

BURNSIDE CHROMATIC POLYNOMIALS OF GROUP-INVARIANT GRAPHS

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Abstract

We introduce the Burnside chromatic polynomial of a graph that is invariant under a group action. This is a generalization of the Q -chromatic function Zaslavsky introduced for gain graphs. Given a group \mathfrak{G} acting on a graph G and a \mathfrak{G} -set X , a proper X -coloring is a function with no monochromatic edge orbit. The set of proper colorings is a \mathfrak{G} -set which induces a polynomial function from the Burnside ring of \mathfrak{G} to itself. In this paper, we study many properties of the Burnside chromatic polynomial, answering some questions of Zaslavsky.

Keywords: chromatic polynomial, Burnside ring, gain graph, polynomial function.

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1. INTRODUCTION

In [11], Zaslavsky introduces the Q -chromatic function of a gain graph. Given a gain graph Σ with gain group \mathfrak{G} , suppose that \mathfrak{G} acts on a finite set Q on the right. A Q -coloring is a coloring $f : V \rightarrow Q$ such that there is no edge $e = (u, v)$ with $f(u) = f(v)\mathbf{g}$, where \mathbf{g} is the gain on the edge e . Then the Q -chromatic function, which we denote by $P(\Sigma, Q)$, counts the number of Q -colorings. At the end of [11], Zaslavsky asks for an interpretation when we evaluate $P(\Sigma, Q)$ at ‘negative numbers’, noting that it is not clear what the correct generalization of ‘negative number’ should be. In this paper we answer Zaslavsky’s question. We also generalize Zaslavsky’s results from gain graphs to \mathfrak{G} -invariant graphs and to a more general notion of coloring.

The first key insight is to view the Q -chromatic function as a polynomial function between two rings. The codomain is the ring of integers. The domain of the Q -chromatic function is the Burnside ring of a group \mathfrak{G} , which is defined as follows. If \mathfrak{G} acts on a pair of sets X and X' on the right, then we say X and X' are equivalent if there is a bijection $f : X \rightarrow X'$ such that $f(x)\mathfrak{g} = f(x)\mathfrak{g}$ for all $\mathfrak{g} \in \mathfrak{G}$ and $x \in X$. Then the Burnside ring is the set of formal differences of equivalence classes of the resulting equivalence relation. We show that the function $P(\Sigma, Q)$ can be extended in a unique way to a polynomial function on the entire Burnside ring. Thus we can define the notion of ‘evaluating at a negative argument’ for the Q -chromatic function.

Moreover, there is a combinatorial interpretation for $P(\Sigma, -Q)$: up to sign it counts the number of acyclic colorings from $V(\Sigma)$ to Q . An orientation of an ordinary graph is *acyclic* if it has no directed cycle. Given a function $f : V(\Sigma) \rightarrow Q$, an edge e is *satisfied* if $f(u) = f(v)\mathfrak{g}$. An acyclic coloring is a pair (f, O) , where $f : V(\Sigma) \rightarrow Q$ is a coloring, and O is an acyclic orientation of the *satisfied* edges of Σ with respect to f .

Theorem 1. *Let Σ be a gain graph on n vertices with finite gain group \mathfrak{G} , and let Q be a finite set that \mathfrak{G} acts on. Then $(-1)^n P(\Sigma, -Q)$ is the number of acyclic Q -colorings.*

We know that there is a natural ring homomorphism Fix from the Burnside ring to the integers (for the precise definition see Section 2). Is there a polynomial function B from the Burnside ring to itself, such that $\text{Fix} \circ B$ is the Q -chromatic function? Is there a generalization of ‘coloring’ where the set of colorings also comes with a group action, and where we recover the usual proper colorings as fixed points? The goal of this paper is to introduce such a generalization.

This leads to our second key insight: replace gain graphs with their derived covers. The *derived cover* [1, 3] is a \mathfrak{G} -invariant graph $\tilde{\Sigma}$ that is naturally associated to a given gain graph Σ . Derived covers arise in the study of universal covers, as described in [4]. Since \mathfrak{G} acts on the vertices of $\tilde{\Sigma}$ and on Q , we can construct a group action on the set of functions from $V(\tilde{\Sigma})$ to Q .

Let G be a graph which is equipped with a group action by a finite group \mathfrak{G} . Fix a \mathfrak{G} -set X of colors on which \mathfrak{G} acts. Given a function $f : V \rightarrow X$, an edge orbit O is *monochrome* if $f(u) = f(v)$ for every edge $(u, v) \in O$. A proper X -coloring is a function $f : V \rightarrow X$ for which there is no monochrome edge orbit. The group \mathfrak{G} also acts on the set $C(G, X)$ of all X -colorings. If X and X' are equivalent, $C(G, X)$ and $C(G, X')$ are also equivalent. Therefore we can view the map $X \mapsto C(G, X)$ as a function $B_{\mathfrak{G}}(G, x)$ from the Burnside ring of \mathfrak{G} to itself. We call this function the *Burnside chromatic polynomial* of G . This new invariant is the central character in this article.

We can obtain the Q -chromatic function from the Burnside chromatic polynomial of $\tilde{\Sigma}$ by applying a linear map. Given any set X on which \mathfrak{G} acts, we let

$\text{fix}(X)$ be the number of elements of X fixed by *every* element of \mathfrak{G} . This gives rise to a linear map from the Burnside ring to \mathbb{Z} . Then for any gain graph Σ we have $P(\Sigma, Q) = \text{Fix} \circ B_{\mathfrak{G}}(\widetilde{\Sigma^{-1}}, Q)$. Here Σ^{-1} is the gain graph obtained by inverting all the gains.

We review the necessary terminology about group actions, the Burnside ring, and polynomial functions in Section 2. In Section 3 we discuss some of the theory of \mathfrak{G} -invariant graphs and gain graphs. We prove our results regarding derived covers. In Section 4, we define the Burnside chromatic polynomial. We prove that it satisfies a deletion and contraction recurrence. We also prove an inclusion-exclusion formula in Theorem 11 which generalizes one part of Theorem 4 of Zaslavsky [11]. We hold off on the exact formula until that section, as the terms on the right hand side involve generalizations of power functions.

Zaslavsky [11] also introduces fundamentally closed subgraphs of a gain graph, and uses this notion to define the lattice of flats for a gain graph. He notes that almost nothing is known about the lattice of fundamentally closed subgraphs. We define a closure operator for subsets of vertices of a \mathfrak{G} -invariant graph G . We refer to the closed elements as \mathfrak{G} -flats. The \mathfrak{G} -flats form a lattice. We prove a Möbius-inversion formula in Theorem 13 which generalizes another part of Theorem 4 of Zaslavsky [11]. We also show in Theorem 14 that the lattice of fundamentally closed subgraphs of a gain graph Σ is isomorphic to the lattice of \mathfrak{G} -flats of $\widetilde{\Sigma}$. We investigate these lattices and show that there are examples where \mathfrak{G} is a cyclic group and the lattices fail to be atomic, graded, or semi-modular.

Finally in Section 6, we prove a combinatorial reciprocity result regarding $\text{Fix} \circ B_{\mathfrak{G}}(\Sigma, Q)$. Recall that an *acyclic orientation* of an ordinary graph G is an orientation O that does not contain a directed cycle. Given a set X that \mathfrak{G} acts on, we define an *acyclic X -coloring* to be a pair (κ, O) where $\kappa : V \rightarrow X$ is a \mathfrak{G} -invariant function and O is a \mathfrak{G} -invariant acyclic orientation of the subgraph of monochrome edges.

Theorem 2. *Let G be a \mathfrak{G} -invariant graph, and let X be a \mathfrak{G} -set. Then we have $(-1)^{|V|} B_{\mathfrak{G}}^{\text{inv}}(G, -X)$ is the number of acyclic X -colorings.*

2. GROUP ACTIONS AND THE BURNSIDE RING

Let \mathfrak{G} be a finite group. If \mathfrak{G} acts on a set X , then we refer to X as a \mathfrak{G} -set. We focus on left group actions for this section. The *orbit* of a point x under \mathfrak{G} is the set $\{gx : g \in \mathfrak{G}\}$. We let $\mathfrak{G}(x)$ be the orbit of x under \mathfrak{G} . We let X/\mathfrak{G} be the collection of orbits of X under the action of \mathfrak{G} . Given $x \in X$, the *stabilizer* of x is the group $\{g \in \mathfrak{G} : gx = x\}$. Given $g \in \mathfrak{G}$, we let $\text{Fix}(g)$ be the set of fixed points of X under g . A group action is *free* if all of the stabilizer subgroups are trivial.

Finally, a subset $T \subseteq X$ is a *transversal* if $|O \cap T| = 1$ for every $O \in X/\mathfrak{G}$.

Let \mathfrak{G} be a group and let X and Y be finite \mathfrak{G} -sets. A function $f : X \rightarrow Y$ is an *equivariant map* if $f(\mathfrak{g}x) = \mathfrak{g}f(x)$ for all $x \in X$. Then f is an isomorphism if it is an equivariant bijection.

We discuss how group actions lift to multisubsets. These facts are used when we define \mathfrak{G} -invariant graphs. Let $\left(\left(\begin{smallmatrix} X \\ 2 \end{smallmatrix}\right)\right)$ be the collection of multisubsets of X of size two. If \mathfrak{G} acts on X , then \mathfrak{G} acts on $\left(\left(\begin{smallmatrix} X \\ 2 \end{smallmatrix}\right)\right)$ via $\mathfrak{g}\{a, b\} = \{\mathfrak{g}a, \mathfrak{g}b\}$. Also, given a function $f : X \rightarrow Y$, we let $\bar{f} : \left(\left(\begin{smallmatrix} X \\ 2 \end{smallmatrix}\right)\right) \rightarrow \left(\left(\begin{smallmatrix} Y \\ 2 \end{smallmatrix}\right)\right)$ be defined by $\bar{f}(\{a, b\}) = \{f(a), f(b)\}$. If f is \mathfrak{G} -equivariant, then so is \bar{f} . Similarly, if f is a bijection, then so is \bar{f} .

