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A DECOMPOSITION FOR DIGRAPHS WITH MINIMUM OUTDEGREE 3 HAVING NO VERTEX DISJOINT CYCLES OF DIFFERENT LENGTHS

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Abstract

We say that a digraph $D = (V, A)$ admits a good decomposition $D = D_1 \cup D_2 \cup D_3$ if $D_1 = (V_1, A_1), D_2 = (V_2, A_2)$ and $D_3 = (V_3, A_3)$ are such subdigraphs of D that $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$, $V_2 \neq \emptyset$ but V_1 may be empty, D_1 is the subdigraph of D induced by V_1 and is an acyclic digraph, D_2 is the subdigraph of D induced by V_2 and is a strong digraph and D_3 is a subdigraph of D, every arc of which has its tail in V_1 and its head in V_2. In this paper, we show that a digraph $D = (V, A)$ with minimum outdegree 3 has no vertex disjoint directed cycles of different lengths if and only if D admits a good decomposition $D = D_1 \cup D_2 \cup D_3$, where $D_1 = (V_1, A_1), D_2 = (V_2, A_2)$ and $D_3 = (V_3, A_3)$ are such that D_2 has minimum outdegree 3 and no vertex disjoint directed cycles of different lengths and for every vertex $v \in V_1$, $d^+_{D_1 \cup D_3}(v) \geq 3$. Moreover, when such a good decomposition for D exists, it is unique. By these results, the investigation of digraphs with minimum outdegree 3 having no vertex disjoint directed cycles of different lengths can be reduced to the investigation of strong such digraphs. Further, we classify strong digraphs with minimum outdegree 3 and girth 2 having no vertex disjoint directed cycles of different lengths.

Keywords: digraph with minimum outdegree 3, vertex disjoint cycles, cycles of different lengths, acyclic digraph, strong digraph.

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1. Introduction

In this paper, the term digraph always means a *finite simple digraph*, i.e., a digraph that has a finite number of vertices, no loops and no multiple arcs. Unless otherwise indicated, our graph-theoretic terminology will follow [1].

Let D be a digraph. Then the vertex set and the arc set of D are denoted by $V(D)$ and $A(D)$ (or by V and A for short), respectively. A vertex $v \in V$ is called an *outneighbor* of a vertex $u \in V$ if $(u, v) \in A$. We denote the set of all outneighbors of u by $N_D^+(u)$. The *outdegree* of $u \in V$, denoted by $d_D^+(u)$, is $|N_D^+(u)|$. The *minimum outdegree* of D is $\min\{d_D^+(u) \mid u \in V\}$. Similarly, a vertex $w \in V$ is called an *inneighbor* of a vertex $u \in V$ if $(w, u) \in A$. We denote the set of all inneighbors of u by $N_D^-(u)$. The *indegree* of $u \in V$, denoted by $d_D^-(u)$, is $|N_D^-(u)|$.

Let $D = (V, A)$ be a digraph. Then we write uv for an arc $(u, v) \in A$ for short. By a *cycle* and a *path* in D we always mean a directed cycle and a directed path, respectively. By *disjoint cycles* in D we always mean vertex disjoint cycles. The *girth* of D is the length of a shortest cycle in D. For a subset $W \subseteq V$, the subdigraph of D induced by W is denoted by $D[W]$.

For a natural number k, all integers modulo k are $0, 1, 2, \ldots, k-1$. A digraph $D = (V, A)$ is called *k-regular* if $d_D^+(v) = d_D^-(v) = k$ for every vertex $v \in V$. A digraph is called *acyclic* if it has no cycles. If $C = v_0, v_1, \ldots, v_{m-1}, v_0$ is a cycle of length m in D and $v_i, v_j \in V(C)$, then v_iCv_j denotes the sequence $v_i, v_{i+1}, v_{i+2}, \ldots, v_j$, where all indices are taken modulo m. We will consider v_iCv_j both as a path and as a vertex set. Similar notation as described above for a cycle is also used for a path. A digraph $D = (V, A)$ is called *strong* if for every pair x, y of distinct vertices in D there exist both a path from x to y and a path from y to x. A digraph with only one vertex is considered to be strong. A *strong component* of a digraph D is a maximal induced subdigraph of D which is strong. A strong component of D is called *trivial* if it consists of only one vertex; otherwise, it is called *nontrivial.* The *strong component digraph* $SC(D)$ of D is obtained by contracting each strong component of D into a single vertex and deleting any parallel arcs obtained in this process. In other words, if $C_1, C_2, \ldots, C_t$ are the strong components of D, then $V(SC(D)) = \{v_1, v_2, \ldots, v_t\}$ and $A(SC(D)) = \{v_iv_j \mid \text{There is an arc } u_iu_j \in A \text{ with } u_i \in V(C_i) \text{ and } u_j \in V(C_j)\}$. Then $SC(D)$ is acyclic and therefore there exists a vertex in $SC(D)$ with outdegree 0. A strong component C of D is called *terminal* if the corresponding vertex in $SC(D)$ of C has outdegree 0. Thus, any digraph D has a terminal strong component.

