METRIC DIMENSION AND DIAMETER
IN BIPARTITE GRAPHS

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Abstract

Let G be a connected graph and W a set of vertices of G. If every vertex of G is determined by its distances to the vertices in W, then W is said to be a resolving set. The cardinality of a minimum resolving set is called the metric dimension of G. In this paper we determine the maximum number of vertices in a bipartite graph of given metric dimension and diameter. We also determine the minimum metric dimension of a bipartite graph of given maximum degree.

Keywords: metric dimension, resolving set, diameter, maximum degree, bipartite graph.

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1. Introduction

If G is a connected graph, then a set W of vertices of G is a *resolving set* if every vertex of G is uniquely identified by its distances to the vertices in W. The minimum cardinality of a resolving set of G is called the *metric dimension* of G. These notions were introduced in papers by Slater [11] and Harary and Melter [5] and studied extensively over the past decades (see [2,4,9,10,12–14] for some recent results).

The *diameter* of a graph is the largest of the distances between its vertices. The relationship between the metric dimension, the diameter and the order of graphs was first investigated by Chartrand, Poisson and Zhang [3] and Khuller, Raghavachari and Rosenfield [8], who observed the following upper bound on the order of a graph of given metric dimension k and diameter D.

Proposition 1 [3, 8]. *For every connected graph of order n, diameter D and metric dimension k,*

$$n \leq D^k + k.$$

Unfortunately this bound is sharp only for $D = 2$. The problem of determining the maximum order of a graph of given diameter and metric dimension was solved by Hernando, Mora, Pelayo, Seara and Wood [7].

Theorem 2 [7]. *For every graph of diameter D, metric dimension k and order n we have*

$$n \leq \left(\left\lfloor \tfrac{2D}{3} \right\rfloor + 1 \right)^k + k \sum_{i=1}^{\left\lceil \frac{D}{3} \right\rceil} , (2i-1)^{k-1}.$$

and this bound is sharp.

It is natural to ask if the bound in Theorem 2 can be improved if we restrict ourselves to certain graph classes. Foucaud, Mertzios, Naserasr, Parreau, and Valicov [6] showed that this is indeed the case; specifically for interval graphs and permutation graphs they improved the bound in Theorem 2 to $\mathcal{O}(Dk^2)$, and for unit interval graphs, bipartite permutation graphs, and cographs, it was further improved to $\mathcal{O}(Dk)$. The maximum order of a tree of given diameter D and metric dimension k was determined in [1], this value is roughly $\frac{1}{8}D^2k$. For outerplanar graphs the maximum order is $\mathcal{O}(D^2k)$, and for graphs with a tree decomposition of width w and length ℓ, the order cannot exceed $\mathcal{O}(kD^2(2\ell + 1)^{3w+1})$ (see [1]).

This paper is concerned with improving the bound in Theorem 2 for bipartite graphs. We determine the maximum order of a bipartite graph of given diameter and metric dimension, thus improving the bound in Theorem 2 by a factor of roughly $\frac{1}{2^k}$ for this graph class. Our main result reads as follows.

Theorem 3. *The order n of a bipartite graph of diameter D and metric dimension k satisfies*

$$n \leq \begin{cases} k \sum_{i=0}^{(D-3)/3} (i+1)^{k-1} + \left(\tfrac{D+3}{3} \right)^k + \left(\tfrac{D}{3} \right)^k & \text{if } D \equiv 0 \pmod 3, \\ k \sum_{i=0}^{(D-4)/3} (i+1)^{k-1} + 2 \left(\tfrac{D+2}{3} \right)^k & \text{if } D \equiv 1 \pmod 3, \\ k \sum_{i=0}^{(D-2)/3} (i+1)^{k-1} + 2 \left(\tfrac{D+1}{3} \right)^k & \text{if } D \equiv 2 \pmod 3, \end{cases}$$

and this bound is sharp.

We also relate the metric dimension to the *maximum degree*, defined as the largest of the degrees of the vertices, in bipartite graphs. Chartrand, Poisson and Zhang [3] proved that the metric dimension of a graph of maximum degree Δ is at least $\log_3(\Delta(G) + 1)$. We prove in this paper that for bipartite graphs this bound can be improved to $\lceil \log_2(\Delta(G)) \rceil$, and that this bound is sharp.