We define the Burnside ring of a group \mathfrak{G} . We let $B\mathfrak{G}_+$ be the set of isomorphism classes of finite right \mathfrak{G} -sets. The isomorphism classes form an additive monoid under $[X] + [Y] = [X \sqcup Y]$, where \sqcup is disjoint union. We let $B\mathfrak{G}$ be the Grothendieck group of $B\mathfrak{G}_+$, which is the abelian group of formal differences of elements of $B\mathfrak{G}_+$. The abelian group $B\mathfrak{G}$ becomes a ring with multiplication induced by $[X][Y] = [X \times Y]$, where X and Y are right \mathfrak{G} -sets. This ring is called the *Burnside ring*, and was introduced by Solomon [7].

Let M be a commutative monoid and R be a commutative ring. We define what it means for a function from M to R to be a polynomial. Given m and $n \in M$, and a function $f : M \rightarrow R$, we let $\Delta_m(f)(n) = f(n + m) - f(n)$. We refer to Δ_m as the *mth difference operator*. A function is a polynomial of degree 0 if it is constant. Then a function $f : M \rightarrow R$ is a *polynomial of degree $d > 0$* if $\Delta_m(f)$ is a polynomial of degree $d - 1$ for all $m \in M$. Let $\mathbf{P}(M, R)$ be the set of polynomial functions from M to R . Then $\mathbf{P}(M, R)$ is also a ring under pointwise multiplication and addition. Given a ring homomorphism $\varphi : R \rightarrow S$ and a polynomial function $f : M \rightarrow R$, we let $\hat{\varphi}(f) = \varphi \circ f$. We see that $\hat{\varphi}(f)$ is a polynomial function from M to S . Moreover $\hat{\varphi} : \mathbf{P}(M, R) \rightarrow \mathbf{P}(M, S)$ is a ring homomorphism.

Definition 3. A common construction we use frequently is the following. Let \mathfrak{G} and \mathfrak{H} be groups. For every \mathfrak{G} -set X , suppose we have an \mathfrak{H} -set $F(X)$. Moreover, given isomorphic \mathfrak{G} -sets X and Y , suppose that $F(X) \simeq F(Y)$. Then there is an induced function $F_{\mathfrak{G}} : B\mathfrak{G}_+ \rightarrow B\mathfrak{H}$ given by $F_{\mathfrak{G}}([X]) = [F(X)]$ for all $[X] \in B\mathfrak{G}_+$. We call $F_{\mathfrak{G}}$ the function induced by F .

In general, $\mathfrak{H} = \mathfrak{G}$ or will be the trivial group.

Now we discuss an important example of a polynomial function from $B\mathfrak{G}_+$ to $B\mathfrak{G}$, which generalizes the power function. Let N be a fixed left \mathfrak{G} -set, and given a right \mathfrak{G} -set X , let $F(X) = X^N$ be the set of all functions $f : N \rightarrow X$. This is a right \mathfrak{G} -set with action given by $f^{\mathfrak{g}}(n) = f(\mathfrak{g}n)\mathfrak{g}$. If X is isomorphic to another \mathfrak{G} -set Z , then $F(X) \simeq F(Z)$. Hence there is an induced function $F_{\mathfrak{G}}$.

We will denote this function by x^N , in analogy with the classical power function. It is a function in the variable x , and we can substitute any element $[X] \in B\mathfrak{G}_+$ in for x .

Proposition 4. *The function x^N is a polynomial of degree $|N|$.*

Proof. Let k be a positive integer. Given a sequence Y_1, \dots, Y_k of \mathfrak{G} -sets, we let $\Delta_{Y_1, \dots, Y_k}(X_N) = \Delta_{Y_1} \circ (\Delta_{Y_2, \dots, Y_k}(X^N))$. Observe that $(\Delta_Y(X^N))$ is the class of functions $f : N \rightarrow X \sqcup Y$ such that $f^{-1}(Y) \neq \emptyset$. By induction on k , we observe that $\Delta_{Y_1, \dots, Y_k}(X^N)$ is the class of the functions $f : N \rightarrow X \sqcup Y_1 \sqcup \dots \sqcup Y_k$ such that $f^{-1}(Y_i) \neq \emptyset$ for all i . In particular, when $k = |N|$, then $f^{-1}(X) = \emptyset$, so that $\Delta_{Y_1, \dots, Y_n}(x^N)$ is a polynomial of degree 0. Hence induction on $|N| - k$ allows us to conclude that $\Delta_{Y_1, \dots, Y_k}(x^N)$ is a polynomial of degree $|N| - k$. Therefore x^N is a polynomial of degree $|N|$. ■

We define a ring homomorphism which we use later when specializing the Burnside chromatic polynomial. Given a right \mathfrak{G} -set X , let $X_{\text{inv}} = |\bigcup_{\mathfrak{g} \in \mathfrak{G}} \text{Fix}(\mathfrak{g})|$. If X and Y are isomorphic right \mathfrak{G} -sets, then $X_{\text{inv}} \simeq Y_{\text{inv}}$. Moreover, for disjoint right \mathfrak{G} -sets X and Y , we have $(X \cup Y)_{\text{inv}} = X_{\text{inv}} \cup Y_{\text{inv}}$, and $(X \times Y)_{\text{inv}} = X_{\text{inv}} \times Y_{\text{inv}}$. Let the function $\text{fix} : B\mathfrak{G} \rightarrow \mathbb{Z}$ be defined by $\text{fix}([X] - [Y]) = |X_{\text{inv}}| - |Y_{\text{inv}}|$, where X and Y are \mathfrak{G} -sets. Then fix is a ring homomorphism, which we refer to as the *projection map*.

Another important example of a polynomial function is $\text{fix} \circ x^N : B\mathfrak{G}_+ \rightarrow \mathbb{Z}$, where fix is the projection map. We will use x_{inv}^N to denote this *invariant* power function. Both functions x^N and x_{inv}^N appear in formulas in the sequel.

We will construct several more examples of polynomial functions, and prove identities about them. Our main proof technique relies on the following lemma.

Lemma 5. *Let $F_{\mathfrak{G}} : B\mathfrak{G}_+ \rightarrow B\mathfrak{H}$ be a function induced by F and $E_{\mathfrak{G}} : B\mathfrak{G}_+ \rightarrow B\mathfrak{H}$ be a function induced by E . Suppose that, for every \mathfrak{G} -set X , we have $F(X) \simeq G(X)$ as \mathfrak{H} -sets. Then $F_{\mathfrak{G}} = E_{\mathfrak{G}}$. Moreover, if $F_{\mathfrak{G}}$ is a polynomial function, then so is $E_{\mathfrak{G}}$.*

Proof. Clearly for $[X] \in B\mathfrak{G}_+$, we have $F_{\mathfrak{G}}([X]) = [F(X)] = [E(X)] = E_{\mathfrak{G}}([X])$. ■

One way to prove an equality of polynomials $p(x) = q(x)$ combinatorially is to find sets $P(n)$ and $Q(n)$ such that $p(n) = |P(n)|$ and $q(n) = |Q(n)|$ for all positive integers n , along with a bijection between $P(n)$ and $Q(n)$ for all n . Essentially Lemma 5 is the \mathfrak{G} -analog for our polynomial invariants: interpret both sides of the identity as being induced functions, and find \mathfrak{H} -invariant bijections $F(X) \simeq G(X)$ for every \mathfrak{G} -set X . In most examples, either \mathfrak{H} is the trivial group, or $\mathfrak{H} = \mathfrak{G}$.

Finally we discuss how to extend a polynomial function $f : B\mathfrak{G}_+ \rightarrow R$ into a polynomial function $f : B\mathfrak{G} \rightarrow R$. Given a commutative cancellative

monoid M , the *Grothendieck group* $G(M)$ is the smallest abelian group containing M . Elements of the Grothendieck group $G(M)$ are equivalence classes of the form $a - b$, where $a, b \in M$. The equivalence relation is given by requiring $a - b = c - d$ whenever $a + d = b + c$, where $a, b, c, d \in M$. The Grothendieck group of the Burnside monoid is the Burnside ring $B\mathfrak{G}$. Given a polynomial function $f : M \rightarrow R$, there is a way to extend it to a polynomial function $\tilde{f} : G(M) \rightarrow R$. Given $a, b \in M$, we know that the b th difference operator $\Delta_b(f)$ is a polynomial of lower degree than f . Hence by induction we may assume there exists a polynomial function $g : G(M) \rightarrow R$ such that $g(m) = \Delta_b(f)(m)$ for all $m \in M$. We set $\tilde{f}(a - b) = f(a) - g(a - b)$. Induction on the degree of f can be used to prove that \tilde{f} is well-defined. Thus, we have a way to talk about evaluating a polynomial on *negative* \mathfrak{G} -sets. This corresponds to computing $f(-[X])$ where X is a \mathfrak{G} -set.

We also have the following theorem.

Theorem 6. *Let M be a commutative cancellative monoid, R be a commutative ring, and let d be a nonnegative integer. Let $f : M \rightarrow \mathbb{Z}$ and $g : M \rightarrow \mathbb{Z}$ be polynomial functions of degree d . Let $\tilde{f} : G(M) \rightarrow \mathbb{Z}$ and $\tilde{g} : G(M) \rightarrow \mathbb{Z}$ be their extensions. Then $f = g$ if and only if $\tilde{f} = \tilde{g}$.*

In particular, we always prove polynomial identities by restricting to $B\mathfrak{G}_+$.

