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IDENTIFYING CODES IN THE DIRECT PRODUCT OF A PATH AND A COMPLETE GRAPH

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Abstract

Let G be a simple, undirected graph with vertex set V. For any vertex $v \in V$, the set $N[v]$ is the vertex v and all its neighbors. A subset $D \subseteq V(G)$ is a dominating set of G if for every $v \in V(G)$, $N[v] \cap D \neq \emptyset$. And a subset $F \subseteq V(G)$ is a separating set of G if for every distinct pair $u, v \in V(G)$, $N[u] \cap F \neq N[v] \cap F$. An identifying code of G is a subset $C \subseteq V(G)$ that is dominating as well as separating. The minimum cardinality of an identifying code in a graph G is denoted by $\gamma^{ID}(G)$. The identifying codes of the direct product $G_1 \times G_2$, where G_1 is a complete graph and G_2 is a complete/ regular/ complete bipartite graph, are known in the literature. In this paper, we find $\gamma^{ID}(P_n \times K_m)$ for $n \geq 3$, and $m \geq 3$ where P_n is a path of length n, and K_m is a complete graph on m vertices.

Keywords: identifying code, direct product, path, complete graph.

2010 Mathematics Subject Classification: 05C69, 05C76, 68R99.

1. Introduction

Identifying codes are studied in the various scientific disciplines such as information theory, electrical engineering, mathematics, and computer science to design efficient and reliable networks. This typically involves the removal of redundancy and the correction or detection of errors in the networks. An identifying code C is a dominating set having the property that any two vertices of the graph have distinct neighborhoods in C, thus every vertex is uniquely identified by its neighbors within the dominating set. The notion of an identifying code was introduced by Karpovsky *et al.* [18] with the original motivation of achieving fault diagnosis for multiprocessor systems. Numerous papers dealt with identifying codes Balbuena *et al.* [1], Ben-Haim and Litsyn [2], Bertrand *et al.* [3], Bertrand *et al.* [4], Chen *et al.* [5], Cohen *et al.* [6], Foucaud [9], Foucaud *et al.* [10], Gravier *et al.* [12], Honkala and Lobstein [15], Janson and Laihonen [16], Junnila and Laihonen [17], Laifenfeld and Trachtenberg [20], Laihonen and Moncel [21], and Xu *et al.* [28]. Several results about different types of product graphs are known (see Feng and Wang [7], Feng *et al.* [8], Goddard and Wash [11], Gravier *et al.* [13], Hedetniemi [14], Kim and Kim [19], Rall and Wash [25]). Identifying codes of direct product of graphs are studied for $K_m \times K_n$ by Rall and Wash [24], and $K_m \times G$, where G is a regular/ complete bipartite graph by Lu *et al.* [23]. For more references, we direct the reader to extensive bibliography maintained by Lobstein [22].

In this work, for a path P_n on n vertices, and a complete graph K_m, on m vertices, we find $\gamma^{ID}(P_n \times K_m)$ for $n \geq 3$, and $m \geq 3$. As $P_2 = K_2$ and identifying codes of $K_m \times K_n$ are studied by Rall and Wash [24], we assume that $n \geq 3$. While studying identifying codes in the direct product of a path and a complete graph, we prove a set of necessary and sufficient conditions that are used to determine whether a given set is an identifying code.

2. Preliminaries

A graph G is an ordered pair $(V(G), E(G))$ comprising a set of vertices or nodes $V(G)$ together with a set $E(G)$ of edges, which are two-element subsets of $V(G)$. Given a vertex $v \in V(G)$, its closed neighborhood, denoted by $N[v]$, is made up of the node v together with all its neighbors and its open neighborhood is denoted by $N(v) = N[v] \setminus \{v\}$. For $S \subseteq V(G)$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$. Similarly, the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $D \subseteq V(G)$ is dominating if for any $v \in V(G) \setminus D$ there exists a vertex $u \in D$ such that the edge $uv \in E(G)$. A set $D \subseteq V(G)$ is said to be separating if for any two distinct vertices $u, v \in V(G)$, $N[u] \cap D \neq N[v] \cap D$. The symmetric difference of two sets A and B is the set $(A \setminus B) \cup (B \setminus A)$ and it is denoted by $A \triangle B$. Thus, $A \triangle B$ is the set of all those elements that belong either to A or to B but not to both. Two vertices u, v of a graph G are said to be separated by a subset D of $V(G)$ if the intersection of D with the symmetric difference of their neighborhood is non-empty, that is, $(N[u] \cap D) \triangle (N[v] \cap D) = (N[u] \triangle N[v]) \cap D \neq \emptyset$. If a set D is dominating as well as separating, then we say that D is an identifying code of G. Naturally, identifying codes exist in a graph if and only if the graph is twin-free, that is, for any two distinct vertices $u, v \in V(G)$, $N[u] \neq N[v]$. The minimum cardinality of an identifying code in a graph G is denoted by $\gamma^{ID}(G)$. Given an identifying code C, vertices of C are called codewords.

There is no identifying code in K_m, as it is not twin-free. A bipartite graph is a graph whose vertices can be divided into two disjoint and independent sets V_1 and V_2 such that every edge connects a vertex in V_1 to one in V_2. Vertex sets V_1 and V_2 are usually called the parts/ cells of the graph.

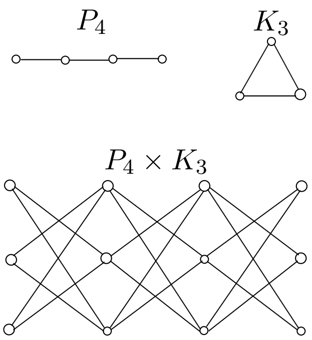

Figure 1. The graphs P_4 and K_3 and their direct product $P_4 \times K_3$.

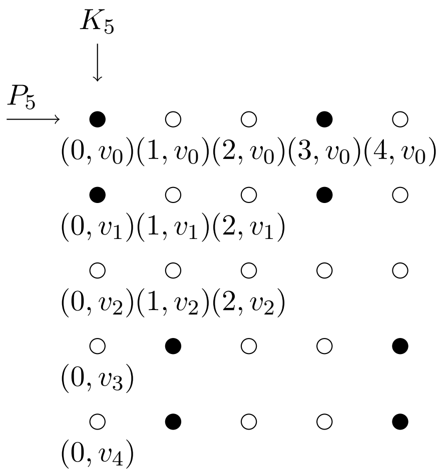

Figure 2. An identifying code of $P_5 \times K_5$.