Researchers have been interested in conditions for the existence of disjoint cycles in digraphs since long ago. Among many results about this problem, we mention here the following nice one that gives a condition to guarantee the existence of two disjoint cycles in a digraph.

Theorem 1 (Thomassen [10]). *Every digraph with minimum outdegree at least three contains two disjoint cycles.*

In the above result, we do not care if the lengths of disjoint cycles are equal or not. In recent years, many researchers were interested in conditions to guarantee the existence of disjoint cycles of different lengths in digraphs. In [4], Henning and Yeo have posed the following conjecture.

Conjecture 2 (Henning & Yeo [4]). *Every digraph with minimum outdegree at least four contains two disjoint cycles of different lengths.*

This conjecture has been verified for several classes of digraphs in [3, 4, 6] before it has been proved completely by Lichiardopol in [5].

Theorem 3 (Lichiardopol [5]). *Every digraph with minimum outdegree at least four contains two disjoint cycles of different lengths.*

In this paper, we investigate the structure of digraphs having no disjoint cycles of different lengths. On account of Theorems 1 and 3, it is natural and reasoned to restrict our investigation to digraphs with minimum outdegree 3. In Section 2, we will prove a decomposition theorem for digraphs with minimum outdegree 3 having no disjoint cycles of different lengths. Before the formulating our results, we define a notion called a *good decomposition* for a digraph. Namely, we say that a digraph $D = (V, A)$ admits a good decomposition $D = D_1 \cup D_2 \cup D_3$ if $D_1 = (V_1, A_1), D_2 = (V_2, A_2)$ and $D_3 = (V_3, A_3)$ are such subdigraphs of D that $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$, $V_2 \neq \emptyset$ but V_1 may be empty, D_1 is the subdigraph of D induced by V_1 and is an acyclic digraph, D_2 is the subdigraph of D induced by V_2 and is a strong digraph and D_3 is a subdigraph of D, every arc of which has its tail in V_1 and its head in V_2. Further, we say that a good decomposition $D = D_1 \cup D_2 \cup D_3$ of a digraph D is *unique* if for any other good decomposition $D = D_1' \cup D_2' \cup D_3'$ of D we always have $D_1' = D_1$, $D_2' = D_2$ and $D_3' = D_3$.

Theorem 4. *A digraph $D = (V, A)$ with minimum outdegree 3 has no vertex disjoint cycles of different lengths if and only if D admits a good decomposition $D = D_1 \cup D_2 \cup D_3$, where $D_1 = (V_1, A_1), D_2 = (V_2, A_2)$ and $D_3 = (V_3, A_3)$ are such that D_2 has minimum outdegree 3 and no vertex disjoint cycles of different lengths and for every vertex $v \in V_1$, $d^+_{D_1 \cup D_3}(v) \geq 3$. Moreover, when such a good decomposition for D exists, it is unique.*

By Theorem 4, the investigation of digraphs with minimum outdegree 3 having no vertex disjoint directed cycles of different lengths can be reduced to the investigation of strong such digraphs. Further in this paper, we will get a classification for strong digraphs with minimum outdegree 3 and girth 2 having no vertex disjoint cycles of different lengths. In [7], for every integer $n \geq 2$ we have defined the digraph $D^2_{2n} = \big(V(D^2_{2n}), A(D^2_{2n})\big)$ as follows. The vertex set $V(D^2_{2n}) = \{u_i, v_i \,|\, i = 0, 1, \ldots, n-1\}$ and the arc set $A(D^2_{2n}) = \{u_iv_i, v_iu_i, u_iu_{i+1}, u_iv_{i+1}, v_iu_{i+1},\ v_iv_{i+1} \,|\, i = 0, 1, \ldots, n-1\}$, where $i + 1$ is always taken modulo n.