Proof. We prove the result by induction on degree. Suppose that $f = g$. Let $x \in B\mathfrak{G}$. Then there exists y and $z \in B\mathfrak{G}_+$ such that $x = y - z$. Then $\Delta_z(f)$ has an extension to $G(M)$, which we denote by h . We have $\tilde{f}(y - z) = f(y) - h(y - z)$. We see that $\Delta_z(f) = \Delta_z(g)$, so by induction h is also an extension for $\Delta_z(g)$. Thus $\tilde{f}(y - z) = \tilde{g}(y - z)$.

The converse direction is immediate. ■

3. \mathfrak{G} -INVARIANT GRAPHS AND THEIR RELATION TO GAIN GRAPHS

In this section we define \mathfrak{G} -invariant graphs, review the definition of gain graphs, describe the derived cover construction, and show that derived covers are \mathfrak{G} -invariant graphs. We show that the derived covers of switching-equivalent gain graphs are isomorphic as \mathfrak{G} -invariant graphs. Moreover, a \mathfrak{G} -invariant graph G is the derived cover of a gain graph if and only if the group action is free on both the vertices and edges of G . Finally, we discuss deletion and contraction for \mathfrak{G} -invariant graphs and for gain graphs.

Recall that a *multigraph* G consists of a vertex set $V(G)$, an edge set $E(G)$ and an endpoint mapping $v_G : E(G) \rightarrow \left(\binom{V(G)}{2}\right)$ which sends an edge to its multiset of endpoints. A *link* is an edge with two distinct endpoints, while a *loop* has only one distinct endpoint.

Let \mathfrak{G} be a group. Then a \mathfrak{G} -invariant graph is a graph G where $V(G)$ and $E(G)$ are \mathfrak{G} -sets and $v_G : E(G) \rightarrow \left(\binom{V}{2}\right)$ is an equivariant map. An *isomorphism* $\varphi : G \rightarrow G'$ between \mathfrak{G} -invariant graphs G and G' is a pair of \mathfrak{G} -invariant bijections $\varphi_V : V(G) \rightarrow V(G')$ and $\varphi_E : E(G) \rightarrow E(G')$ such that, if $e \in E(G)$ has endpoints u and v , then $\varphi_E(e)$ has endpoints $\varphi_V(u)$ and $\varphi_V(v)$. Equivalently, $v_{G'} \circ \varphi_E = \bar{\varphi}_V \circ v_G$.

Naturally every graph is \mathfrak{G} -invariant when \mathfrak{G} is the trivial group. Given a graph G , let $\mathfrak{S}(G)$ be the set of automorphisms of G . Then G is also $\mathfrak{S}(G)$ -invariant. These form two classes of examples. A third class of examples comes from derived covers of gain graphs.

A *gain graph* consists of an underlying graph $|\Sigma|$ equipped with a *gain function* σ from an orientation of E to a group \mathfrak{G} . A gain graph is a graph where each edge is given an orientation and a gain from \mathfrak{G} . Loops are defined to have two possible orientations. Also we define $\sigma(e^{-1}) = \sigma(e)^{-1}$, where we view e^{-1} as reversing the orientation on e .

There is an operation on gain graphs called *switching*. Given a gain graph Σ and a function $\eta : V \rightarrow \mathfrak{G}$, we define Σ^η to be a gain graph with $|\Sigma^\eta| = |\Sigma|$ and with gain function σ^η defined by setting $\sigma^\eta(e) = \eta(u)^{-1}\sigma(e)\eta(w)$ for an edge e directed from u to w . Then Σ^η is the *switching* of Σ with respect to η . There are many papers which study properties or invariants of gain graphs, and often the property or invariant is preserved by switching operations. For instance, there is the notion of balance in signed graphs [5], the study of frustration index [2], and the chromatic polynomial of a gain graph [?]. One philosophy in the study of gain graphs is that the ‘interesting’ invariants or properties are those that are invariant under switching equivalence.

To any gain graph Σ we associate a natural \mathfrak{G} -invariant graph $\tilde{\Sigma}$, known in the literature as the *derived cover*. The vertex set of $\tilde{\Sigma}$ is $\mathfrak{G} \times V(\Sigma)$, and the edge set is $\mathfrak{G} \times E(\Sigma)$, where in both cases the group action is by left multiplication on the first coordinate. Given an edge e , oriented from a vertex u to a vertex w , we define $v_{\tilde{\Sigma}}(\mathfrak{g}, e) = \{(\mathfrak{g}, u), (\mathfrak{g}\sigma(e), w)\}$. The derived cover is a \mathfrak{G} -invariant graph. Our definition suggests that each orientation on e gives a different set of edges, however we also require that $(\mathfrak{g}, e^{-1}) = (\mathfrak{g}\sigma(e)^{-1}, e)$ for every edge e and every $\mathfrak{g} \in \mathfrak{G}$.

Let Σ be a gain graph, and let σ be a switching function. We show that Σ and Σ^η are isomorphic as \mathfrak{G} -invariant graphs.

Proposition 7. *Let Σ be a gain graph, and let $\eta : V \rightarrow \mathfrak{G}$. Define $\varphi_V : V(\tilde{\Sigma}) \rightarrow V(\tilde{\Sigma}^\eta)$ by $\varphi_V(\mathfrak{g}, v) = (\mathfrak{g}\eta(v), v)$ for $v \in V$. For an edge e oriented from u to w , let $\varphi_E(\mathfrak{g}, e) = (\mathfrak{g}\eta(u), e)$. Then the pair (φ_V, φ_E) is an isomorphism $\varphi : \tilde{\Sigma} \rightarrow \tilde{\Sigma}^\eta$ of \mathfrak{G} -invariant graphs.*

Proof. Observe that φ_V is an isomorphism of \mathfrak{G} -sets. Similarly, φ_E is an isomor-

phism of \mathfrak{G} -sets. Thus we need to show that $v_{\widetilde{\Sigma}^\eta} \circ \varphi_E = \overline{\varphi}_V \circ v_\Sigma$. Let $e \in E(\Sigma)$ be an edge oriented from u to v and let $\mathfrak{g} \in \mathfrak{G}$. Then $\varphi_E(\mathfrak{g}, e) = (\mathfrak{g}\eta(u), e)$ with endpoints $(\mathfrak{g}\eta(u), u)$ and $(\mathfrak{g}\eta(u)\sigma^\eta(e), v)$. However, by definition of $\sigma^\eta(e)$, the last vertex is equal to $(\mathfrak{g}\sigma(e)\eta(v), v)$. The endpoints of (\mathfrak{g}, e) are (\mathfrak{g}, u) and $(\mathfrak{g}\sigma(e), v)$, which are mapped by φ_V to $(\mathfrak{g}\eta(u), v)$ and $(\mathfrak{g}\sigma(e)\eta(v), v)$. Thus $v_{\widetilde{\Sigma}^\eta} \circ \varphi_E = \overline{\varphi}_V \circ v_{E(\widetilde{\Sigma})}$. ■

In some sense, this suggests that many ‘interesting’ gain graph invariants under switching equivalence might have extensions to \mathfrak{G} -invariant graph invariants.

We can classify which \mathfrak{G} -invariant graphs are derived covers of gain graphs.

Theorem 8. *Let \mathfrak{G} be a group and let G be a \mathfrak{G} -invariant graph. Then there exists a gain graph Σ such that $G \cong \widetilde{\Sigma}$ if and only if \mathfrak{G} acts freely on $V(G)$ and $E(G)$.*

Proof. First, suppose there exists a gain graph Σ such that $G \cong \widetilde{\Sigma}$. By construction, \mathfrak{G} acts freely on $V(\widetilde{\Sigma})$ and $E(\widetilde{\Sigma})$, so \mathfrak{G} also acts freely on $V(G)$ and $E(G)$.

Now suppose that \mathfrak{G} acts freely on $V(G)$ and $E(G)$. Let $V(\Sigma)$ be a transversal of $V(G)$ under \mathfrak{G} . Let $E/\mathfrak{G} = \{E_1, \dots, E_m\}$. Given an orbit $E_i \in E/\mathfrak{G}$, there exists at least one $e \in E_i$ such that $v_G(e) \cap V(\Sigma) \neq \emptyset$. For each orbit E_i , choose $e_i \in E_i$ such that $v_G(e_i) \cap V(\Sigma) \neq \emptyset$. Since \mathfrak{G} acts freely on $V(G)$, and $V(\Sigma)$ is a transversal, there exists at least one choice of e_i . Let $E(\Sigma) = \{e_1, \dots, e_m\}$. Thus we have defined the vertices and edges of Σ .

Now we define the endpoint function v_Σ and the gain function σ . Given $e \in E(\Sigma)$, we know $v_G(e) = \{v, w\}$ where $v \in V(\Sigma)$. Since \mathfrak{G} acts freely on $V(G)$, there exists a unique $\mathfrak{g} \in \mathfrak{G}$ such that $\mathfrak{g}^{-1}w \in V(\Sigma)$. Then set $v_\Sigma(e) = \{v, \mathfrak{g}^{-1}w\}$ and set $\sigma(e) = \mathfrak{g}$. Note that we have oriented e from v to w .

We claim that G is isomorphic to $\widetilde{\Sigma}$ as a \mathfrak{G} -invariant graph. We define the map $f_V : V(\widetilde{\Sigma}) \rightarrow V(G)$ by $f_V(\mathfrak{g}, v) = \mathfrak{g} \cdot v$. Similarly, we define the map $f_E : E(\widetilde{\Sigma}) \rightarrow E(G)$ by $f_E(\mathfrak{g}, e) = \mathfrak{g} \cdot e$. By construction, both f_V and f_E are isomorphisms of \mathfrak{G} -sets.