Given two graphs G and H, the direct product $G \times H$ (see Figure 1) is the graph whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edge set is

$$E(G \times H) = \{(g_1, h_1)(g_2, h_2) : g_1 g_2 \in E(G) \text{ and } h_1 h_2 \in E(H)\}.$$

By $d_G(u, v)$, we denote the distance between the vertices u and v in a graph G. Let $V(K_m) = \{v_0, v_1, \ldots, v_{m-1}\}$ and $V(P_n) = \{0, 1, 2, \ldots, n-1\}$. Let D be a subset of $V(P_n \times K_m)$. For $0 \leq i \leq n-1$ and $0 \leq j \leq m-1$, we define $C_i = \{(i, v) : v \in V(K_m)\}$ (i^{th} column), $R'_j = \{(i, v_j) : i \in V(P_n)\}$ (j^{th} row) (see Figure 2), $D_i = C_i \cap D$, $R_j = R'_j \cap D$, and $r_i(D)$ as follows.

$r_0(D) = \{v : \{(0, v), (1, v)\} \cap D \neq \emptyset\}$,

$r_i(D) = \{v : \{(i-1, v), (i, v), (i+1, v)\} \cap D \neq \emptyset\}$, for $1 \leq i \leq n-2$,
$r_{n-1}(D) = \{v : \{(n-2, v), (n-1, v)\} \cap D \neq \emptyset\}$.

Note that in all the given figures, the codewords are indicated by black circles. The graphs with edges are illustrated in Figure 1. For ease, in all the remaining figures, edges are avoided. For undefined terminology and notation, see West [27].

3. Necessary and Sufficient Conditions on a Set D to be an Identifying Code of $P_n \times K_m$

Before we proceed, we need some additional definitions, as well as some easy but useful lemmas.

Lemma 1. *A connected bipartite graph on at least three vertices is a twin-free graph.*

Proof. Let G be a connected bipartite graph with $|V(G)| \geq 3$ and $V(G) = V_1 \cup V_2$. If $u, v \in V_i$ (for $i = 1, 2$), then $N[u] \neq N[v]$. Let $u \in V_1$ and $v \in V_2$. If $N[u] = N[v] = \{u, v\}$, then G would have more than two components. Now, suppose there exists a vertex w, with $w \neq u, v$, such that $w \in N[u]$. Since G is bipartite, $w \notin N[v]$. Thus, G is twin-free. ∎

Weichsel [26] proved the following result.

Theorem 2 [26]**.** *Assume that G and H are finite, nontrivial connected graphs in which loops are admitted. If at least one of G and H has an odd cycle, then the direct product of G and H is connected.*

By using Theorem 2, we state the following result.

Lemma 3. *If G is a nontrivial, connected graph on at least three vertices such that it has an odd cycle, then the direct product of G and P_n is connected.*

Lemma 4. *The direct product of any graph G on m vertices and a path P_n is a bipartite graph.*

Proof. Since P_n is bipartite, the graph $P_n \times G$ is bipartite. ∎

Lemma 5. *If G is a nontrivial connected graph on at least three vertices such that G has an odd cycle, then the direct product of G and a path P_n admits an identifying code.*

Proof. By Lemmas 3 and 4, $P_n \times G$ is a connected and bipartite graph. Also, by Lemma 1, it is a twin-free graph. Hence, $P_n \times G$ admits an identifying code. ∎

Lemma 6. *The direct product of a complete graph K_m (with $m \geq 3$) and a path P_n admits an identifying code.*

Proof. The proof follows by Lemma 5. ■

Lemma 7. *If a subset D of $V(P_n \times K_m)$ is such that $|D_i| \geq 2$, for all $0 \leq i \leq n-1$, then D is dominating.*

Proof. Any vertex, say (k, v_j), of $V(P_n \times K_m)$ is adjacent to $(k-(+)1, v_q)$ for $0 \leq k \leq n-1$, $0 \leq q, j \leq m-1$, and $q \neq j$. Therefore, $|D_i| \geq 2$ for $0 \leq i \leq n-1$ gives $D = \bigcup_{i=0}^{n-1} D_i$ as a dominating set of $P_n \times K_m$. ■

In the following result, we provide a necessary condition on a set D to be an identifying code of $P_n \times K_m$.

Proposition 3.1. If D is an identifying code of $P_n \times K_m$ (with $n, m \geq 3$), then $|r_i(D)| \geq m-1$ for $0 \leq i \leq n-1$.

Proof. Assume that $|r_0(D)| < m-1$, that is, there exist at least two vertices, say $v_1, v_2 \in V(K_m)$, such that $(0, v_1), (1, v_1), (0, v_2), (1, v_2) \notin D$. Then, $N[(0, v_1)] \cap D = N[(0, v_2)] \cap D$, which is a contradiction to the fact that D is identifying. Thus, $|r_0(D)| \geq m-1$. By using a similar argument, we can prove that $|r_{n-1}(D)| \geq m-1$.

Now, assume that $|r_i(D)| < m-1$ for some $1 \leq i \leq n-2$, that is, there exist at least two vertices, say $v_1, v_2 \in V(K_m)$, such that $(i-1, v_1), (i, v_1), (i+1, v_1), (i-1, v_2), (i, v_2), (i+1, v_2) \notin D$. Then, $N[(i, v_1)] \cap D = N[(i, v_2)] \cap D$, which is a contradiction to the fact that D is identifying. Thus, $|r_i(D)| \geq m-1$ for $0 \leq i \leq n-1$. ■

Now, we provide a sufficient condition on a set D to be an identifying code of $P_n \times K_m$.

Proposition 3.2. If $D \subset V(P_n \times K_m)$ (for $n \geq 4$, $m \geq 3$) is such that $|D_i| \geq 2$ and $|r_i(D)| \geq m-1$ for every $0 \leq i \leq n-1$, then D is an identifying code of $P_n \times K_m$.