2. Notation

In this paper G denotes a connected graph with vertex set V. A graph G is *bipartite* if its vertex set can be partitioned into two sets V_1 and V_2 such that every edge joins a vertex in V_1 to a vertex in V_2; these two sets are referred to as the *partite sets* of G. The *order* of G, i.e., the number of vertices of G, is denoted by n.

The *neighbourhood* $N(v)$ of a vertex v is the set of all vertices that are adjacent to v, and the *degree* of v is defined as $|N(v)|$. The *maximum degree* of G, denoted by $\Delta(G)$, is the largest of the degrees of the vertices of G.

The *distance* $d(u,v)$ between two vertices u and v of G is the length of a shortest (u,v)-path in G.

If W is a k-set $\{w_1, w_2, \ldots, w_k\}$ of vertices of G with an imposed ordering, then the *metric representation* of a vertex v with respect to W is the k-tuple $(d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$. If W is a resolving set of G, then every vertex of G has a unique metric representation. The metric dimension, i.e., the minimum cardinality of a resolving set of G, is denoted by k.

If a and b are integers with $a \leq b$, then $[a,b]$ denotes the set of all integers x with $a \leq x \leq b$. For a set M and a positive integer ℓ, we denote the set of ℓ-tuples of elements of M by M^ℓ.

3. Maximum Order of a Bipartite Graph of Given Metric Dimension and Diameter

In this section we prove the main result of our paper, Theorem 4. Our proof is similar to the proof of Theorem 2 in [7], but requires additional arguments.

We split the proof of our main result into two parts. We first prove the upper bound on the order, and then we construct bipartite graphs that show that this bound is sharp.

Theorem 4. *For every bipartite graph of diameter D, metric dimension k and order n we have*

$$n \leq \begin{cases} k \sum\limits_{i=0}^{(D-3)/3} (i+1)^{k-1} + \left(\frac{D+3}{3}\right)^k + \left(\frac{D}{3}\right)^k & \text{if } D \equiv 0 \pmod 3, \\[4mm] k \sum\limits_{i=0}^{(D-4)/3} (i+1)^{k-1} + 2\left(\frac{D+2}{3}\right)^k & \text{if } D \equiv 1 \pmod 3, \\[4mm] k \sum\limits_{i=0}^{(D-2)/3} (i+1)^{k-1} + 2\left(\frac{D+1}{3}\right)^k & \text{if } D \equiv 2 \pmod 3. \end{cases}$$

Proof. Let G be a bipartite graph of order n, diameter D and metric dimension k with partite sets V_1 and V_2. Let $W = \{w_1, \ldots, w_k\}$ be a resolving set. Without loss of generality, assume $w_1, w_2, \ldots, w_a \in V_1$ and $w_{a+1}, w_{a+2}, \ldots, w_k \in V_2$. Note that $a = 0$ if $W \subseteq V_2$, and $a = k$ if $W \subseteq V_1$. For each vertex $v \in W$ and $i \in \mathbb{N} \cup \{0\}$ we define $N_i(v) = \{x \in V(G) \mid d(v, x) = i\}$. We fix an $s \in \{0, 1, \ldots, D\}$ whose value will be specified later, and we partition V into two sets R and S defined as follows.

$$R = \{v \in V \mid d(v, w_i) \leq s \text{ for some } i\},$$

$$S = \{v \in V \mid d(v, w_i) > s \text{ for all } i\}.$$

We bound the cardinalities of R and S separately. We first prove that

$$(1) \qquad\qquad |R| \leq k \sum_{i=0}^{s} (i+1)^{k-1}.$$

For $1 \leq p \leq k$ consider $N_i(w_p)$ and let $x, y \in N_i(w_p)$. There exist paths of length i from x to w_p and from y to w_p, thus $d(x, y) \leq 2i$. Consider w_q for $1 \leq q \leq k$ and $w_q \neq w_p$. Then $|d(x, w_q) - d(y, w_q)| \leq 2i$. Since all $x \in N_i(w_p)$ are in the same partite set, the distance $d(x, w_q)$ has the same parity for all $x \in N_i(w_p)$, so we have at most $\left\lceil \frac{2i+1}{2} \right\rceil = i + 1$ possible values for the distance to w_q. Hence,

$$|N_i(w_p)| \leq (i+1)^{k-1}.$$

Thus

$$(2) \qquad\qquad |R| \leq \sum_{p=1}^{k} \sum_{i=0}^{s} |N_i(w_p)| \leq k \sum_{i=0}^{s} (i+1)^{k-1},$$

which is (1).