Let $e \in E(\Sigma)$. Then $v_G(e) = \{v, w\}$ where $v \in V(\Sigma)$. There exists a unique $\mathfrak{h} \in \mathfrak{G}$ with $\mathfrak{h}^{-1}w \in V(\Sigma)$. As an edge of $\widetilde{\Sigma}$, the edge (\mathfrak{g}, e) has endpoints (\mathfrak{g}, v) and $(\mathfrak{g}\sigma(e), \mathfrak{h}^{-1}w)$. Clearly $f_V(\mathfrak{g}, v) = \mathfrak{g}v$. Moreover, $f_E(\mathfrak{g}, e) = \mathfrak{g} \cdot e$, which has endpoints $\mathfrak{g}v$ and $\mathfrak{g}w$. It remains to show that $\mathfrak{g}w = \mathfrak{g}\sigma(e)\mathfrak{h}^{-1}w$. By definition, $\sigma(e) = \mathfrak{h}$ and the result follows. Thus (f_V, f_E) is an isomorphism of \mathfrak{G} -invariant graphs. ■

We review deletion and contraction for gain graphs. For a gain graph Σ and an edge e , $\Sigma - e$ is defined by deleting the edge e , but not its endpoints. Contraction for gain graphs is defined up to switching operations. For an edge e with $\sigma(e) = 1$, we define Σ/e by identifying the endpoints of e , and then deleting

the edge e . Given a non-loop edge e , if $\sigma(e) \neq 1$, then we first choose a switching function η such that $\sigma^\eta(e) = 1$. Then $\Sigma/e = \Sigma^\eta/e$. Note that the resulting graph depends on the choice of switching function η . However, up to switching equivalence, Σ/e is well-defined. We do not contract loops in this paper.

We define the operations of deletion and contraction for edge orbits. The definition is given regardless of whether or not the edge is a link or a loop. Our definition also does not require performing switching operations, unlike the definition for gain graphs. Let G be a \mathfrak{G} -invariant graph. Given an edge orbit $\mathfrak{G}(e) \in E/\mathfrak{G}$, we define the *deletion* $G - \mathfrak{G}(e)$ to be the \mathfrak{G} -invariant graph $(V, E \setminus \mathfrak{G}(e), v|_{E \setminus \mathfrak{G}(e)})$, where $v|_{E \setminus \mathfrak{G}(e)}$ denotes restriction. Note that we are deleting an entire orbit. For a gain graph Σ , and an edge $e \in E(\Sigma)$, we have $\widetilde{\Sigma - e} = \widetilde{\Sigma} - \mathfrak{G}(1, e)$.

Given a set $S \subseteq E/\mathfrak{G}$, we can define contraction with respect to S . First we define a symmetric relation \sim . We declare that $u \sim v$ if the vertices u and v are both endpoints of an edge e where $\mathfrak{G}(e) \in S$. Then we take the transitive closure of \sim to get an equivalence relation that we also denote by \sim . Define V/S to be the set of equivalence classes of \sim , and let $[v]$ be the equivalence class of $v \in V$. We define $E \setminus S$ to be the set of edges e for which $\mathfrak{G}(e) \notin S$. For $e \in E \setminus S$, if $v_G(e) = \{u, v\}$, then we define $v_{G/S}(e) = \{[u], [v]\}$. We define the *contraction* of G with respect to S , denoted by G/S , to be the \mathfrak{G} -invariant graph $(V/S, E \setminus S, v_{G/S})$. Given a gain graph Σ , and a non-loop edge e , we have $\widetilde{\Sigma/e} = \widetilde{\Sigma}/\mathfrak{G}(1, e)$.

In Figure 2, we show an example of deletion and contraction. Both come from the graph in Figure 1, whose vertices are $0, 1, 2, 3$. The underlying group is \mathbb{Z}_4 , acting via rotation.

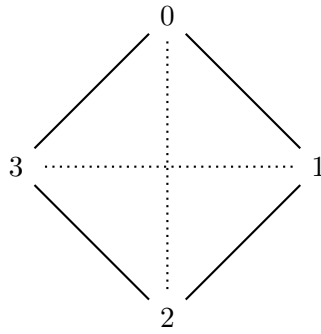


Figure 1. A \mathbb{Z}_4 -invariant graph G , with a marked edge orbit $[e]$ that is indicated by dashed lines.

Finally, a *loop orbit* is an orbit $O \subseteq E$ such that every edge in O is a loop. We will see that loop orbits play the same role for the Burnside chromatic polynomial that loops play for the chromatic polynomial.

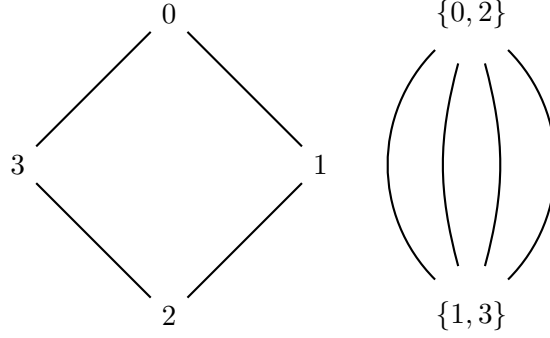


Figure 2. The deletion $G - \mathfrak{G}(e)$, and contraction $G/\mathfrak{G}(e)$.

4. CHROMATIC POLYNOMIAL

Now we define the chromatic polynomial of a \mathfrak{G} -invariant graph G , which generalizes the Q -chromatic function of Zaslavsky. The idea is to find a polynomial invariant whose domain and codomain are the Burnside ring, just as the ordinary chromatic polynomial has domain \mathbb{Z} and codomain \mathbb{Z} . Given a set X , and a function $\kappa : V \rightarrow X$, we say an edge e is *monochromatic* under κ if $\kappa(u) = \kappa(v)$, where u and v are the endpoints of e . We say an edge orbit $\mathfrak{G}(e)$ is *monochromatic* if every edge in the orbit is monochromatic. The function $\kappa : V \rightarrow X$ is a *proper coloring* if there are no monochromatic edge *orbits* under κ . It is possible for a coloring to be proper even if there are some monochromatic edges.

Suppose that X is a right \mathfrak{G} -set. Then the set $B(G, X)$ of proper colorings has a natural right \mathfrak{G} -action given by letting $\kappa^{\mathfrak{g}}(v) = \kappa(\mathfrak{g}v)\mathfrak{g}$ for any $\mathfrak{g} \in \mathfrak{G}$ and any $\kappa \in B(G, X)$.

Let X' be another \mathfrak{G} -set, and let $\varphi : X \rightarrow X'$ be an isomorphism. Given a coloring $\kappa \in B(G, X)$, it follows that $\varphi \circ \kappa \in B(G, X')$. Moreover, the two \mathfrak{G} -sets $B(G, X)$ and $B(G, X')$ are isomorphic. We let $B_{\mathfrak{G}}(G, x)$ denote the induced function from $B\mathfrak{G}_+$ to $B\mathfrak{G}$. We call $B_{\mathfrak{G}}(G, x)$ the *chromatic polynomial* due to the following theorem.

Theorem 9. *Let G be a \mathfrak{G} -invariant graph. Then $B_{\mathfrak{G}}(G, x)$ is a polynomial function of x . Moreover, $B_{\mathfrak{G}}(G, x)$ has degree 0 if and only if there is an orbit of E consisting entirely of loops. Otherwise $B_{\mathfrak{G}}(G, x)$ has degree $|V|$.*

Moreover, $B_{\mathfrak{G}}(G, x)$ can be computed recursively. If there are no edges, then $B_{\mathfrak{G}}(G, x) = x^{V(G)}$. Otherwise

$$B_{\mathfrak{G}}(G, x) = B_{\mathfrak{G}}(G - O, x) - B_{\mathfrak{G}}(G/O, x)$$

for any orbit $O \in E(G)/\mathfrak{G}$.

Proof. We prove the result by induction on the number of edges of G . First suppose that there are no edges. Then every function is a proper coloring and $B_{\mathfrak{G}}(G, x) = x^{V(G)}$.

Now we prove the deletion-contraction recurrence. Let X be a right \mathfrak{G} -set, and let $O \in E/\mathfrak{G}$. Observe that $B(G, X)$ is a \mathfrak{G} -stable subset of $B(G - O, X)$. We claim that $B(G/O, X)$ is isomorphic to $B(G - O, X) \setminus B(G, X)$ as a right \mathfrak{G} -set. Then by Lemma 5 we have $B_{\mathfrak{G}}(G/O, x) = B_{\mathfrak{G}}(G - O, x) - B_{\mathfrak{G}}(G, x)$, and the result follows.

Let $\kappa \in B(G/O, X)$. We define $\varphi(\kappa) \in B(G - O, X)$ as follows. For $v \in V(G)$, let $[v]$ be the equivalence class of v in $V(G/O)$. Then let $\varphi(\kappa)(v) = \kappa([v])$. We claim that $\varphi(\kappa)$ is a proper coloring of $G - O$. If not, then there is a monochromatic edge orbit O' in $G - O$. Let $e \in O'$ with endpoints u and v . Then $\varphi(\kappa)(u) = \varphi(\kappa)(v)$. However, this implies that $\kappa([u]) = \kappa([v])$, and that e is a monochromatic edge in G/O . Thus O' is a monochromatic edge orbit in G/O , contradicting the fact that κ is a proper coloring of G/O .

We also claim that $\varphi(\kappa)$ is *not* a proper coloring of G . This is because O is monochromatic under κ . Thus we have a function $\varphi : B(G/O, X) \rightarrow B(G - O, X) \setminus B(G, X)$. This function is a bijection, since any coloring κ of G that is proper other than the monochromatic orbit edge-orbit O induces a proper coloring of the contraction G/O . Moreover, φ is an isomorphism of right \mathfrak{G} -sets.