Proof. By using Lemma 7, D is dominating. Because $|D_i| \geq 2$, columns C_{i+1} and C_{i-1} (if they exist) are dominated by D, thus two vertices in different columns are separated if $n \geq 4$. Now, assume that there exist two vertices (i, y) and (i, v) such that $N[(i, y)] \cap D = N[(i, v)] \cap D \neq \emptyset$ for $0 \leq i \leq n-1$. Since $N[(i, y)] \cap D = N[(i, v)] \cap D$ and (i, y) and (i, v) are non-adjacent, both $(i, y), (i, v) \notin D$. Also, $(i-1, y), (i+1, y), (i-1, v), (i+1, v) \notin D$ (if they exist). Therefore, $|r_i(D)| < m-1$, which is a contradiction to our assumption that $|r_i(D)| \geq m-1$. Thus, D is an identifying code of $P_n \times K_m$. ■

4. Construction of an Identifying Code of $P_n \times K_m$

In this section, we construct an identifying code of $P_n \times K_m$ that gives us an upper bound on $\gamma^{ID}(P_n \times K_m)$.

For non-negative integers n, m, t, b, q with $m, n \geq 5$, and $2 \leq b \leq m-3$, we define $D_{3q} = \{(3q, v_j) : 0 \leq j \leq b-1\}$, and $D_{3q+1} = \{(3q+1, v_j) : b+1 \leq j \leq m-1\}$, so that $|D_{3q} \cup D_{3q+1}| = m-1$. Using this, D is constructed in the following manner.

1. For $n = 6$, $D = \{(1, v_i), (4, v_i) : 1 \leq i \leq m-1\} \cup \{(2, v_1), (3, v_1)\}$ (see Figure 3).
2. For $n = 3t, t \geq 3$, $D = \bigcup_{q=0}^{t-1} \left(D_{3q} \cup D_{3q+1}\right) \cup \{(n-4, v_0)\} \cup \left(\bigcup_{j=0}^{b-1} \{(n-1, v_j)\}\right)$ (see Figure 4).
3. For $n = 3t+1$, $t \geq 2$, $D = \bigcup_{q=0}^{t-1} (D_{3q} \cup D_{3q+1}) \cup \{(n-2, v_0), (n-2, v_1)\} \cup \left(\bigcup_{j=3}^{m-1} \{(n-1, v_j)\}\right)$ (see Figure 5).
4. For $n = 3t+2$, $t \geq 1$, $D = \bigcup_{q=0}^{t} \left(D_{3q} \cup D_{3q+1}\right)$ (see Figure 2, and 6).

Figures 2–6 illustrate the construction, for $m = 5$, and different values of n. Note that in the above construction of D, b is the only parameter on which we will play. For D to be dominating, we need b to be between 2 and $m-3$. Later, we shall see that the most interesting case is taking b as small as possible, that is, $b = 2$. We will prove that D defined as above is an identifying code of $P_n \times K_m$.

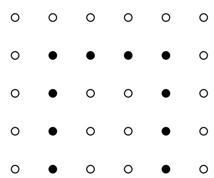

Figure 3. An identifying code of $P_6 \times K_5$ with 10 codewords.

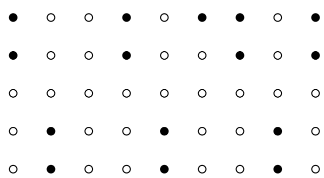

Figure 4. An identifying code of $P_9 \times K_5$ with $b = 2$.

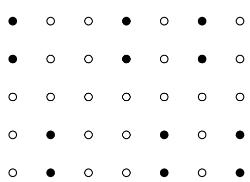

Figure 5. An identifying code of $P_7 \times K_5$.

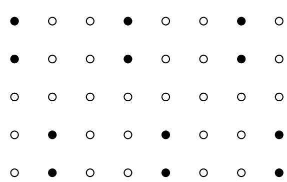

Figure 6. An identifying code of $P_8 \times K_5$.

Theorem 8. *For $m, n \geq 5$, the set D is an identifying code of $P_n \times K_m$. Hence, for $m, n \geq 5$, and $t > 0$,*

$$\gamma^{ID}(P_n \times K_m) \leq \begin{cases} 2m & \text{if } n = 6, \\ t(m-1)+3 & \text{if } n = 3t,\ t \geq 3, \\ (t+1)(m-1) & \text{if } n = 3t+1,\ t \geq 2, \\ (t+1)(m-1) & \text{if } n = 3t+2,\ t \geq 1. \end{cases}$$

Proof. It is easy to check that $\{(1, v_i), (4, v_i) : 1 \leq i \leq m-1\} \cup \{(2, v_1), (3, v_1)\}$ is an identifying code of $P_6 \times K_m$ (see Figure 3) of cardinality $2m$. Therefore, $\gamma^{ID}(P_6 \times K_m) \leq 2m$.

From now, assume that $n \neq 6$. First, we prove that all vertices of columns C_0, C_1, and C_2 are dominated and separated by D_0, D_1, and D_3. Note that $D_2 = \emptyset$.

Since $2 \leq b \leq m-3$, $|D_0|$, $|D_1| \geq 2$, which implies that all vertices of C_1 are dominated by D_0 and all vertices of C_0, and C_2 are dominated by D_1. Otherwise, if $b = 1$, then $(1, v_0)$ is not dominated by D. Similarly, if $b = m-2/\ m-1$, then D is not identifying.

Now, we prove that any two vertices of $C_0 \cup C_1 \cup C_2$, say (i, v_j), and (k, v_f), are separated (for $0 \leq i, k \leq 2$, and $0 \leq j, f \leq m-1$).

Case 1. $i = k$. If one or both of $\{(i, v_j),\ (k, v_f)\}$ lies in D, then they are separated since they are non-adjacent to each other. Therefore, assume that both are not in D.

Case 1.1. $i = k = 0$. If $b \leq f < j \leq m-1$, then $\{(1, v_j)\} \subseteq (N[(0, v_j)] \triangle N[(0, v_f)]) \cap D$.

Case 1.2. $i = k = 1$. If $0 \leq f < j \leq b$, then $\{(0, v_f)\} \subseteq (N[(1, v_j)] \triangle N[(1, v_f)]) \cap D$.

Case 1.3. $i = k = 2$. Similarly, the separation is done by D_1 or D_3. This proves that any two vertices in one column are separated by D.