We now bound $|S|$ as follows

$$(3) \qquad\qquad |S| \leq \begin{cases} 2\left(\frac{D-s}{2}\right)^k & \text{if } D - s \text{ is even,} \\[3mm] \left(\frac{D-s+1}{2}\right)^k + \left(\frac{D-s-1}{2}\right)^k & \text{if } D - s \text{ is odd.} \end{cases}$$

Let $v \in S$. Then $d(v, w_p) \geq s + 1$ for all $1 \leq p \leq k$. Moreover, since $w_1, \ldots, w_a \in V_1$ and $w_{a+1}, \ldots, w_k \in V_2$, the distances $d(w_1, v), \ldots, d(w_a, v)$ are all even and the distances $d(w_{a+1}, v), \ldots, d(w_k, v)$ are all odd, or vice versa. Hence, the metric representation $(x_1, \ldots, x_k)$ of v satisfies

$$(4) \qquad x_1 \equiv x_2 \equiv \cdots \equiv x_a, \quad x_{a+1} \equiv x_{a+2} \equiv \cdots \equiv x_k, \quad \text{and} \quad x_a \not\equiv x_{a+1} \pmod 2.$$

Define

$$M = \left\{ (x_1, x_2, \ldots, x_k) \in [s+1, D]^k \mid (x_1, \ldots, x_k) \text{ satisfies } (4) \right\}.$$

Denoting by $r(S|W)$ the set of metric representations of the vertices of S, it follows from the above that $r(S|W) \subseteq M$, and consequently

$$(5) \qquad\qquad\qquad\qquad |S| = |r(S|W)| \leq |M|.$$

In order to bound $|M|$ we consider the following cases.

Case 1. $D - s$ is even. There are $D - s$ possible values for x_1. Once x_1 has been chosen, there are $\frac{D-s}{2}$ values for each of $x_2, x_3, \ldots, x_k$. Thus $|M| = 2\left(\frac{D-s}{2}\right)^k$, and by (5),

$$|S| \leq 2\left(\frac{D-s}{2}\right)^k,$$

and (3) follows.

Case 2. $D - s$ is odd. The set $[s + 1, D]$ contains $D - s$ possible values for x_1. Of these, $\frac{D-s+1}{2}$ are of the same parity as $s + 1$, and $\frac{D-s-1}{2}$ are of the same parity as $s + 2$. Hence, if $x_1 \equiv s + 1 \pmod 2$, there are $\frac{D-s+1}{2}$ choices for each of $x_1, x_2, \ldots, x_a$, and $\frac{D-s-1}{2}$ choices for each of $x_{a+1}, x_{a+2}, \ldots, x_k$, which yields a total of $\left(\frac{D-s+1}{2}\right)^a \left(\frac{D-s-1}{2}\right)^{k-a}$ choices for $(x_1, \ldots, x_k)$. Similarly, if $x_1 \equiv s + 2 \pmod 2$, we have a total of $\left(\frac{D-s-1}{2}\right)^a \left(\frac{D-s+1}{2}\right)^{k-a}$ choices for $(x_1, \ldots, x_k)$. Therefore,

$$(6) \quad |M| = \left(\frac{D-s+1}{2}\right)^a \left(\frac{D-s-1}{2}\right)^{k-a} + \left(\frac{D-s-1}{2}\right)^a \left(\frac{D-s+1}{2}\right)^{k-a}.$$

The last expression is maximised for $a \in \{0, 1, \ldots, k\}$ if $a = 0$ or $a = k$. Substituting this we obtain $|M| \leq \left(\frac{D-s+1}{2}\right)^k + \left(\frac{D-s-1}{2}\right)^k$ and thus, by (5),

$$|S| \leq \left(\frac{D-s+1}{2}\right)^k + \left(\frac{D-s-1}{2}\right)^k,$$

and (3) also follows in this case.