The deletion-contraction recurrence implies that $B_{\mathfrak{G}}(G, x)$ is a polynomial. If G has a loop orbit O , then $B(G, X) = \emptyset$ for any \mathfrak{G} -set X , as any X -coloring would leave O monochrome. Hence $B_{\mathfrak{G}}(G, x) = 0$ in that case. So suppose G has no loop orbit, and let $O \in E/\mathfrak{G}$. Then $B_{\mathfrak{G}}(G - O, x)$ also has no loop orbit, and hence by induction has degree $|V|$. Similarly, $B_{\mathfrak{G}}(G/O, x)$ will have degree that is strictly less than $|V|$, since V/O has at least one fewer vertex. Thus $B_{\mathfrak{G}}(G, x)$ also has degree $|V|$. ■

There is a specialization of the Burnside chromatic polynomial, which we call the *\mathfrak{G} -invariant chromatic polynomial*. We let $B^{\text{inv}}(G, X)$ be the set of \mathfrak{G} -invariant proper colorings of G . This comes with an action by the trivial group, so there is an induced function $B_{\mathfrak{G}}^{\text{inv}}(G, x) : B\mathfrak{G}_+ \rightarrow \mathbb{Z}$. By the way \mathfrak{G} acts on $B(G, X)$, we also have $B_{\mathfrak{G}}^{\text{inv}}(G, x) = \text{Fix} \circ B_{\mathfrak{G}}(G, x)$. The reader can verify that $B_{\mathfrak{G}}^{\text{inv}}(G, x)$ is a polynomial function. Moreover, if G has no loop orbit, then $B_{\mathfrak{G}}^{\text{inv}}(G, x)$ has degree $|V(G)/\mathfrak{G}|$. Given a right \mathfrak{G} -set Q , we see that Q is also a left \mathfrak{G} -set with action $\mathfrak{g}x = x\mathfrak{g}^{-1}$ for $\mathfrak{g} \in \mathfrak{G}$ and $x \in Q$. Then a coloring $f \in B_{\mathfrak{G}}(G, Q)$ is in $B_{\mathfrak{G}}^{\text{inv}}(G, Q)$ if and only if it is \mathfrak{G} -invariant with respect to the left action on Q .

Now we discuss how we can obtain Zaslavsky's Q -chromatic function from the Burnside chromatic polynomial. Let Σ be a gain graph with gain function σ , let Q be a right \mathfrak{G} -set, and let $f : V(\Sigma) \rightarrow Q$. We define Σ^{-1} to be the gain graph with gain function $\sigma^{-1}(e) = (\sigma(e))^{-1}$ for all $e \in E$. Naturally, given a

function $f : V(\Sigma) \rightarrow Q$, we can define another function $\tilde{f} : V(\tilde{\Sigma}) \rightarrow Q$, given by $\tilde{f}(\mathfrak{g}, u) = f(u)\mathfrak{g}$. Then \tilde{f} is a \mathfrak{G} -invariant function. Moreover, f is a proper coloring if and only if \tilde{f} is a proper coloring. Finally, let h be a \mathfrak{G} -invariant proper coloring of $\widetilde{\Sigma^{-1}}$. Define $\underline{h} : V(\Sigma) \rightarrow Q$ by $\underline{h}(v) = h(1, v)$. Then $h = (\underline{h})$, and \underline{h} is a proper coloring of Σ . Thus we have proven the following result.

Proposition 10. *Let Σ be a gain graph with gain group \mathfrak{G} . Let Q be a finite right \mathfrak{G} -set. Then $P(\Sigma, Q) = B_{\mathfrak{G}}^{\text{inv}}(\Sigma^{-1}, Q)$.*

The next result follows from a standard inclusion-exclusion argument. It generalizes Theorem 4 in [11].

Theorem 11. *Let G be a \mathfrak{G} -invariant graph. Then*

$$B_{\mathfrak{G}}(G, x) = \sum_{A \subseteq E/\mathfrak{G}} (-1)^{|A|} x^{V/A}$$

and

$$B_{\mathfrak{G}}^{\text{inv}}(G, x) = \sum_{A \subseteq E/\mathfrak{G}} (-1)^{|A|} x_{\text{inv}}^{V/A}.$$

Zaslavsky found a similar formula for $P(\Sigma, Q)$ when Σ is a gain graph. His formula is essentially given by applying our formula when $G = \Sigma$, and replacing $x^{V/A}$ with the number of fixed points of $Q^{V/A}$. We say essentially because he replaces $Q^{V/A}$ with the number of fixed points of $Q^{V(\Sigma)}$ under an action of the *fundamental group*, see [11] for details. However, in both cases one is counting the number of functions $f : V(\Sigma) \rightarrow Q$ where the edges of $A \subseteq E(\Sigma)$ are monochrome.

Proof. Given $A \subseteq E/\mathfrak{G}$, let $C(G, A, X)$ be the set of functions $\kappa : V \rightarrow X$ such that an edge orbit $O \in E/\mathfrak{G}$ is monochrome if and only if $O \in A$. If X and Y are isomorphic right \mathfrak{G} -sets, then $C(G, A, X) \simeq C(G, A, Y)$. Let $C_{\mathfrak{G}}(G, A, x)$ denote the induced function from $B\mathfrak{G}_+$ to $B\mathfrak{G}$.

Given $A \subseteq E/\mathfrak{G}$, let

$$C_A(G, X) = \bigcup_{B: A \subseteq B \subseteq E/\mathfrak{G}} C(G, B, X).$$

Then $C_A(G, X)$ is also a \mathfrak{G} -set. Moreover, if X and Y are isomorphic \mathfrak{G} -sets, then $C_A(G, X) \simeq C_A(G, Y)$. We claim that $X^{V/A} \simeq C_A(G, X)$ as right \mathfrak{G} -sets. Given $f : V/A \rightarrow X$, we define $\varphi(f) : V \rightarrow X$ by requiring $\varphi(f)(v) = f([v])$, where $[v]$ is the equivalence class of v . Let $B \subseteq E/\mathfrak{G}$ be the set of edge orbits of G that are monochrome under f . We observe that $A \subseteq B$, and thus $\varphi(f) \in C(G, B, X)$. Moreover, for $\mathfrak{g} \in \mathfrak{G}$, $\varphi(f\mathfrak{g}) = \varphi(f)\mathfrak{g}$. Thus we have defined a homomorphism of \mathfrak{G} -sets between $X^{V/A}$ and $C_A(G, X)$. The function φ has an inverse: given a set

$B \subseteq E/\mathfrak{G}$ containing A , and $f \in C(G, B, X)$, observe that f is constant on the equivalence classes of V/A . Hence $f = \varphi(h)$ for a unique $h : V/A \rightarrow X$.

Therefore by Lemma 5, we have

$$x^{V/A} = \sum_{B:A \subseteq B \subseteq E/\mathfrak{G}} C_{\mathfrak{G}}(G, B, x).$$

Then the principle of inclusion-exclusion implies that

$$C_{\mathfrak{G}}(G, A, x) = \sum_{B:A \subseteq B \subseteq E/\mathfrak{G}} (-1)^{|B|-|A|} x^{V/B}.$$

We obtain our result from observing that $B_{\mathfrak{G}}(G, x) = C_{\mathfrak{G}}(G, \emptyset, x)$.

The proof that

$$B_{\mathfrak{G}}^{\text{inv}}(G, x) = \sum_{A \subseteq E/\mathfrak{G}} (-1)^{|A|} x_{\text{inv}}^{V/A}$$

is similar. Since we have shown that $X^{V/A} \simeq C_A(G, X)$ as right \mathfrak{G} -sets, it follows that $\text{Fix}(X^{V/A}) \simeq \text{Fix}(C_A(G, X))$ as sets. Then we have

$$x_{\text{inv}}^{V/A} = \sum_{B:A \subseteq B \subseteq E/\mathfrak{G}} \text{fix}(C_{\mathfrak{G}}(G, B, x)).$$

The principle of inclusion-exclusion implies that

$$\text{Fix}(C_{\mathfrak{G}}(G, \emptyset, x)) = \sum_{A \subseteq E/\mathfrak{G}} (-1)^{|A|} x_{\text{inv}}^{V/A}.$$

Then we observe that a fixed point of $C_{\mathfrak{G}}(G, \emptyset, X)$ is a \mathfrak{G} -invariant function with no monochrome edge orbit. Hence $\text{Fix}(C_{\mathfrak{G}}(G, \emptyset, x)) = B_{\mathfrak{G}}^{\text{inv}}(G, x)$. ■

5. LATTICE OF FLATS

In this section, we discuss a generalization of the lattice of flats of a graph. We call our generalization the lattice of \mathfrak{G} -flats of the \mathfrak{G} -invariant graph G , and denote it by $L_{\text{cl}_{\mathfrak{G}}}(G)$. We derive a formula for $B_{\mathfrak{G}}(G, x)$ in terms of the Möbius function of $L_{\text{cl}_{\mathfrak{G}}}(G)$. We also prove that $L_{\text{cl}_{\mathfrak{G}}}(\tilde{\Sigma})$ is isomorphic to the lattice of *fundamentally closed* flats introduced by Zaslavsky. All undefined terminology about posets, lattices, and Möbius functions can be found in Stanley [8] or in Rota [6].

A *closure operator* on the boolean lattice is a function $\text{cl} : 2^X \rightarrow 2^X$ that satisfies all of the following conditions.