Case 2. $i \neq k$.

Case 2.1. Assume that $d_{P_n}(i, k) = 1$. $(i, v_j) \in C_i$ and $(k, v_f) \in C_k$ for $i, k \in \{0, 1, 2\}$. If $(i, v_j) \in C_0$ and $(k, v_f) \in C_1$, then they are separated since $|r_0(D)| = m-1$. If $(i, v_j) \in C_1$ and $(k, v_f) \in C_2$, then they are separated since (k, v_f) is adjacent to vertices in D_3 that are non-adjacent to (i, v_j).

Case 2.2. Assume that $d_{P_n}(i, k) = 2$. Assume also that $(i, v_j) \in C_0$ and $(k, v_f) \in C_2$. They are separated since (k, v_f) is adjacent to vertices in D_3, which are non-adjacent to (i, v_j).

Thus, all vertices of columns C_0, C_1, C_2 are dominated and separated by D_0, D_1, D_3. By continuing the idea applied above, all vertices of C_3, C_4, C_5 are identified by D_3, D_4, D_6, and so on. It is easy to see that these groups of 3 columns must also be separated from one another (C_2 from C_3 or C_2 from C_4, for instance).

Case A. $n = 3t$, $t \geq 3$. We have similarly applied the same idea to C_0, $C_1, C_2, \ldots, C_{n-6}$, C_{n-5}, C_{n-4}. Here, the vertices in the last column C_{n-1} are selected as $\bigcup_{j=0}^{b-1}\{(n-1, v_j)\}$ and for separation of C_{n-1} and C_{n-3} one vertex is selected as $(n-4, v_0)$. Thus, $D = \bigcup_{q=0}^{t-1} (D_{3q} \cup D_{3q+1}) \cup \{(n-4, v_0)\} \cup \big(\bigcup_{j=0}^{b-1}\{(n-1, v_j)\}\big)$ and $|D| = t(m-1) + b + 1$. In this case, we get the best value of $|D|$ for $b = 2$, that is, $t(m-1) + 3$. Thus, $\gamma^{ID}(P_n \times K_m) \leq t(m-1) + 3$.

Case B. $n = 3t + 1$, $t \geq 2$. We have similarly applied the same idea to $C_0, C_1, C_2, \ldots, C_{n-4}$, C_{n-3}, C_{n-2}. Here, for the purpose of separation we modify the selection of vertices of the last two columns C_{n-2} and C_{n-1} as $\{(n-2, v_0), (n-2, v_1)\}$ and $\bigcup_{j=3}^{m-1}\{(n-1, v_j)\}$. So, $D = \bigcup_{q=0}^{t-1} \big(D_{3q} \cup D_{3q+1}\big) \cup \{(n-2, v_0), (n-2, v_1)\} \cup \big(\bigcup_{j=3}^{m-1}\{(n-1, v_j)\}\big)$. Thus, $|D| = (t+1)(m-1)$. Thus, $\gamma^{ID}(P_n \times K_m) \leq (t+1)(m-1)$.

Case C. $n = 3t + 2$, $t \geq 1$. Here also, we have similarly applied the same idea to $C_0, C_1, C_2, \ldots, C_{n-5}$, C_{n-4}, C_{n-3}. To identify columns C_{n-2} and C_{n-1}, the vertices of $D_{3t} \cup D_{3t+1}$ are selected. So, $D = \bigcup_{q=0}^{t}(D_{3q} \cup D_{3q+1})$. Thus, $|D| = (t+1)(m-1)$. Thus, $\gamma^{ID}(P_n \times K_m) \leq (t+1)(m-1)$. ■

5. Lower Bounds on the Size of an Identifying Code of $P_n \times K_m$

In this section, we study lower bounds on the size of an identifying code of $P_n \times K_m$ (for $m \geq 5$, and $n \geq 5$). We frequently use Proposition 3.1 to find a lower bound on $|D_i|$ (for $0 \leq i \leq n-1$).

Theorem 9. *For $m, n \geq 5$, and $t > 0$, $\gamma^{ID}(P_n \times K_m) \geq (t+1)(m-1)$ if $n = 3t+1$, and $n = 3t+2$.*

Proof. Let D be an identifying code of $P_n \times K_m$.

When $n = 3t + 1$, we write $n = 2 + 3(t-1) + 2$. If we make a bunch of three consecutive columns together, then there are $t - 1$ such bunches for $t > 1$. Moreover, there are four extra columns, say two in the beginning and two at the end. Now, by Proposition 3.1, $|r_0(D)| \geq m-1$ gives $|D_0| + |D_1| \geq m-1$, $|r_{n-1}(D)| \geq m-1$ gives $|D_{n-2}| + |D_{n-1}| \geq m-1$ and also, we get $|D_{3h-1}| + |D_{3h}| + |D_{3h+1}| \geq m-1$(for $1 \leq h \leq t-1$). Thus, when $n = 3t+1$ and $n \geq 7$, $\gamma^{ID}(P_n \times K_m) \geq (m-1) + (t-1)(m-1) + (m-1) = (t+1)(m-1)$.

Now, consider $n = 3t+2$. In this case, we have t bunches (of three consecutive columns together) and two extra columns in the beginning. Now, by Proposition 3.1, $|r_0(D)| \geq m-1$ gives $|D_0| + |D_1| \geq m-1$, and also, we get $|D_{3h-1}| + |D_{3h}| + |D_{3h+1}| \geq m-1$(for $1 \leq h \leq t$). Thus, when $n = 3t+2$ and $n \geq 5$, $\gamma^{ID}(P_n \times K_m) \geq (m-1) + t(m-1) = (t+1)(m-1)$. ■

Figures 7–16 illustrate identifying codes of $P_{15} \times K_6$.

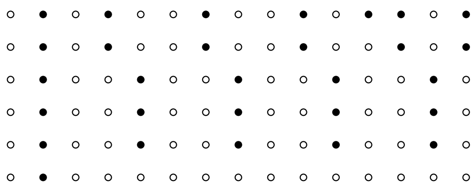

Figure 7. An identifying code of $P_{15} \times K_6$ with 29 codewords.