We now choose the value of s to be $s = \left\lfloor \frac{D-2}{3} \right\rfloor$. Then

$$
D - s = \begin{cases}
2\frac{D}{3} + 1 & \text{if } D \equiv 0 \pmod 3, \\
2\frac{D-1}{3} + 2 & \text{if } D \equiv 1 \pmod 3, \\
2\frac{D-2}{3} + 2 & \text{if } D \equiv 2 \pmod 3.
\end{cases}
$$

Substituting this value for s into (1) and (3), and noting that $D - s$ is odd if $D \equiv 0 \pmod 3$, and even if $D \equiv 1, 2 \pmod 3$, we get

$$
n = |R| + |S| = \begin{cases}
k \sum\limits_{i=0}^{(D-3)/3} (i+1)^{k-1} + \left(\frac{D+3}{3}\right)^k + \left(\frac{D}{3}\right)^k & \text{if } D \equiv 0 \pmod 3, \\
k \sum\limits_{i=0}^{(D-4)/3} (i+1)^{k-1} + 2\left(\frac{D+2}{3}\right)^k & \text{if } D \equiv 1 \pmod 3, \\
k \sum\limits_{i=0}^{(D-2)/3} (i+1)^{k-1} + 2\left(\frac{D+1}{3}\right)^k & \text{if } D \equiv 2 \pmod 3,
\end{cases}
$$

as desired.   ∎

Theorem 4 is sharp. To show this, we construct for given $k, D \in \mathbb{N}$ with $k \geq 2$ and $D \geq 3$ a graph $G_{k,D}$ that attains the bound in Theorem 4. Define $\mathbb{Z}_k^*$ to be the set of all k-tuples of integers in which all coordinates have the same parity. Define integers A and B by

$$
A = \left\lfloor \frac{D-2}{3} \right\rfloor + 1 \quad \text{and} \quad B = 2\left\lfloor \frac{D-2}{3} \right\rfloor + 2 = 2A.
$$

Note that B is even for all D. Let

$$
Q = [A, D]^k \cap \mathbb{Z}_k^*.
$$

For each $1 \leq i \leq k$ and $0 \leq r \leq A - 1$, let

$$
P_{i,r} = \{(x_1, \ldots, x_{i-1}, r, x_{i+1}, \ldots, x_k) \in \mathbb{Z}_k^* \mid B - r \leq x_j \leq B + r \text{ for } j \neq i\},
$$
$$
P_i = \bigcup_{r=0}^{A-1} P_{i,r},
$$
$$
P = \bigcup_{i=1}^{k} P_i.
$$

Define $G_{k,D}$ to be the graph with vertex set $V(G) = P \cup Q$, where two vertices $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ are adjacent if and only if $|x_i - y_i| = 1$, for all $i \in \{1, 2, \ldots, k\}$.

From the definition of adjacency it is clear that $G_{k,D}$ is bipartite. We show in a sequence of lemmas that the graph $G_{k,D}$ has metric dimension k, maximum degree D, and that its order equals the upper bound in Theorem 4.

The graphs $G_{2,6}$, $G_{2,7}$ and $G_{2,8}$ are shown in Figure 1. The "square" shaded region in $G_{2,6}$, $G_{2,7}$ and $G_{2,8}$ represents Q in our construction, and the "triangle" shaded regions represent P_1 and P_2. The solid vertices form the metric basis $\{w_1, w_2\}$.
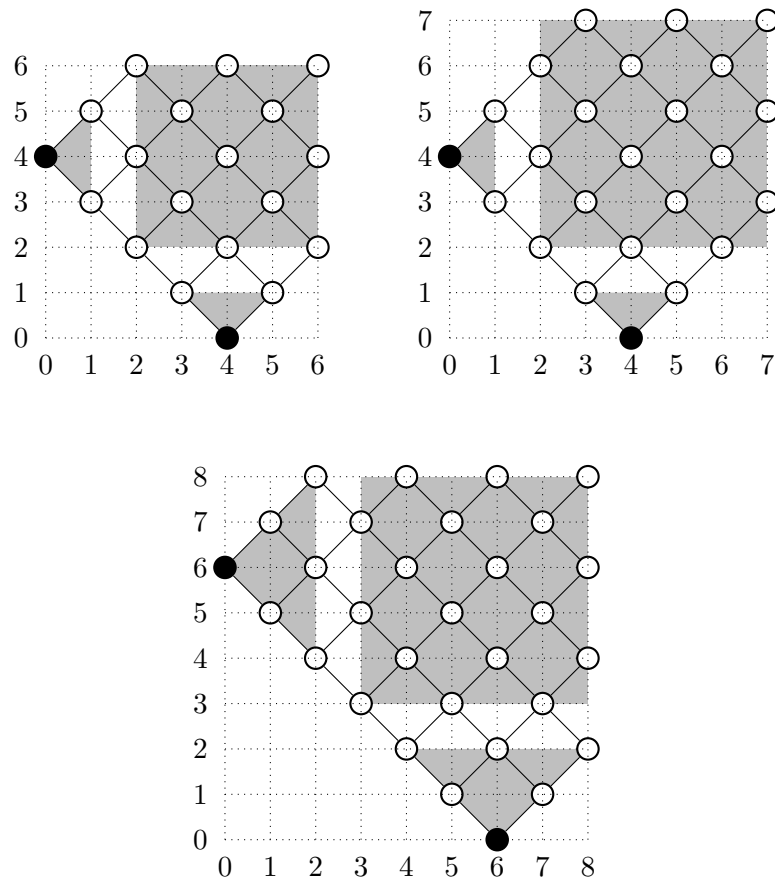