The digraph D_4^2 is the complete digraph on 4 vertices. The digraph D_8^2 is illustrated in Figure 1.

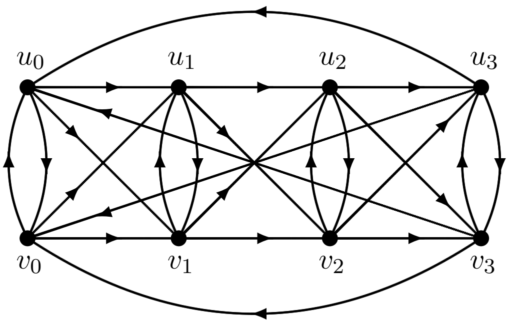

Figure 1. The digraph D_8^2.

It has been proved in [7] that for any integer $n \geq 2$, the digraph D_{2n}^2 is a 3-regular digraph with girth 2 having no vertex disjoint cycles of different lengths. For more results about digraphs having no vertex disjoint cycles of different lengths, the reader can see the papers [2, 8] and [9].

The following result will be proved in Section 3.

Theorem 5. *Let $D = (V, A)$ be a strong digraph with minimum outdegree 3 and girth 2. Then D is a digraph having no vertex disjoint cycles of different lengths if and only if D is isomorphic to a digraph D_{2n}^2 for some integer $n \geq 2$.*

2. Proof of Theorem 4

In this section, we will prove Theorem 4. First, we prove its necessity. So, let $D = (V, A)$ be a digraph with minimum outdegree 3 having no disjoint cycles of different lengths. Then, we have the following claims.

Claim 6. *Every terminal strong component of D is nontrivial.*

This claim is trivial because the minimum outdegree of D is 3.

Claim 7. *D has a unique nontrivial strong component.*

Proof. Suppose, on the contrary, that D has at least two nontrivial strong components and let C_1 and C_2 be two of them. By Claim 6, without loss of generality, we may assume that C_1 is terminal. Let $P = v_1, v_2, \ldots, v_k$ be a longest path in C_1. Then, since C_1 is a terminal strong component, every outneighbor of v_k in D must be in $V(P)$. Therefore, since the minimum outdegree of D is 3, there exist $i, j \in \{1, 2, \ldots, k-1\}$ with $i \neq j$ such that both $v_k v_i \in A$ and $v_k v_j \in A$ hold. It follows that $Q_1 = v_i, v_{i+1}, \ldots, v_k, v_i$ and $Q_2 = v_j, v_{j+1}, \ldots, v_k, v_j$ are two cycles of different lengths in C_1. Further, since C_2 is nontrivial and strong, it is clear that C_2 has a cycle Q_3. Therefore, either Q_1 and Q_3 or Q_2 and Q_3 are two disjoint cycles of different lengths in D, a contradiction. ■

Let $C_1, C_2, \ldots, C_t$ be strong components of D. Since any digraph has a terminal strong component, by Claims 6 and 7, without loss of generality, we may assume that C_t is the only nontrivial strong component of D which must be terminal and the other strong components $C_1, C_2, \ldots, C_{t-1}$ of D are nonterminal and trivial. Let $V_1 = V(C_1) \cup V(C_2) \cup \cdots \cup V(C_{t-1})$, $V_2 = V(C_t) = V \setminus V_1$, $D_1 = (V_1, A_1) = D[V_1]$ and $D_2 = (V_2, A_2) = C_t = D[V_2]$. Further, let $A_3 = A \setminus (A_1 \cup A_2)$ and $D_3 = (V_3, A_3)$ be the subdigraph of D, arc-induced by A_3. Then, since the strong component digraph $SC(D)$ of D is acyclic, it is clear that $D = D_1 \cup D_2 \cup D_3$ is a good decomposition of D. Furthermore, since D has minimum outdegree 3 and no disjoint cycles of different lengths, the subdigraph D_2 must have minimum outdegree 3 and no disjoint cycles of different lengths and for every vertex $v \in V_1$, $d^+_{D_1 \cup D_3}(v) \geq 3$. The necessity of Theorem 4 is proved.