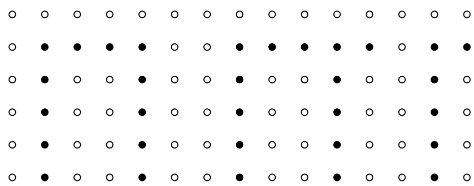

Figure 8. An identifying code of $P_{15} \times K_6$ with 31 codewords.

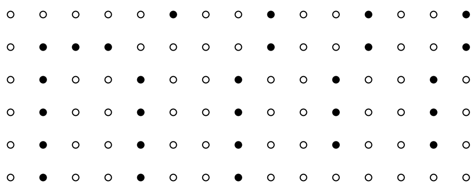

Figure 9. An identifying code of $P_{15} \times K_6$ with 28 codewords.

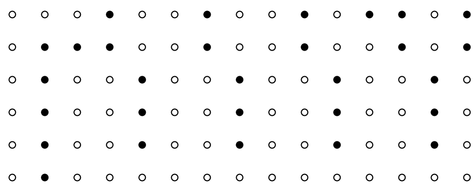

Figure 10. An identifying code of $P_{15} \times K_6$ with 29 codewords.

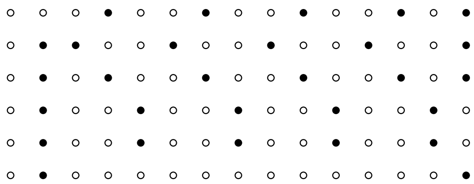

Figure 11. An identifying code of $P_{15} \times K_6$ with 29 codewords.

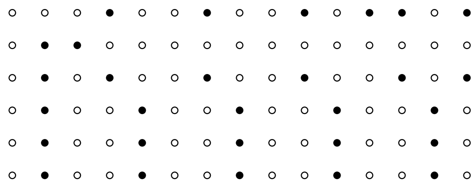

Figure 12. An identifying code of $P_{15} \times K_6$ with 29 codewords.

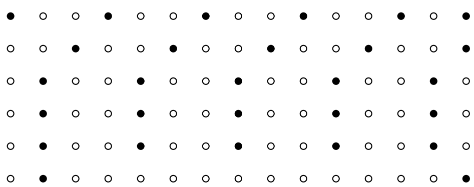

Figure 13. An identifying code of $P_{15} \times K_6$ with 28 codewords.

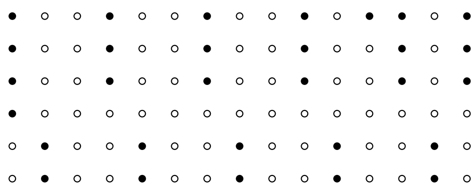

Figure 14. An identifying code of $P_{15} \times K_6$ with 30 codewords.

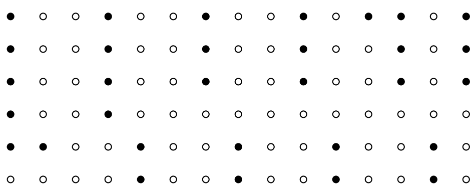

Figure 15. An identifying code of $P_{15} \times K_6$ with 31 codewords.

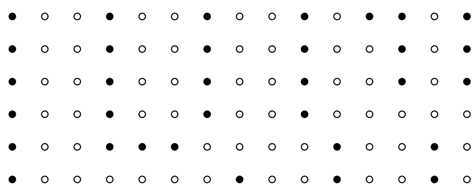

Figure 16. An identifying code of $P_{15} \times K_6$ with 33 codewords.

Now, in the following result, we provide different identifying codes when $n = 3t$, for $t \geq 2$. This constructive method also gives us the lower bound on the size of an identifying code in the direct product $P_n \times K_m$(for $m \geq 5$, and $n \geq 6$).

Theorem 10. *For $m, n \geq 5$ and positive integer t, in the direct product $P_n \times K_m$, $\gamma^{ID}(P_6 \times K_m) \geq 2m$ and $\gamma^{ID}(P_n \times K_m) \geq t(m-1)+3$ when $n = 3t$, for $n \geq 9$.*

Proof. Let D be an identifying code of $P_n \times K_m$. Therefore, by Proposition 3.1, $|r_i(D)| \geq m-1$ for $0 \leq i \leq n-1$. By using the idea applied in Theorem 9, $|D| \geq t(m-1)$. To prove that $|D| \geq t(m-1)+3$, it is enough to prove, for instance, that in at least three bunches of three consecutive columns, each bunch contains at least m codewords. Here, we consider the cases, depending upon the cardinality of D_0 and, then, we will move on step by step towards $|D_i|$, for $1 \leq i \leq n$.

Case 1. If $|D_0| = 0$, then $|D_1| \geq m-1$. If $|D_1| = m$ (see Figure 7 for an example), then $D_2 = \emptyset$. To separate columns C_0 and C_2, at least two codewords must lie in D_3. Since $|r_3(D)| \geq m-1$, $|D_4|$ must be greater than $m-3$. Thus, from column C_5 onward, we apply the idea used in Theorem 8. If $n = 3t$, for $n \geq 9$, then $|D| \geq (0+m+0)+(2+m-3+0)+(2+m-3+0)+\cdots+(2+m-3+1)+(2+m-3+2) = t(m-1)+4$. Thus, in this case, the first bunch and the last two bunches have at least m codewords and the remaining $t-3$ bunches have at least $m-1$ codewords. If $n = 6$, then $|D| \geq (0+m+1)+(2+m-3+2) = 2m+2$.

Now, let $|D_1| = m-1$. Thus, if $(1, v_0) \notin D$, to dominate it at least one vertex of type $(2, v_j)$, for $j \neq 0$, must lie in D, say $(2, v_1)$. Therefore, $|D_2| \geq 1$. Now, to separate columns C_0 and C_2, D_3 must be non-empty.

Case 1.1. Assume that $(3, v_1) \in D$. Thus, $|D_3| \geq 1$. Now, we need $|r_3(D)| \geq m-1$ and $|r_4(D)| \geq m-1$. So, if $|D_3| = 1$, we get $|D_4| \geq m-2$ and $|D_5| \geq 0$. Similarly, if $|D_3| \geq 2$, we get $|D_4| \geq m-3$ and $|D_5| \geq 0$. Now, to dominate $(4, v_1)$, at least one of $(3, v_f), (4, v_1), (5, v_f)$ for $0 \leq f \leq m-1$ and $f \neq 1$ must lie in D. If $(4, v_1) \in D$ (see Figure 8), then in the first three bunches, there are m codewords when $n \geq 9$. So, $|D| \geq t(m-1)+3$. And if $n = 6$, then $|D| \geq 2m$.