Figure 1. The graphs $G_{2,6}$, $G_{2,7}$ and $G_{2,8}$.

Lemma 5. *Let $G_{k,D}$ be the graph constructed above. Then*

$$|V(G_{k,D})| = \begin{cases} k \sum\limits_{i=0}^{(D-3)/3} (i+1)^{k-1} + \left(\frac{D+3}{3}\right)^k + \left(\frac{D}{3}\right)^k & \text{if } D \equiv 0 \pmod 3, \\[3mm] k \sum\limits_{i=0}^{(D-4)/3} (i+1)^{k-1} + 2\left(\frac{D+2}{3}\right)^k & \text{if } D \equiv 1 \pmod 3, \\[3mm] k \sum\limits_{i=0}^{(D-2)/3} (i+1)^{k-1} + 2\left(\frac{D+1}{3}\right)^k & \text{if } D \equiv 2 \pmod 3. \end{cases}$$

Proof. Let $i \in [1,k]$ be fixed. Then the sets $P_{i,0}, P_{i,1}, \ldots, P_{i,A-1}$ are disjoint since each vertex in $P_{i,r}$ has r in the ith coordinate. Also, for distinct $i, j \in [1,k]$ the sets P_i and P_j are disjoint since each vertex in P_j has an ith coordinate of at least $B - r \geq B - (A-1) = B - A + 1$, and each vertex in P_i has an ith coordinate of at most $r \leq A - 1 < B - (A-1) = B - A + 1$. Finally, P and Q are disjoint since each coordinate of each vertex in P has a value less than A, while the vertices in Q have each coordinate at least A. Hence

$$(7) \qquad |V(G_{k,D})| = |P| + |Q| = \sum_{i=1}^{k} \sum_{r=0}^{A-1} |P_{i,r}| + |Q|.$$

To determine the number of vertices in $P_{i,r}$ observe that since B is even, $[B - r, B + r]$ contains $r + 1$ integers of the same parity as r. Hence, for all $i \in [1,k]$ and $r \in [0, A-1]$,

$$(8) \qquad |P_{i,r}| = (r+1)^{k-1}.$$

The same reasoning as in the computation of $|M|$ in the proof of Theorem 4 yields the cardinality of Q as

$$(9) \qquad |Q| = \begin{cases} 2\left(\frac{D-A+1}{2}\right)^k & \text{if } D - A + 1 \text{ is even}, \\[3mm] \left(\frac{D-A+2}{2}\right)^k + \left(\frac{D-A}{2}\right)^k & \text{if } D - A + 1 \text{ is odd}. \end{cases}$$

Depending on the value of D modulo 3, we obtain the following parities for $D - A + 1$:

If $D \equiv 0 \pmod 3$, then $A = \frac{D}{3}$, and so $D - A + 1 = 2\frac{D}{3} + 1$, which is odd.
If $D \equiv 0 \pmod 3$, then $A = \frac{D-1}{3}$, and so $D - A + 1 = 2\frac{D+2}{3}$, which is even.
If $D \equiv 0 \pmod 3$, then $A = \frac{D+1}{3}$, and so $D - A + 1 = 2\frac{D+1}{3}$, which is even.

Combining equations (7), (8)) and (9) we now obtain the desired value of $|V(G_{k,D})|$ in all three cases. ∎

The next two lemmas, which are very similar to lemmas in [7], prove that G has diameter D. The first lemma is proved by an exhaustive analysis of the different cases (i) $x \in Q$ and $y \in P$, (ii) $x \in P$ and $y \in Q$, (iii) $x \in Q$ and $y \in Q$, (iv) $x \in P$ and $y \in P$. We leave the details of this proof to the reader.