Next, we prove the sufficiency. So, let a digraph D admit a good decomposition $D = D_1 \cup D_2 \cup D_3$, where $D_1 = (V_1, A_1), D_2 = (V_2, A_2)$ and $D_3 = (V_3, A_3)$ are such that D_2 has minimum outdegree 3 and no vertex disjoint cycles of different lengths and for every vertex $v \in V_1$, $d^+_{D_1 \cup D_3}(v) \geq 3$. Then it is clear that the minimum outdegree of D is 3. If C is a cycle in D, then since D_1 is acyclic and D_3 is a digraph every arc of which has its tail in V_1 and its head in V_2, it is not difficult to see that C must be a cycle in D_2. Thus, if C_1 and C_2 are two disjoint cycles in D, then they are two disjoint cycles in D_2 which is a digraph having no disjoint cycles of different lengths. It follows that C_1 and C_2 have the same length and therefore D must be a digraph having no disjoint cycles of different lengths. The sufficiency of Theorem 4 is proved.

Finally, we prove the uniqueness of the decomposition. Let $D = D_1 \cup D_2 \cup D_3$ be a good decomposition of D. Then, it is clear that D_2 is the unique nontrivial strong component of D. Hence, if $D = D_1' \cup D_2' \cup D_3'$ is another good decomposition of D, then we must have $D_2' = D_2$ because D_2' also is a nontrivial strong component of D. It follows that $D_1' = D_1$ and therefore $D_3' = D_3$. Thus, a good decomposition for D is unique.

The proof of Theorem 4 is complete.

3. Proof of Theorem 5

In this section, we will prove Theorem 5. Let $D = (V, A)$ be a strong digraph with minimum outdegree 3 and girth 2. If D is isomorphic to a digraph D^2_{2n} for some integer $n \geq 2$, then by Theorem 1 in [7] D is a digraph having no disjoint cycles of different lengths. Thus, it remains to prove the converse. So, we suppose that D is a digraph having no disjoint cycles of different lengths. We consider the following two cases separately.

Case 1. D has a vertex u which lies on two different cycles $C_1 = u, v_1, u$ and $C_2 = u, v_2, u$ of length 2.

In this case, we consider the digraph $D_u = D - u$. Let $P_1 = u_1, u_2, \ldots, u_t$ be a maximal path in D_u with $u_1 = v_1$. By maximality of P_1, all outneighbors of u_t in D_u must be in $V(P_1)$. Since the minimum outdegree of D is 3, the minimum outdegree of D_u is at least 2. So, u_t has at least two outneighbors in $V(P_1)$ and therefore $t \geq 3$. If u_t has three outneighbors in $V(P_1)$, say u_i, u_k and u_j with $1 \leq i < k < j \leq t-1$, then the cycles $C_3 = u_k P_1 u_t, u_k$ and C_1 are two disjoint cycles of different lengths in D, a contradiction. Thus, u_t has exactly two outneighbors in $V(P_1)$, say u_i and u_j with $1 \leq i < j \leq t-1$. But u_t must have at least 3 outneighbors in D because the minimum outdegree of D is 3. It follows that we also must have $u_t u \in A$. Further, let $C_4 = u_i P_1 u_t, u_i$ and $C_5 = u_j P_1 u_t, u_j$. Then, if $i > 1$, then C_4 and C_1 are two disjoint cycles of different lengths in D, a contradiction. So, $i = 1$. Now, if $i = 1$ but $j < t-1$, then C_5 and C_1 are two disjoint cycles of different cycles in D, a contradiction again. Thus, we can conclude that if $P_1 = u_1, u_2, \ldots, u_t$ is a maximal path in D_u with $u_1 = v_1$, then $u_1 = v_1, u_{t-1}$ and u are all outneighbors of u_t in D.