If say $(5, v_0)$ (see Figure 9) lies in D, then the first two bunches have m codewords. Then, $D_6 = \emptyset$. As $|r_6(D)|, |r_7(D)| \geq m-1$, $|D_7| \geq m-2$ and $|D_8| \geq 1$, say $\{(7, v_i) : 2 \leq i \leq m-1\} \subseteq D$ and $(8, v_0) \in D$. To dominate $(7, v_0)$ either $D_6 \neq \emptyset$ or $|D_8| \geq 2$, which implies that third bunch also contains m codewords when $n \geq 9$. Therefore, $|D| \geq t(m-1)+3$. And if $n = 6$, then $|D| \geq 2m$.

If say $(3, v_0)$ (see Figure 10) lies in D, $|D_3| \geq 2$ and we need $|D_4| \geq m-3$. For $n = 6$, we select $(5, v_0), (5, v_1) \in D$, which implies that $|D| \geq 2m+1$. Thus, for $n \neq 9$, if $(3, v_0)$ lies in D, then only $m-3$ vertices of D_4 are enough to identify vertices of columns C_3, C_4, C_5. Now, continue in this manner. Thus, from column

C_6 onward we apply the idea used in Theorem 8. We get that if $n = 3t$, for $n \geq 9$, then $|D| \geq (0+m-1+1)+(2+m-3+0)+(2+m-3+0)+\cdots+(2+m-3+1)+(2+m-3+2) = t(m-1)+4$.

Case 1.2. If $(3, v_1) \notin D$, then to separate columns C_0 and C_2, at least two codewords must lie in D_3, say $(3, v_0), (3, v_2) \in D$. Since $|r_3(D)| \geq m-1$, $|D_4| \geq m-4$. If $|D_4| = m-4$, say $\{(4, v_i) : 3 \leq i \leq m-2\} \subset D$, then for $m \geq 6$, one of $(5, v_1), (5, v_{m-1})$ must lie in D since $|r_4(D)| \geq m-1$ (see Figure 11), say $(5, v_1) \in D$. Similarly, we get the pattern $(0+(m-1)+1)+(2+(m-4)+1)+(2+(m-4)+1)+\cdots+(2+(m-4)+1)+(2+(m-4)+4) = t(m-1)+4$. Therefore, $|D| \geq t(m-1)+4$. If $m = 5$, then $|D_4| = m-3$, if $|D_4| = m-4 = 1$, that is $(4, v_3) \in D$, then $(3, v_3)$ and $(1, v_0)$ are not separated by D. If $|D_4| = m-3$, say $\{(4, v_i) : 3 \leq i \leq m-1\} \subset D$, then $D_5 = \emptyset$ (see Figure 12). In a similar way, we get the pattern $(0+(m-1)+1)+(2+(m-3)+0)+(2+(m-3)+0)+\cdots+(2+(m-3)+0)+(2+(m-3)+1)+(2+(m-3)+2) = t(m-1)+4$. Therefore, $|D| \geq t(m-1)+4$.

Case 2. Consider $|D_0| = 1$ (see Figure 13). Assume that $(0, v_0) \in D$. Then, $|D_1| \geq m-2$, say $\{(1, v_i) : 2 \leq i \leq m-1\} \subseteq D$.

Case 2.1. To dominate $(1, v_0)$, say $(2, v_1) \in D$. Therefore, $|D_2| \geq 1$. Now, to separate vertices of columns C_0 and C_2, $|D_3| \geq 1$, say $(3, v_0)$, lie in D. Now onward by continuing the idea of Theorem 8 we get, $|D| \geq (1+m-2+1)+(1+m-2+1)+\cdots+(1+m-2+1)+(1+m-2+3)$. Therefore, $|D| \geq tm+2 \geq t(m-1)+4$ for all $t \geq 2$. We will get D with same cardinality if $(3, v_1)$ is selected instead of $(3, v_0)$. If we select $(3, v_f)$, $f \notin \{0, 1\}$, then to separate C_0 and C_2, at least two vertices of C_3 must lie in D. In this case, $|D| \geq (1+m-2+1)+(2+m-3+0)+\cdots+(2+m-3+0)+(2+m-3+1)+(2+m-3+2) = t(m-1)+4$.

Case 2.2. To dominate $(1, v_0)$, if $(1, v_0) \in D$, then $|D_1| \geq m-1$. Then, to separate vertices of columns C_0 and C_2, $(3, v_0)$ must lie in D. Now onward by continuing the idea of Theorem 8 we get, $|D| \geq (1+m-1+0)+(1+m-1+0)+\cdots+(1+m-1+0)+(1+m-1+1)+(1+m-1+0)$. Therefore, $|D| \geq tm+1 \geq t(m-1)+3$ for all $t \geq 2$.

Case 3. $|D_0| = b$, for $2 \leq b \leq m-3$ $(m \neq 5)$, and b is an integer.

In this case, we get D as in Case A of Theorem 8 where $|D| = t(m-1)+b+1 \geq t(m-1)+3$.

Case 4. If $|D_0| = m-2$, then $|D_1| \geq 1$ (see Figure 14). Assume that $\{(0, v_j) : 0 \leq j \leq m-3\} \subseteq D$ and $(1, v_{m-2}) \in D_1$. To dominate $(0, v_{m-2})$, one more vertex of C_1 must lie in D. Therefore, $|D_1| \geq 2$. Since $|r_1(D)|, |r_2(D)|, |r_3(D)| \geq m-1$, we obtain $|D_2| \geq 0$, $|D_3| \geq m-3, |D_4| \geq 2$, and $|D_5| \geq 0$. From now on, by continuing the idea of Theorem 8 we get $|D| \geq (m-2+2+0)+(m-3+2+0)+\cdots+(m-3+2+1)+(m-3+2+m-3)$. Therefore, $|D| \geq (t+1)(m-1) \geq t(m-1)+4$.

