then the directed tree T is obtained from the orientation of a tree of order $n \in \{5, 6, 7\}$ or $n \in \{6, 7, 8, 9\}$, respectively. In all the possible cases, we have $L_2^t(T) + L_2^t(T^{-1}) \leq 16n/9$.

Subcase 2.2.2. $q < p$. Let x_p be the unique vertex of T_{1p} which is adjacent with the vertex x. Since both T_{1p} and $T - V(T_{1p})$ have at least three vertices, it follows by the induction hypothesis that $L_2^t(T_{1p}) + L_2^t(T_{1p}^{-1}) \leq 16|V(T_{1p})|/9$ and $L_2^t(T - V(T_{1p})) + L_2^t(T^{-1} - V(T_{1p})) \leq 16(n - |V(T_{1p})|)/9$. Therefore,

$$\begin{aligned} L_2^t(T) + L_2^t(T^{-1}) &\leq L_2^t(T_{1p}) + L_2^t(T - V(T_{1p})) + L_2^t\big(T_{1p}^{-1}\big) \\ &+ L_2^t\big(T^{-1} - V(T_{1p})\big) \leq \frac{16n}{9}. \end{aligned} \tag{21}$$

Indeed, in all possible cases we have proved the inequality (15) for a directed tree T.

Since D is connected, it has a spanning directed tree T. Moreover, $L_2^t(D) \leq L_2^t(T)$ and $L_2^t(D^{-1}) \leq L_2^t(T^{-1})$. We now have

$$L_2^t(D) + L_2^t(D^{-1}) \leq L_2^t(T) + L_2^t(T^{-1}) \leq \frac{16n}{9} \tag{22}$$

by (15), as desired.

In what follows, we show that the upper bound is sharp. Let D' be an arbitrary connected digraph on the set of vertices $V(D') = \{v_1, \ldots, v_{n'}\}$. For every $1 \leq i \leq n'$, we add four directed paths $P_{i1} : x_{11}^i, x_{12}^i$, $P_{i2} : x_{21}^i, x_{22}^i$, $P_{i3} : x_{31}^i, x_{32}^i$ and $P_{i4} : x_{41}^i, x_{42}^i$, and four arcs $(v_i, x_{11}^i), (v_i, x_{21}^i), (x_{31}^i, v_i), (x_{41}^i, v_i)$. Let R be the obtained digraph. It is easy to observe that $|V(R)| = 9n'$ and $B = \{x_{11}^i, x_{12}^i, \ldots, x_{41}^i, x_{42}^i\}_{i=1}^{n'}$ is both an $L_2^t(R)$-set and an $L_2^t(R^{-1})$-set. Thus, $L_2^t(R) + L_2^t(R^{-1}) = 16n' = 16|V(R)|/9$. This completes the proof. ∎

Maximizing $L_2^t(D)L_2^t(D^{-1})$ subject to $L_2^t(D) + L_2^t(D^{-1}) = 16n/9$, we have $L_2^t(D) = L_2^t(D^{-1}) = 8n/9$. Therefore, we have the following upper bound for the product of $L_2^t(D)$ and $L_2^t(D^{-1})$. Furthermore, the bound is sharp for the digraph R defined in the proof of Theorem 11.

Corollary 12. *For any connected digraph D of order $n \geq 3$, $L_2^t(D)L_2^t(D^{-1}) \leq 64n^2/81$.*

6. Concluding Remarks

Note that the digraph parameters L_2^t and γ_2^t are the same for both the complete biorientation $cb(K_n)$ of the complete graph K_n of order $n \geq 2$ and the digraph R introduced in the proof of Theorem 11. Moreover, we proved that $L_2^t(T) \leq \gamma_2^t(T)$ for all nontrivial directed tree T. So, it is natural to present the following open problem.

Problem 1. Does the inequality $L_2^t(D) \leq \gamma_2^t(D)$ hold for any digraph D with no isolated vertices?

Mojdeh *et al.* [21] proved that $\rho(T) = \gamma(T)$, for all directed trees T. Although such a result does not hold for $L_2^t(T)$ and $\gamma_2^t(T)$, one can ask for the family of all directed trees for which these two parameters are the same. Indeed, we pose the following problem.

Problem 2. Characterize the directed trees T for which $L_2^t(T) = \gamma_2^t(T)$.

Acknowledgements

Thanks are due to the anonymous referees for providing us with useful comments on this paper which helped to improve its presentation.

This research work has been supported by a research grant from the University of Mazandaran.