Lemma 6. *Let $G_{k,D}$ be as defined above. For two distinct k-tuples of integers $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$, define $z(x,y)$ to be the k-tuple $z = (z_1, \ldots, z_k)$ with*

$$z_i = \begin{cases} x_i + 1 & \text{if } x_i \leq y_i \text{ and } x_i \neq D, \\ x_i - 1 & \text{if } x_i > y_i \text{ and } x_i \neq D, \\ x_i - 1 & \text{if } x_i = D. \end{cases}$$

Then $z(x,y) \in V(G_{k,D})$ whenever $x, y \in V(G_{k,D})$.

With the use of Lemma 6 we show that G has diameter D.

Lemma 7. *For any two vertices $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ of $G_{k,D}$ we have*

$$d(x,y) = \max\{|y_i - x_i| \,\big|\, i \in [1,k]\} \leq D.$$

Proof. Let $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ be two vertices of $G_{k,D}$. We have $d(x,y) \geq \max_i |y_i - x_i|$ since every two vertices on this (x,y)-path have their i^{th} coordinates differing by at most 1, for all i.

We prove the converse inequality, $d(x,y) \leq \max_i |y_i - x_i|$, by induction on $\max_i |y_i - x_i|$. If $\max_i |y_i - x_i| = 1$, then x and y are adjacent by the construction of $G_{k,D}$, so $d(x,y) = 1 = \max_i |y_i - x_i|$. Assume that $\max_i |y_i - x_i| \geq 2$. Then it is easy to see from the definition of $z = (z_1, \ldots, z_k)$ that $\max_i |y_i - z_i| = \max_i |y_i - x_i| - 1$. Since $z \in V(G_{k,D})$ by Lemma 6, we have $d(z,y) \leq \max_i |y_i - x_i| - 1$. Since x and z are adjacent, we conclude $d(x,y) \leq \max_i |y_i - x_i|$, as desired. ∎

Lemma 7 implies that D is the diameter of G. Indeed, since the vertices of $G_{k,D}$ are k-tuples from $[0,D]^k$, Lemma 7 yields that the diameter of $G_{k,D}$ is at most D. On the other hand, the two vertices $(0, B, \ldots, B)$ and $(D, D, \ldots, D)$ are at distance D, so the diameter of $G_{k,D}$ is at least D.

For the final lemma, we will show that $G_{k,D}$ has metric dimension k.

Let $W = \{w_1, \ldots, w_k\}$, where w_i is the vector whose i^{th} position equals 0, and the remaining positions equal B. The following lemma shows that W is a resolving set of $G_{k,D}$.

Lemma 8. *For $i \in \{1, 2, \ldots, k\}$ let w_i be as defined above. Then for every vertex $x = (x_1, \ldots, x_k)$ of $G_{k,D}$ we have $d(x, w_i) = x_i$.*

Proof. Let $w_{i,j}$ be the jth coordinate of w_i, that is $w_{i,i} = 0$ and $w_{i,j} = B$ for $j \neq i$. Then, by Lemma 7,

$$\begin{aligned} d(x, w_i) &= \max\big\{|w_{i,j} - x_j| \,\big|\, j \in [1,k]\big\} \\ &= \max\big\{\max\{|w_{i,i} - x_i|\}, \max\{|w_{i,j} - x_j| \,\big|\, i,j \in [1,k], j \neq i\}\big\} \\ &= \max\big\{x_i, \max\{|B - x_j| \,\big|\, i,j \in [1,k], j \neq i\}\big\}. \end{aligned}$$

An exhaustive case-analysis now shows that $|B - x_j| \leq x_i$ for all $j \neq i$. It thus follows that $\max\{x_i, \max\{|B - x_j| \,\big|\, i, j \in [1, k], j \neq i\} = x_i$ and the lemma follows. ∎

Since W is a resolving set for $G_{k,D}$ of cardinality k, it follows that the metric dimension of $G_{k,D}$ is at most k. On the other hand, since the bound in Theorem 4 is strictly increasing in k, and since the graph $G_{k,D}$ attains this bound, we conclude that its metric dimension equals k. This concludes our proof that the bound in Theorem 4 is sharp.