1. For $A \subseteq B \subseteq X$, we have $\text{cl}(A) \subseteq \text{cl}(B)$.
2. For $A \subseteq X$, we require $A \subseteq \text{cl}(A)$.
3. For $A \subseteq X$, we require $\text{cl}^2(A) = \text{cl}(A)$.

Given a closure operator cl , a subset A is *closed* if $\text{cl}(A) = A$. Let L_{cl} be the set of closed subsets, called *flats*, ordered by inclusion. Then L_{cl} is a lattice. We let μ be the Möbius function of the resulting lattice. We mention a result from Rota [6] which we will use.

Proposition 12. *Let cl be a closure operator on 2^X , and let A and B be closed subsets of X . Then*

$$\mu(A, B) = \sum_{C: A \subseteq C \subseteq B} (-1)^{|C| - |A|}.$$

Now we define a closure operator on E/\mathfrak{G} . Given a set $S \subseteq E/\mathfrak{G}$, an S -loop is an edge orbit $\mathfrak{G}(e) \in E/\mathfrak{G} \setminus S$ such that $[V/(S \cup \mathfrak{G}(e))] = [V/S]$. The *closure* of S is the set

$$\text{cl}_{\mathfrak{G}} S = \{\mathfrak{G}(e) : \mathfrak{G}(e) \text{ is an } S\text{-loop}\}.$$

This is a closure operator on the power set of E/\mathfrak{G} . We refer to the closed sets as \mathfrak{G} -flats. The minimum element, denoted by $\hat{0}$, is the closure of the empty set. The following is a formula for the Burnside chromatic polynomial of a \mathfrak{G} -invariant graph in terms of the lattice of flats $L_{\text{cl}_{\mathfrak{G}}}(G)$. It generalizes the corresponding formula of Zaslavsky.

Theorem 13. *Let G be a \mathfrak{G} -invariant graph with no loop orbits. Then*

$$B_{\mathfrak{G}}(G, x) = \sum_{A \in L_{\text{cl}_{\mathfrak{G}}}(G)} \mu(\hat{0}, A) x^{V/A}$$

and

$$B_{\mathfrak{G}}^{\text{inv}}(G, x) = \sum_{A \in L_{\text{cl}_{\mathfrak{G}}}(G)} \mu(\hat{0}, A) x_{\text{inv}}^{V/A}$$

Proof. We only prove the first equation. The second equation has a similar proof. Observe that, for $A \subseteq E/\mathfrak{G}$, $V/A = V/\text{cl}_{\mathfrak{G}}(A)$. Thus,

$$B_{\mathfrak{G}}(G, x) = \sum_{A \subseteq E/\mathfrak{G}} (-1)^{|A|} x^{V/A} = \sum_{A \in L(G)} \sum_{B: \text{cl}_{\mathfrak{G}}(B) = A} (-1)^{|B|} x^{V/A}.$$

Thus it suffices to show that $\sum_{B \subseteq E/\mathfrak{G}: \text{cl}_{\mathfrak{G}}(B) = A} (-1)^{|B|} = \mu(\hat{0}, A)$. The result follows from Proposition 12. ■

We discuss fundamental closure for gain graphs, which is denoted by fcl , and show how fundamental closure for a gain graph Σ is equivalent to the closure operator we just defined on $\tilde{\Sigma}$. To prevent confusion, we use cl to refer to graphic closure of a graph. For a walk $W = (e_1, \dots, e_k)$ in σ , we let $\sigma(W) = \sigma(e_1) \cdots \sigma(e_k)$. Let $A \subseteq E(\Sigma)$ and let A_1, \dots, A_m be the connected components of the graph $(V(\Sigma), A)$. For each connected component A_i , fix a rooted spanning tree T_i with root vertex r_i . For any $e \in \text{cl } A$, there exists an $i = i(e)$ such that the endpoints of e belong to A_i . We will use $i(e)$ when we need to make the dependence of i on e explicit. There is a unique minimal closed walk W_e in $T_i \cup \{e\}$ from r_i to r_i that contains both endpoints of e . Let $H(r_i, T_i) = \langle \sigma(W_e) : e \in E(A_i) \setminus E(T_i) \rangle$. Then fundamental closure is defined by

$$\text{fcl}(A) = \{e \in \text{cl } A : \sigma(W_e) \in H(r_{i(e)}, T_{i(e)})\}.$$

Zaslavsky proved that $\text{fcl}(A)$ does not depend on the choice of rooted spanning trees.

Zaslavsky shows that the fundamentally closed subsets of a gain graph form a lattice, which we denote by $L_{\text{fcl}}(\Sigma)$. We show that the resulting lattice is isomorphic to $L_{\text{cl}_{\mathfrak{G}}}(\tilde{\Sigma})$.

Theorem 14. *For a set $A \subseteq E(\Sigma)$, let $f(A) = (\mathfrak{G} \times A)/\mathfrak{G}$. Then $f : 2^{E(\Sigma)} \rightarrow 2^{E(\tilde{\Sigma})}/\mathfrak{G}$ is a bijection. Moreover, $f(\text{fcl}(A)) = \text{cl}_{\mathfrak{G}}(f(A))$, and hence $L_{\text{fcl}}(\Sigma) \simeq L_{\text{cl}_{\mathfrak{G}}}(\tilde{\Sigma})$ as lattices.*

Proof. As in the definition of fundamental closure, let A_1, \dots, A_m be the components of A . Assume we have a spanning tree T_i and a root vertex r_i in every component A_i . We let F denote the spanning forest that is the union of all the trees T_i . Given $A \subseteq \Sigma$, we let $\tilde{A} = (\mathfrak{G} \times A)/\mathfrak{G}$.

Let $\mathfrak{G}(e) \in \text{cl}_{\mathfrak{G}}(\tilde{A}) \setminus \tilde{A}$. Suppose that $(1, e)$ is directed from $(1, s)$ to $(\sigma(e), t)$ for vertices $s, t \in V(\Sigma)$. Then there exist a sequence v_1, \dots, v_k of vertices in Σ , edges e_1, \dots, e_{k-1} of $\Sigma \setminus \{e\}$, and group elements $\mathfrak{g}_1, \dots, \mathfrak{g}_k$ such that

1. (\mathfrak{g}_i, e_i) has endpoints (\mathfrak{g}_i, v_i) and $(\mathfrak{g}_{i+1}, v_{i+1})$,
2. $(\mathfrak{g}_1, v_1) = (1, s)$, and
3. $(\mathfrak{g}_k, v_k) = (\sigma(e), t)$.

We are assuming, without loss of generality, that each edge e_i is oriented from v_i to v_{i+1} . Let r be the root of the component of A containing s . Letting P be the path e_1, \dots, e_k , we have $\sigma(P) = \sigma(e)$. Observe that the walks W_{e_1}, \dots, W_{e_k} together form a walk W , and $\sigma(W) = \sigma(W_{e_1}) \cdots \sigma(W_{e_k})$. This walk W is W_e if e_i is in F for every e_i . We proceed to show that $\sigma(W) = \sigma(W_e)$ even when that condition is not met.

Since e_i and e_{i+1} share the vertex v_{i+1} , the walk W contains the closed walk L_i from v_{i+1} to r to v_{i+1} , consisting only of edges from the spanning forest F .

Thus $\sigma(L_i) = 1$. Moreover, $W = Re_1L_1e_2L_2\cdots L_{k-1}e_kQ^{-1}$ where R is the path from the root r to v_1 using only edges from F and Q is the path from the root r to v_{k+1} using only edges from F . Then $\sigma(W) = \sigma(R)\sigma(e_1)\cdots\sigma(e_k)\sigma(Q)^{-1}$. However, $\sigma(e_1)\cdots\sigma(e_k) = \sigma(e)$. Hence $\sigma(W) = \sigma(R)\sigma(e)\sigma(Q)^{-1} = \sigma(W_e)$. Thus $e \in \text{fcl}(A)$.

Observe that, given any walk W in Σ between vertices u and v , there is a corresponding walk $\mathbf{g}\widetilde{W}$ in $\widetilde{\Sigma}$ from (\mathbf{g}, u) to $(\mathbf{g}\sigma(W_e), v)$. We utilize this observation to show that $e \in \text{fcl}(A)$ implies that the endpoints of $(1, e)$ lie in the same component of $\widetilde{\Sigma}/\widetilde{A}$. Note that the derived cover of a connected graph is not necessarily connected: if Σ is a gain graph whose gains are all 1, then $\widetilde{\Sigma}$ is a disjoint union of $|\mathfrak{G}|$ copies of Σ .

Now let $e \in \text{fcl}(A) \setminus A$. Assume e is directed from s to t . Let P be the path from the root r to s , and let Q be the path from the root r to t . Since $e \in \text{fcl}(A)$, $\sigma(W_e) = \sigma(W_{e_1})\cdots\sigma(W_{e_k})$ for some edges e_1, \dots, e_k . Thus $\sigma(e) = \sigma(P)^{-1}\sigma(W_{e_1})\cdots\sigma(W_{e_k})\sigma(Q)$. Let $\mathbf{g}_i = \mathbf{g}_{i-1}\sigma(W_{e_i})$, with $\mathbf{g}_0 = \sigma(P)^{-1}$. Then $\mathbf{g}_{i-1}\widetilde{W}_{e_i}$ is a walk from (\mathbf{g}_{i-1}, r) to (\mathbf{g}_i, r) . Thus \widetilde{P}^{-1} , followed by $\mathbf{g}_0\widetilde{W}_{e_1}$, $\mathbf{g}_1\widetilde{W}_{e_2}$, \dots , $\mathbf{g}_{k-1}\widetilde{W}_{e_k}$, and ending with $\mathbf{g}_k\widetilde{Q}$ forms a walk from $(1, s)$ to $(\sigma(e), t)$, using only edges from \widetilde{A} . Hence $\mathfrak{G}(e) \in \text{cl}(\widetilde{A})$.