References

[1] M. Aouchiche and P. Hansen, *A survey of Nordhaus-Gaddum type relations*, Discrete Appl. Math. **161** (2013) 466–546.
https://doi.org/10.1016/j.dam.2011.12.018

[2] S. Arumugam, K. Jacob and L. Volkmann, *Total and connected domination in digraphs*, Australas. J. Combin. **39** (2007) 283–292.

[3] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications, in: Springer Monogr. Math. (Springer-Verlag London Ltd., London, 2007).

[4] C. Berge, The Theory of Graphs and its Applications (Methuen, London, 1962).

[5] G. Chartrand, F. Harary and B. Quan Yue, *On the out-domination and in-domination numbers of a digraph*, Discrete Math. **197/198** (1999) 179–183.
https://doi.org/10.1016/S0012-365X(99)90059-6

[6] X. Chen, G. Hao and L. Volkmann, *Bounds on the signed Roman k-domination number of a digraph*, Discuss. Math. Graph Theory **39** (2019) 67–79.
https://doi.org/10.7151/dmgt.2068

[7] X. Chen, G. Hao and Z. Xie, *A note on Roman domination of digraphs*, Discuss. Math. Graph Theory **39** (2019) 13–21.
https://doi.org/10.7151/dmgt.2067

[8] N.E. Clarke and R.P. Gallant, *On* 2*-limited packings of complete grid graphs*, Discrete Math. **340** (2017) 1705–1715.
https://doi.org/10.1016/j.disc.2016.11.001

[9] J.F. Fink and M.S. Jacobson, *n-domination in graphs*, in: Graph Theory with Applications to Algorithms and Computer Science, Alavi, Chartrand, Lesniak and Wall (Ed(s)), (Wiley, New-York, 1985) 283–300.

[10] Y. Fu, *Dominating set and converse dominating set of a directed graph*, Amer. Math. Monthly **75** (1968) 861–863.
https://doi.org/10.2307/2314337

[11] M. Furuya, *Bounds on the domination number of a digraph and its reverse*, Filomat **32** (2018) 2517–2524.
https://doi.org/10.2298/FIL1807517F

[12] R. Gallant, G. Gunther, B.L. Hartnell and D.F. Rall, *Limited packing in graphs*, Discrete Appl. Math. **158** (2010) 1357–1364.
https://doi.org/10.1016/j.dam.2009.04.014

[13] M.R. Garey and D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness (W.H. Freeman & Co., New York, 1979).

[14] D.S. Hochbaum and D.B. Shmoys, *A best possible heuristic for the k-center problem*, Math. Oper. Res. **10** (1985) 180–184.
https://doi.org/10.1287/moor.10.2.180

[15] G. Hao and J. Qian, *On the sum of out-domination number and in-domination number of digraphs*, Ars Combin. **119** (2015) 331–337.

[16] G. Hao, J. Qian and Z. Xie, *On the sum of the total domination numbers of a digraph and its converse*, Quaest. Math. **42** (2019) 47–57.
https://doi.org/10.2989/16073606.2018.1438531

[17] F. Harary and T.W. Haynes, *Double domination in graphs*, Ars Combin. **55** (2000) 201–213.

[18] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, New York, 1998).

[19] C. Lee, *Domination in digraphs*, J. Korean Math. Soc. **35** (1998) 843–853.

[20] D.A. Mojdeh, B. Samadi and I.G. Yero, *Further results on packing related parameters in graphs*, Discuss. Math. Graph Theory **42** (2022) 333–348.
https://doi.org/10.7151/dmgt.2262

[21] D.A. Mojdeh, B. Samadi and I.G. Yero, *Packing and domination parameters in digraphs*, Discrete Appl. Math. **269** (2019) 184–192.
https://doi.org/10.1016/j.dam.2019.04.008

[22] M. Liedloff, *Finding a dominating set on bipartite graphs*, Inform. Process. Lett. **107** (2008) 154–157.
https://doi.org/10.1016/j.ipl.2008.02.009

[23] E.A. Nordhaus and J.W. Gaddum, *On complementary graphs*, Amer. Math. Monthly **63** (1956) 175–177.
https://doi.org/10.2307/2306658

[24] O. Ore, Theory of Graphs in: Amer. Math. Soc. Colloq. Publ. **38** (Amer. Math. Soc., Providence, 1962).

[25] L. Ouldrabah, M. Blidia and A. Bouchou, *On the k-domination number of digraphs*, J. Comb. Optim. **38** (2019) 680–688.
https://doi.org/10.1007/s10878-019-00405-1

[26] D.B. West, Introduction to Graph Theory, Second Edition (Prentice Hall, USA, 2001).

Received 7 May 2020
Revised 19 November 2020
Accepted 26 November 2020