4. A Lower Bound on the Metric Dimension in Terms of Maximum Degree

In this section we consider a relation between metric dimension and maximum degree. Chartrand *et al.* [3] provided the following lower bound on metric dimension in terms of maximum degree.

Theorem 9 [3]. *Let G be a nontrivial connected graph, with metric dimension k and maximum degree $\Delta(G)$. Then*

$$k \geq \log_3(\Delta(G) + 1),$$

and this bound is sharp.

We now show that for bipartite graphs, Theorem 9 can be improved by a factor of about $\log_2(3) \approx 1.58$.

Theorem 10. *Let G be a connected bipartite graph, with metric dimension k and maximum degree $\Delta(G)$. Then*

$$k \geq \lceil \log_2(\Delta(G)) \rceil,$$

and this bound is sharp.

Proof. Let G be a bipartite graph with partite sets V_1 and V_2, and let $W = \{w_1, \ldots, w_k\}$ be a minimum resolving set. Let v be a vertex of degree $\Delta(G)$ and $N(v)$ be the neighbourhood of v. Without loss of generality, suppose $v \in V_1$. Then $N(v) \subseteq V_2$. Since G is bipartite, it follows that for every vertex $u \in N(v)$,

$$d(u, w_i) \in \{d(v, w_i) - 1, d(v, w_i) + 1\}$$

for $1 \leq i \leq k$. Therefore, $d(u, w_i)$ has a range of two possible numbers. Since the metric representation of u has k entries, there are at most 2^k distinct representations of the vertices in $N(v)$. Hence, $|N(v)| = \Delta(G) \leq 2^k$, and so

$$k \geq \log_2(\Delta(G)).$$

Since k is an integer, the desired bound follows.

We now prove that the bound in Theorem 10 is sharp. Let $\Delta \in \mathbb{N}$ with $\Delta \geq 2$ be given and let $k = \lceil \log_2(\Delta(G)) \rceil$. We construct a bipartite graph G_Δ with maximum degree Δ and metric dimension k.

Let the partite sets of G_Δ be $V_1 = \{w_0, w_1, \ldots, w_{k-1}\} \cup \{v\}$ and $V_2 = \{u_0, \ldots, u_{\Delta-1}\}$. Vertex v is adjacent to every vertex in V_2. For the edges between the vertices in $\{u_0, \ldots, u_{\Delta-1}\}$ and $\{w_0, w_1, \ldots, w_{k-1}\}$ note that each integer $i \in \{0, 1, \ldots, \Delta - 1\}$ has a unique binary representation $i = \sum_{j=0}^{k-1} a_j(i)2^j$ with $a_j(i) \in \{0, 1\}$ for $j = 0, 1, \ldots, k - 1$. We define u_i to be adjacent to w_j if and only if the coefficient $a_j(i)$ of 2^j in the binary representation of i equals 1. So u_0 is not adjacent to any vertex in $\{w_0, w_1, \ldots, w_{k-1}\}$, u_1 is adjacent only to w_0, while u_2 is adjacent only to w_1, and u_3 is adjacent only to w_0 and w_1, and so on. The graph G_5 is shown in Figure 2.
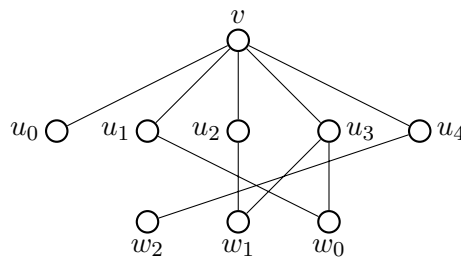

Figure 2. The graph G_5 constructed in the proof of Theorem 10.

Clearly, G_Δ is connected and has maximum degree Δ. We now show that G_Δ has metric dimension k. It follows from Theorem 10 that the metric dimension of G_Δ is at least k. Hence it suffices to prove that the set $W = \{w_0, w_1, \ldots, w_{k-1}\}$ is a resolving set of G_Δ. Since no two numbers share the same binary representation, the vertices $u_0, u_1, \ldots, u_{\Delta-1}$ have different metric representations with respect to W. Since v is the only vertex that has distance 2 from every vertex of W, and since the vertices of W are the only vertices with zeros in their metric representations, it follows that W forms a resolving set. Hence G_Δ has metric dimension k, as desired. ∎

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