If $v_2 \notin V(P_1)$, then C_4 and C_2 are two disjoint cycles of different lengths in D, a contradiction. So, $v_2 \in V(P_1)$. Further, if $v_2 \in V(P_1)$ but $v_2 \notin \{u_{t-1}, u_t\}$, then the cycles $C_6 = v_1 P_1 v_2, u, v_1$ and C_5 are two disjoint cycles of different lengths in D, a contradiction again. Thus, we must have $v_2 \in \{u_{t-1}, u_t\}$. If $v_2 = u_t$, then u, v_1 and v_2 are vertices in D such that $v_2 v_1 \in A$ and u lies on two different cycles u, v_1, u and u, v_2, u of length 2. If $v_2 = u_{t-1}$, then the vertices u_{t-1}, u and u_t are such that $u_t u \in A$ and u_{t-1} lies on two different cycles u_{t-1}, u, u_{t-1} and u_{t-1}, u_t, u_{t-1} of length 2.

In a word, we can find in D vertices u, v_1 and v_2 such that $v_2 v_1 \in A$ and u lies on two different cycles u, v_1, u and u, v_2, u of length 2. For such vertices u, v_1 and v_2, we consider a maximal path $P_2 = w_1, w_2, \ldots, w_\ell$ in D_u with $w_1 = v_2$ and $w_2 = v_1$. By arguments similar to those used in the above two paragraphs, we can show that $\ell \geq 3$, $w_1 = v_2, w_{\ell-1}$ and u are all outneighbors of w_ℓ in D and $v_1 \in \{w_{\ell-1}, w_\ell\}$. It follows that $\ell = 3$, i.e., $P_2 = w_1, w_2, w_3$ with $w_1 = v_2, w_2 = v_1$ and v_1, v_2 and u are all outneighbors of w_3. Now we apply similar arguments to the vertices v_1, u and w_3 in order to get the path $P_2' = w_3, u, v_2'$ such that w_3, u and v_1 are all outneighbors of v_2'. If $v_2' \neq v_2$, then we apply again these arguments to the vertices u, v_1 and v_2' in order to get the path $P_2'' = v_2', v_1, w_3'$ such that v_1, u and v_2' are all outneighbors of w_3'. Therefore, we can get the two disjoint cycles v_2', v_1, w_3', v_2' and u, v_2, u of different lengths in D, a contradiction. So, $v_2' = v_2$

and therefore $v_2w_3 \in A$. By repeating these arguments to the vertices v_2, u, w_3 and then to the vertices w_3, v_1, v_2, we can see that $v_1v_2 \in A$ and $uw_3 \in A$. Now, it is not difficult to see that D is isomorphic to D_4^2 in this case.

Case 2. D has no vertex which lies on two different cycles of length 2. Let $C_0 = u_0, v_0, u_0$ be a cycle of length 2 in D and $D' = D - V(C_0)$. First, we prove the following simple claim which we will frequently and implicitly use later.

Claim 8. *If $P = x_1, x_2, \ldots, x_t, t \geq 2$, is a path in D' and x_t has outneighbors in $V(P)$, then x_{t-1} is the unique outneighbor of x_t in $V(P)$.*

Proof. Let x_t have outneighbors in $V(P)$. If x_t has an outneighbor $x_i \in V(P)$ with $i \neq t-1$, then $H_1 = x_i, x_{i+1}, \ldots, x_t, x_i$ is a cycle in D' of length greater than 2. Therefore, C_0 and H_1 are two disjoint cycles of different lengths in D, a contradiction. Thus, x_{t-1} is the unique outneighbor of x_t in $V(P)$. ■