Thus we have shown that $\mathfrak{G}(e) \in \text{cl}_{\mathfrak{G}}(f(A))$ if and only if $e \in \text{fcl}(A)$. The latter condition is true if and only if $\mathfrak{G}(e) \in f(\text{fcl}(A))$, so we have $\text{cl}_{\mathfrak{G}}(f(A)) = f(\text{fcl}(A))$.

Moreover, if A is fundamentally closed, then $\text{cl}_{\mathfrak{G}}(f(A)) = f(\text{fcl}(A)) = f(A)$, so $f(A)$ is also closed. Similarly, if $f(A)$ is closed, then $f(\text{fcl}(A)) = \text{cl}_{\mathfrak{G}}(f(A)) = f(A)$. Since f is injective, this implies that $\text{fcl}(A) = A$. Thus when we restrict f to the fundamentally closed subsets of Σ , the result is a bijection between $L_{\text{fcl}}(\Sigma)$ and $L_{\text{cl}_{\mathfrak{G}}}(\widetilde{\Sigma})$. Moreover, if $A \subseteq B$, we have $f(A) \subseteq f(B)$. Thus we see that the restriction of f to the lattices of closed sets is an isomorphism. ■

Zaslavsky mentions that very little is known about the lattice of fundamentally closed sets for gain graphs. We mention two families of examples. First, given a finite group \mathfrak{G} , consider the gain graph $\Sigma(G)$ on one vertex which has one loop edge of gain \mathbf{g} for each $\mathbf{g} \in \mathfrak{G}$ *except* the identity element. The fundamentally closed sets of $\Sigma(\mathfrak{G})$ then correspond to subgroups of \mathfrak{G} , so $L_{\text{fcl}}(\Sigma(\mathfrak{G}))$ is the lattice of subgroups of G . Thus the lattice of fundamentally closed sets, and the lattice of \mathfrak{G} -flats generalizes both lattices of subgroups and geometric lattices.

It is known that the lattice of flats of a graph is graded, semimodular, and atomic. However, there are groups \mathfrak{G} whose subgroup lattices fail to have these properties. Thus, these properties do not always hold for the lattice of fundamentally closed edge sets of a gain graph. Of course, one may wonder if the properties hold if \mathfrak{G} is abelian. We give three examples where these conditions fail and where \mathfrak{G} is a cyclic group.

As a first example, let $\mathfrak{G} = \mathbb{Z}_4$, and let $V = \mathbb{Z}_4 \times \{a, b\}$, where \mathfrak{G} acts on the first coordinate. Let $K_{4,4}$ be the complete bipartite graph on V with bipartition given by the second coordinate; two vertices are in the same part if and only if they have the same second coordinate. Let G be the resulting \mathbb{Z}_4 -invariant graph. We show that $L_{\text{cl}\mathfrak{G}}(G)$ is not semimodular.

We let e_i be the edge with endpoints $(0, a)$ and (i, b) . Observe that $V(K_{4,4})$ has two orbits under this action. Let $G' = K_{4,4}/[e_0]$, and observe that \mathbb{Z}_4 acts transitively on $V(K_{4,4}/[e_0])$. We can label the vertices $V(K_{4,4}/[e_0])$ by 0, 1, 2, and 3 such that \mathbb{Z}_4 acts as modular addition. The graph G' appears in Figure 3. There are three edge orbits, $[e_1]$, $[e_2]$, and $[e_3]$. We see that $G'/[e_1]$ has only one vertex, and many loops. Hence $\{[e_0], [e_1]\}$ is *not* a \mathbb{Z}_4 -flat, because $K_{4,4}/\{[e_0], [e_1]\} \simeq K_{4,4}/\{E\}$.

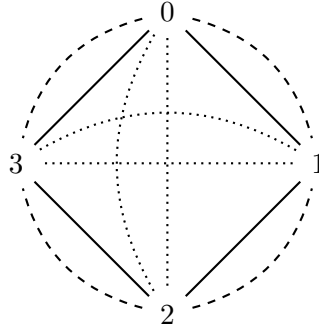


Figure 3. The graph G' . The solid edges are in $[e_1]$, the dotted edges are in $[e_2]$, and the dashed edges are in $[e_3]$.

On the other hand, in G' the edges of $[e_2]$ correspond to diagonals of the 4-cycle. There are thus two components of $G'/[e_2]$ and no loops. We observe that $\{[e_0], [e_2]\}$ is a \mathbb{Z}_4 -flat. Moreover, $K_{4,4}/[e_i]$ is isomorphic to $K_{4,4}/[e_0]$. Hence, $\{[e_i], [e_{i+2}]\}$ is a \mathbb{Z}_4 -flat for all i . We obtain the lattice on the left in Figure 4.

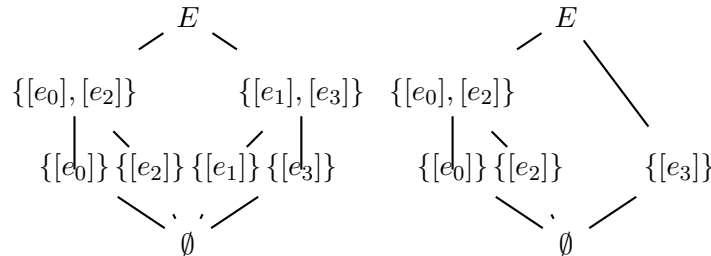


Figure 4. The lattice of \mathfrak{G} -flats for two different graphs.

This lattice is graded, but the ranks of $\{[e_1]\}$ and $\{[e_2]\}$ are both one, while the rank of their join is 3. Thus, the lattice fails to be semi-modular.

Now we give an example of a graph G whose lattice of \mathbb{Z}_4 -flats fails to be graded. Let $G = K_{4,4} - [e_1]$. Again, the graph $G/\{[e_0], [e_3]\}$ contains loops and is not a \mathbb{Z}_4 -flat. We obtain the lattice on the right in Figure 4. The lattice is not graded.

Finally, the lattice of subgroups of \mathbb{Z}_4 is not atomic. It would be interesting to determine what conditions on the group \mathfrak{G} force the lattice $L_{\text{cl}\mathfrak{G}}(G)$ to be graded for every \mathfrak{G} -invariant graph G .

6. A COMBINATORIAL RECIPROCITY THEOREM

In this section we prove a combinatorial reciprocity theorem for \mathfrak{G} -invariant chromatic polynomials similar to Stanley's theorem for acyclic orientations [9]. Unfortunately, we do not have any combinatorial reciprocity result for $B_{\mathfrak{G}}(G, -x)$. However we do have a reciprocity result for $(-1)^{|V/\mathfrak{G}|} B_{\mathfrak{G}}^{\text{inv}}(G, -x)$. Here we view $B^{\text{inv}}(G, X)$ as the set of \mathfrak{G} -invariant X -colorings.

Recall that an *acyclic orientation* of an ordinary graph H is an orientation O of H that does not contain a directed cycle. Given a \mathfrak{G} -invariant graph G and a \mathfrak{G} -set X , we define an *acyclic X -coloring* to be a pair (κ, O) where $\kappa : V \rightarrow X$ is a \mathfrak{G} -invariant function and O is a \mathfrak{G} -invariant acyclic orientation of the subgraph of monochrome edges of κ . Let $A(G, X)$ be the set of acyclic X -colorings. Then if X and Y are isomorphic as \mathfrak{G} -sets, we have $A(G, X) \simeq A(G, Y)$. Thus there is an induced function $A_{\mathfrak{G}}(G, x) : B\mathfrak{G}_+ \rightarrow \mathbb{Z}$. This function is a polynomial function in x . By definition, $A(G, X)$ is the set of acyclic X -colorings of G , and by construction $A_{\mathfrak{G}}(G, [X])$ is the *number* of acyclic X -colorings.

We state our combinatorial reciprocity theorem.

Theorem 15. *Let G be a \mathfrak{G} -invariant graph, and let X be a \mathfrak{G} -set. Then $A_{\mathfrak{G}}(G, x)$ is a polynomial function and*

$$(-1)^{|V/\mathfrak{G}|} B_{\mathfrak{G}}^{\text{inv}}(G, -x) = A_{\mathfrak{G}}(G, x).$$

First, observe that Theorem 2 follows from Theorem 15 by choosing $x \in B\mathfrak{G}_+$, as in that case $A_{\mathfrak{G}}(G, x)$ is the number of acyclic colorings. By Theorem 6, we only need to prove the result for $x \in B\mathfrak{G}_+$. Our proof relies on an identity that is similar to the binomial theorem. For a \mathfrak{G} -invariant graph G and $S \subseteq V/\mathfrak{G}$, we define the induced subgraph G_S as follows. The vertex set of G_S is $\bigcup_{[v] \in S} [v]$ and the edge set of G_S is $\{f \in E : v_G(f) \subseteq V(G_S)\}$. We also let $S^c = (V/\mathfrak{G}) \setminus S$.