Case 5. If $|D_0| = m - 1$, then $|D_1| \geq 0$ (see Figure 15). Assume that $\{(0, v_j) : 0 \leq j \leq m-2\} \subseteq D$. To dominate $(0, v_{m-1})$, say $(1, v_{m-2})$, which lies in D, therefore, $|D_1| \geq 1$. Now, the fact that $|r_2(D)|, |r_3(D)|, |r_4(D)| \geq m - 1$ gives $|D_3| \geq m - 2, |D_4| \geq 1$, and $|D_5| \geq 0$ (for $n \neq 6$). Assume that $\{(3, v_j) : 0 \leq j \leq m - 3\} \subseteq D$ and $(4, v_{m-2}) \in D$. To separate $(2, v_{m-2})$ and $(4, v_{m-1})$ and to dominate $(3, v_{m-2})$, $(4, v_{m-1})$ must lie in D. Therefore, we get $|D_4| \geq 2$. For $n = 6$, we choose $\{(5, v_j) : 0 \leq j \leq m - 4\} \subseteq D$ and $(2, v_0)$ to make D identifying. For $n \geq 9$, $|r_7(D)| \geq m - 1$ and we obtain $|D_6| \geq m - 3$, $|D_7| \geq 2$, and $|D_8| \geq 0$. From now on, we apply the idea used in Theorem 8. We get the following results. If $n = 3t$, for $n \geq 9$, then $|D| \geq (m-1+1+0) + (m-2+2+0) + (m-3+2+0) + \cdots + (m-3+2+1) + (m-3+2+m-3)$. Thus, $|D| \geq t(m-1) + m \geq t(m-1) + 5$.

If $n = 6$, then $|D| \geq (m-1+1+1) + (m-2+2+m-3)$. Thus, $|D| \geq 3m - 2 \geq 2m$.

Case 6. Consider $|D_0| = m$ (see Figure 16). Since $|r_2(D)| \geq m - 1$ and $|D_1| = 0, |D_2| = 0$, $|D_3| \geq m - 1$, say $\{(3, v_j) : 0 \leq j \leq m-2\} \subseteq D$.

Case 6.1. If $|D_3| = m$, then $D_4 = \emptyset$, and to separate columns C_2 and C_4, we need $|D_5| \geq 2$. Since $|r_5(D)| \geq 5$, $|D_6| \geq m - 3$. From now on, we apply the idea used in Theorem 8. We get the following results. If $n = 3t$, for $n \geq 9$, then $|D| \geq (m+0+0)+(m+0+2)+(m-3+0+2)+\cdots+(m-3+0+2)+(m-3+0+m) = t(m-1) + 2 + m$.

Case 6.2. If $|D_3| = m - 1$, then for $n = 6$, we choose $\{(3, v_j) : 0 \leq j \leq m-2\}$, $\{(5, v_j) : 0 \leq j \leq m - 2\}$, and $(4, v_{m-2})$ in D. Thus, $|D| \geq m + (m-1) + 1 + (m-1) \geq 2m$.

For $n = 3t$, for $n \geq 9$, to dominate $(3, v_{m-1})$, say $(4, v_{m-2}) \in D$. To separate columns C_2 and C_4, say $(5, v_{m-2}) \in D$ (here, we proceed as in Case 1.2). So, $|D| \geq (m+0+0) + (m-1+1+1) + (m-2+1+0) + (m-2+2+0) + (m-3+2+0) + \cdots + (m-3+2+0) + (m-3+2+1) + (m-3+2+m-3)$. Therefore $|D| \geq (t+1)(m-1) + 3$.

If $n = 6$, then $|D| \geq (m+0+0) + (m-1+1+m)$. Therefore, $|D| \geq 3m$.

After comparing all the cases, we observe that in Case 3 (for $n \geq 9$), if we take $b = 2$, we get the smallest value of $|D|$. Hence, $\gamma^{ID}(P_n \times K_m) \geq t(m-1)+3$ when $n = 3t$, for $n \geq 9$. Similarly, by using Case 1.1, we conclude that $\gamma^{ID}(P_6 \times K_m) \geq 2m$. ■

6. IDENTIFYING CODES OF $P_3 \times K_m$ AND $P_4 \times K_m$

In this section, we study the identifying codes of $P_n \times K_m$ for small values of n. While studying identifying codes of $P_3 \times K_m$, we will discuss the necessary and sufficient conditions as well.

First, we prove a necessary condition for a subset D of $V(P_3 \times K_m)$, for $m \geq 3$, to be an identifying code.

Theorem 11. *For $m \geq 3$, if a subset D of $V(P_3 \times K_m)$ is identifying, then all the sets R_j are non-empty. Moreover, at most one row with one codeword, say R_i, is such that $(0, v_i) \in R_i$ or $(2, v_i) \in R_i$ and, hence, $|R_j| \geq 2$, (for $0 \leq j \leq m-1$, and $j \neq i$). Thus, $\gamma^{ID}(P_3 \times K_m) \geq 2m-1$.*

Proof. Assume that there exists one R_i, say R_0, such that $|R_0| = 0$, that is, $(0, v_0), (1, v_0), (2, v_0) \notin D$. Then, $N[(0, v_0)] \cap D = N[(2, v_0)] \cap D$. Therefore, all R_i must be non-empty. In fact, if there is any R_i with $|R_i| = 1$, then it must contain either $(0, v_i)$ or $(2, v_i)$. Otherwise if it is $(1, v_i)$, then, $N[(0, v_i)] \cap D = N[(2, v_i)] \cap D$.

Now, suppose there exist two R_i, say R_1 and R_2, such that $|R_1| = |R_2| = 1$. Then, either $(0, v_1) \in D$ or $(2, v_1) \in D$. Similarly, either $(0, v_2) \in D$ or $(2, v_2) \in D$.

Case 1. If $(0, v_1), (0, v_2) \in D$. That is, $(1, v_1), (2, v_1), (1, v_2), (2, v_2) \notin D$, which gives $N[(2, v_1)] \cap D = N[(2, v_2)] \cap D$.

Case 2. If $(2, v_1), (2, v_2) \in D$. This case is similar to Case 1 by symmetry.