Let W_1 be a terminal strong component of D'. Since the minimum outdegree of D is 3, the minimum outdegree of D' is at least 1. It follows that W_1 must be nontrivial. Let $Q = w_1, w_2, \ldots, w_{\ell-1}, w_\ell$ be a longest path in W_1. Then $\ell \geq 2$ and by maximality of $|V(Q)|$, all outneighbors of w_ℓ in W_1 must be in $V(Q)$. Since W_1 is a terminal strong component of D' and the minimum outdegree of D is 3, by Claim 8, it is not difficult to see that $N_D^+(w_\ell) = \{w_{\ell-1}, u_0, v_0\}$. In particular, we have $d_D^+(w_\ell) = 3$. Since W_1 is strong, we can find a path Q' from w_ℓ to w_1 in W_1, say $Q' = z_1, z_2, \ldots, z_r$ with $z_1 = w_\ell$ and $z_r = w_1$. Since $N_D^+(w_\ell) = \{w_{\ell-1}, u_0, v_0\}$, the unique outneighbor of $w_\ell = z_1$ in D' is $w_{\ell-1}$. So, we necessarily have $z_2 = w_{\ell-1}$. Now we go along the path Q' from z_2 in the direction specified by the direction of its arcs. Let $z_i, i \geq 3$, be the first vertex of Q', which lies in $V(Q) \setminus \{w_{\ell-1}\}$, by this travelling, say $z_i = w_j$. Since Q' is a path, it is clear that $j \neq \ell, \ell-1$. Furthermore, $H_2 = (z_2Q'z_i) \cup (w_jQw_{\ell-1})$ is a cycle in D'. If either $i \neq 3$ or $j \neq \ell-2$, then H_2 and C_0 are two disjoint cycles of different lengths in D, a contradiction. Thus, we must have $z_3 = w_{\ell-2}$. If $\ell \geq 4$, then by similar considerations we can get $z_4 = w_{\ell-3}$, $z_5 = w_{\ell-4}, \ldots,$ $r = \ell$ and $z_r = w_1$. But D has no vertex which lies on two different cycles of length 2 in this case. So, we can conclude that $r = \ell = 2$, i.e., $Q = w_1, w_2$, $Q' = w_2, w_1$ and the length of a longest path in W_1 is 1. So, Q' is also a longest path in W_1 and therefore $N_D^+(w_1) = \{w_2, u_0, v_0\}$. Thus, W_1 is a cycle of length 2. Rename this cycle by $C_1 = u_1, v_1, u_1$. Then it is clear from the above consideration that $N_D^+(u_1) = \{v_1, u_0, v_0\}$ and $N_D^+(v_1) = \{u_1, u_0, v_0\}$.

Now let W_2 be a terminal strong component of $D - V(C_1)$. By arguments similar to those used in the above paragraph, we can show that W_2 is a cycle $C_2 = u_2, v_2, u_2$ with $N_D^+(u_2) = \{v_2, u_1, v_1\}$ and $N_D^+(v_2) = \{u_2, u_1, v_1\}$. If $V(C_2) \cap V(C_0) \neq \emptyset$, say $u_2 = u_0$, then $v_2 = v_0$ because D has no vertex which lies on two different cycles of length 2 in this case. Then it is not difficult to see that D is isomorphic to D_4^2, which is impossible in this case. Thus, $V(C_2) \cap V(C_0) = \emptyset$. Further, we consider a terminal strong component W_3 of $D - V(C_2)$. Again, by arguments similar to those used in the above paragraph, we can show that W_3 is a cycle $C_3 = u_3, v_3, u_3$ with $N_D^+(u_3) = \{v_3, u_2, v_2\}$ and $N_D^+(v_3) = \{u_3, u_2, v_2\}$. We have $V(C_3) \cap V(C_1) = \emptyset$ because every vertex of $V(C_3)$ has outneighbors in $V(C_2)$ whilst any vertex of $V(C_1)$ has no outneighbors in $V(C_2)$. If $V(C_3) \cap V(C_0) \neq \emptyset$, say $u_3 = u_0$, then since D has no vertex which lies on two different cycles of length 2 in this case we must have $v_3 = v_0$, i.e., $V(C_3) = V(C_0)$. Then, it is not difficult to see that in this situaton D is isomorphic to D_6^2. Otherwise, $V(C_3) \cap V(C_0) = \emptyset$ and we can repeat similar arguments as above for C_3 and so on. Since D is finite, there is a natural number $n \geq 3$ such that the following hold for each $i \in \{0, 1, \ldots, n-1\}$.

(i) $C_i = u_i, v_i, u_i$ is a cycle of length 2 and all cycles $C_0, C_1, \ldots, C_{n-1}$ are pairwise disjoint;

(ii) $N_D^+(u_i) = \{v_i, u_{i-1}, v_{i-1}\}$ and $N_D^+(v_i) = \{u_i, u_{i-1}, v_{i-1}\}$, where $i-1$ is always taken modulo n.

Then since D is connected and strong, it is not difficult to see that D is isomorphic to D_{2n}^2.

The proof of Theorem 5 is complete.

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