Proposition 16.

$$B_{\mathfrak{G}}^{\text{inv}}(G, x + y) = \sum_{S \subseteq V/\mathfrak{G}} B_{\mathfrak{G}}^{\text{inv}}(G_S, x) B_{\mathfrak{G}}^{\text{inv}}(G_{S^c}, y).$$

Proof. By Theorem 6, it suffices to prove the result for $x \in B\mathfrak{G}_+$ and $y \in B\mathfrak{G}_+$. We prove the resulting combinatorially by applying Lemma 5, as both sides are induced functions. Given a coloring κ of G with colors in $X \cup Y$, with $X \cap Y = \emptyset$, we define $T = \kappa^{-1}(X)$. Then T is a \mathfrak{G} -invariant subset of V , since κ is \mathfrak{G} -invariant. We let $S = T/\mathfrak{G}$. Then $\kappa|_S$ is a proper \mathfrak{G} -invariant coloring of G_S , and $\kappa|_{S^c}$ is a proper \mathfrak{G} -invariant coloring of G_{S^c} . Thus we have defined a bijection ϕ from $B^{\text{inv}}(G, X \cup Y)$ to $\bigcup_{S \subseteq V/\mathfrak{G}} B^{\text{inv}}(G_S, X) \times B^{\text{inv}}(G_{S^c}, Y)$. ■

Now, fix $X \in B\mathfrak{G}_+$. Then we have

$$B_{\mathfrak{G}}^{\text{inv}}(G, [X] + y) = \sum_{S \subseteq V/\mathfrak{G}} B_{\mathfrak{G}}^{\text{inv}}(G_S, [X]) B_{\mathfrak{G}}^{\text{inv}}(G_{S^c}, y).$$

In particular, if we set $f_X(y) = B_{\mathfrak{G}}([X] + y)$, then $f_X(y)$ is a polynomial function with domain $B\mathfrak{G}_+$. Now we extend the domain of f_X to $B\mathfrak{G}$. Since the equality in Proposition 16 holds for all $y \in B\mathfrak{G}_+$, the identity also holds for all $y \in B\mathfrak{G}$.

Now we set $y = -[X]$. On one hand, we have

$$f_X(-[X]) = B_{\mathfrak{G}}^{\text{inv}}(G, [X] - [X]) = B_{\mathfrak{G}}^{\text{inv}}(G, 0) = 0.$$

On the other hand,

$$f_X(-[X]) = \sum_{S \subseteq V/\mathfrak{G}} B_{\mathfrak{G}}^{\text{inv}}(G_S, [X]) B_{\mathfrak{G}}^{\text{inv}}(G_{S^c}, -[X]).$$

Since $B_{\mathfrak{G}}^{\text{inv}}(\emptyset, [X]) = 1$ for all X , we have proven the following theorem.

Lemma 17. *Let G be a \mathfrak{G} -invariant graph with at least one vertex, and let X be a \mathfrak{G} -set. Then*

$$B_{\mathfrak{G}}(G, -[X]) = - \sum_{S \subseteq V/\mathfrak{G}, S \neq \emptyset} B_{\mathfrak{G}}^{\text{inv}}(G_S, [X]) B_{\mathfrak{G}}^{\text{inv}}(G_{S^c}, -[X]).$$

Lemma 18. *Let $A_{\mathfrak{G}}(G, X)$ be the number of acyclic X -colorings. If G has at least one vertex, then*

$$(1) \quad \sum_{S \subseteq V/\mathfrak{G}} (-1)^{|V/\mathfrak{G}| - |S|} B_{\mathfrak{G}}^{\text{inv}}(G_S, [X]) A_{\mathfrak{G}}(G_{S^c}, [X]) = 0.$$

Proof. Given an acyclic X -coloring (ν, O) of G , let $\mathcal{S}(O)$ denote the set of sources of O . Since O is \mathfrak{G} -invariant, G acts on $\mathcal{S}(O)$. Let $A_S(G, X)$ denote the set of acyclic X -colorings (ν, O) where $S \subseteq \mathcal{S}(O)/\mathfrak{G}$. We prove that $B(G_S, X) \times A(G_{S^c}, X) \simeq A_S(G, X)$.

Fix a set $S \subseteq V/\mathfrak{G}$. Then $B^{\text{inv}}(G_S, X) \times A(G_{S^c}, X)$ is the set of triples (λ, κ, O) where λ is a \mathfrak{G} -invariant X -coloring of G_S and (κ, O) is an acyclic X -coloring of G_{S^c} . Define a function $\nu : V \rightarrow X$ by $\nu(v) = \lambda(v)$ when v is in an orbit of S , and $\nu(v) = \kappa(v)$ otherwise. Then ν is also \mathfrak{G} -invariant. Moreover, any monochrome edge of ν must have an endpoint in S^c , as otherwise we would obtain a monochrome edge with respect to λ . Since the functions are \mathfrak{G} -invariant, this would imply that λ has a monochrome edge orbit, and hence was not a proper X -coloring. Thus every monochrome edge for ν must have at least one endpoint v such that $\mathfrak{G}(v) \notin S$.

Now we extend the orientation O from the monochrome edges of κ to the monochrome edges of ν . It suffices to orient monochrome edges with one endpoint in an orbit of S . Given a monochrome edge e from u to v , where u is in an orbit of S , we orient e from u to v so that $(u, v) \in O'$. We claim that O' is \mathfrak{G} -invariant. Clearly if $(u, v) \in O$ and $\mathfrak{g} \in \mathfrak{G}$, then $(\mathfrak{g}u, \mathfrak{g}v) \in O$, since O is already \mathfrak{G} -invariant. Let $(u, v) \in O' \setminus O$. By the definition of O' , $\mathfrak{G}(u) \in S$ and $\mathfrak{G}(v) \notin S$. Hence $(\mathfrak{g}u, \mathfrak{g}v) \in O'$. Thus O' is \mathfrak{G} -invariant. We have also made all the vertices in S that are endpoints of monochromatic edges into sources. Adding new sources to an acyclic digraph does not create a directed cycle, and hence O' is also acyclic. Therefore (ν, O') is an acyclic X -coloring. Moreover, every vertex v whose orbit is in S is also a source of O' . This is either because v is the endpoint of a monochromatic edge, or because vertices that are not incident to any directed edge are still technically sources. Hence $(\nu, O') \in A_S(G, X)$.

To any triple (λ, κ, O) in $B^{\text{inv}}(G_S, X)A(G_{S^c}, X)$, we have associated a pair $(\nu, O) \in T_S$ where $S \subseteq \mathcal{S}(O)/\mathfrak{G}$. It is easy to see that this construction yields a bijection between $B^{\text{inv}}(G_S, X) \times A(G_{S^c}, X)$ and $A_S(G, X)$.

Hence

$$\begin{aligned}
(2) \quad & \sum_{S \subseteq V/\mathfrak{G}} (-1)^{|V/\mathfrak{G}| - |S|} B_{\mathfrak{G}}(G_S, [X]) A_{\mathfrak{G}}(G_{S^c}, [X]) \\
(3) \quad & = (-1)^{|V/\mathfrak{G}|} \sum_{S \subseteq V/\mathfrak{G}} (-1)^{|S|} |A_S(G, X)| \\
(4) \quad & = (-1)^{|V/\mathfrak{G}|} \sum_{S \subseteq V/\mathfrak{G}} \sum_{(\nu, O) \in A_S(G, X)} (-1)^{|S|} \\
(5) \quad & = (-1)^{|V/\mathfrak{G}|} \sum_{(\nu, O) \in A(G, X)} \sum_{S \subseteq \mathcal{S}(O)/\mathfrak{G}} (-1)^{|S|}.
\end{aligned}$$

Since a fixed $(\nu, O) \in A(G, X)$ must have sources, it follows that $\mathcal{S}(O) \neq \emptyset$. Hence we have

$$\sum_{S \subseteq \mathcal{S}(O)/\mathfrak{G}} (-1)^{|S|} = 0.$$

Hence the inner summation in Equation (5) is zero, so the summation in Equation (2) is also zero. \blacksquare

Proof of Theorem 15. We prove the result by induction on the number of vertices of G . If G has more than one vertex, then we see that

$$\begin{aligned} (-1)^{|V/\mathfrak{G}|} A_{\mathfrak{G}}(G, [X]) &= - \sum_{S \subseteq V/\mathfrak{G}: S \neq \emptyset} (-1)^{|V/\mathfrak{G}| - |S|} B_{\mathfrak{G}}^{\text{inv}}(G_S, [X]) A_{\mathfrak{G}}(G_{S^c}, [X]) \\ &= - \sum_{S \subseteq V/\mathfrak{G}} B_{\mathfrak{G}}^{\text{inv}}(G_S, [X]) B_{\mathfrak{G}}^{\text{inv}}(G_{V/\mathfrak{G} \setminus S}, -[X]) \\ &= B_{\mathfrak{G}}(G, -[X]) \end{aligned}$$

where the first equality follows from Equation (1), the second equality follows from the inductive hypothesis, and third equality follows from Lemma 17. Then we have proven the identity for $[X] \in B\mathfrak{G}_+$. Thus $A_{\mathfrak{G}}(G, x)$ is a polynomial function, and the domain of $A_{\mathfrak{G}}(G, x)$ can be extended to all of $B\mathfrak{G}$. By Theorem 6, we have $(-1)^{|V/\mathfrak{G}|} A_{\mathfrak{G}}(G, x) = B_{\mathfrak{G}}(G, -x)$ for all $x \in B\mathfrak{G}$. ■

Proof Sketch of Theorem 1. Theorem 1 essentially follows from Theorem 2. The key insight is that there is a natural bijection between acyclic orientations of Σ and \mathfrak{G} -invariant acyclic orientations of $\tilde{\Sigma}$. The rest follows from Theorem 2 and the fact that $P(\Sigma, Q) = B_{\mathfrak{G}}(\tilde{\Sigma}^{-1}, Q)$. ■

7. A NEW TANTALIZING QUESTION

In the spirit of Zaslavsky [11], we also will end with a tantalizing question: is there a universal Tutte invariant for the deletion-contraction recurrence for Burnside-chromatic polynomials? What properties does it satisfy? It is no longer a matroid invariant, but certainly there is some interesting generalization.

Another interesting question is to determine which theorems for gain graphs generalize to \mathfrak{G} -invariant graphs. Here we are identifying gain graphs with their derived covers, which is sensible for theorems that hold up to switching equivalence.

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