Case 3. If $(0, v_1), (2, v_2) \in D$. That is, $(1, v_1), (2, v_1), (0, v_2), (1, v_2) \notin D$, which gives $N[(2, v_1)] \cap D = N[(0, v_2)] \cap D$.

Therefore, there is at most one R_i, which contains only one codeword. Hence, all the remaining R_i have at least two elements of D.

Thus, $\gamma^{ID}(P_3 \times K_m) \geq 2m-1$. ■

The above condition is necessary but not sufficient for example, see Figure 18.

We now prove a sufficient condition for a subset D of $V(P_3 \times K_m)$, for $m \geq 3$, to be an identifying code.

Theorem 12. *For $m \geq 3$, if a subset D of $V(P_3 \times K_m)$ is such that for exactly one i, $|R_i| = 1$, which contains either $(0, v_i)$ or $(2, v_i)$, $|R_j| \geq 2$ (for $0 \leq j \leq m-1$, and $j \neq i$) and $|D_f| \geq 2$ for $0 \leq f \leq 2$, then D is identifying.*

Proof. By using Lemma 7, D is dominating. Without loss of generality, assume that $|R_0| = 1$, say $(0, v_0) \in R_0$ and $|R_f| \geq 2$ for $1 \leq f \leq m-1$. Let $(i, v_j), (k, v_l)$ be any two vertices of $P_3 \times K_m$.

Case 1. $i = k$. If both or one of (i, v_j) and (k, v_l) belong to D, then they are separated by D since they are non-adjacent. Therefore, assume that (i, v_j), $(k, v_l) \notin D$.

Case 1.1. $i = k = 0$. For $1 \leq j < l \leq m-1$, $(1, v_l), (1, v_j) \in (N[(i, v_j)] \triangle N[(k, v_l)]) \cap D$.

Case 1.2. $i = k = 2$. For $0 \leq j < l \leq m-1$, $(1, v_l) \in (N[(i, v_j)] \triangle N[(k, v_l)]) \cap D$.

Case 1.3. $i = k = 1$. For $0 \leq j < l \leq m-1$, $(0, v_l), (0, v_j), (2, v_l) \in (N[(i, v_j)] \triangle N[(k, v_l)]) \cap D$.

Case 2. $i \neq k$.

Case 2.1. $d_{P_3}(i, k) = 1$. If $v_j = v_l$, then they are separated since $N[(i, v_j)] \cap N[(k, v_l)] = \emptyset$. If $v_j \neq v_l$, then for $i = 0$ and $k = 1$, one vertex in D_2 separates them. If $v_j \neq v_l$, then for $i = 2$ and $k = 1$, one vertex in D_0 separates them.

Case 2.2. $d_{P_3}(i, k) = 2$. Without loss of generality, assume that $i = 0$ and $k = 2$. If both or one of $(0, v_j)$ and $(2, v_l)$ belong to D, then they are separated by D since they are non-adjacent. Therefore, assume that $(0, v_j), (2, v_l) \notin D$, which implies that $j \neq l$. In this case, $(1, v_j)$ or $(1, v_l) \in (N[(0, v_j)] \triangle N[(2, v_l)]) \cap D$.

Thus, D is separating and, hence, identifying. ■

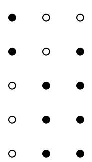

Figure 17. An identifying code of $P_3 \times K_5$.

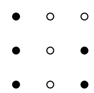

Figure 18. An example showing that a necessary condition for $P_3 \times K_m$ is not sufficient.

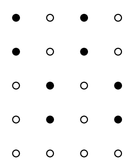

Figure 19. An identifying code of $P_4 \times K_5$.

We now obtain $\gamma^{ID}(P_3 \times K_m)$ and $\gamma^{ID}(P_4 \times K_m)$.

Theorem 13. *For $m \geq 3$, $\gamma^{ID}(P_3 \times K_m) = 2m - 1$.*

Proof. By Theorem 11, $\gamma^{ID}(P_3 \times K_m) \geq 2m - 1$. For $m = 3$, $D = \{(0, v_0), (0, v_1), (1, v_1), (1, v_2), (2, v_2)\}$ is an identifying code of $P_3 \times K_3$ of cardinality 5.

For $m \geq 4$, by using Theorem 12, it can be easily observed that, a subset $D = \{(0, v_0), (0, v_1), (1, v_2), \ldots, (1, v_{m-1}), (2, v_1), (2, v_2), \ldots, (2, v_{m-1})\}$ is an identifying code of $P_3 \times K_m$ of cardinality $2m - 1$. Thus, $\gamma^{ID}(P_3 \times K_m) = 2m - 1$ (see Figure 17). ■

Theorem 14. *For $m \geq 5$, $\gamma^{ID}(P_4 \times K_m) = 2m - 2$.*

Proof. In $P_4 \times K_m$, by Proposition 3.1, the conditions $|r_0(D)| \geq m - 1$ and $|r_3(D)| \geq m - 1$ give $\gamma^{ID}(P_4 \times K_m) \geq 2m - 2$. Also, by Proposition 3.2, the subset $D = \{(0, v_0), (0, v_1), (2, v_0), (2, v_1)\} \cup \bigcup_{i=2}^{m-2} \{(1, v_i), (3, v_i)\}$ is an identifying code of $P_4 \times K_m$ of cardinality $2m - 2$. Thus, $\gamma^{ID}(P_4 \times K_m) = 2m - 2$ (see Figure 19). ■

Remark. For $m \geq 5$, the number of codewords required to identify $P_4 \times K_m$ is one less than that needed to identify $P_3 \times K_m$.

7. IDENTIFYING CODES OF $P_n \times K_4$ AND $P_n \times K_3$.

In this section, we provide identifying codes of $P_n \times K_4$ and necessary condition on a subset D to be an identifying code of $P_n \times K_4$ and $P_n \times K_3$. By using this condition, we derive a lower bound on $\gamma^{ID}(P_n \times K_4)$ and that of $P_n \times K_3$.

In the following result, we provide a subset D of $V(P_4 \times K_4)$. It can be easily observed that the set D is dominating and separating.

Theorem 15. $\gamma^{ID}(P_4 \times K_4) = 7$.

Proof. By using Proposition 3.1, $\gamma^{ID}(P_4 \times K_4) \geq 6$. It can be easily verified that a set of any six vertices of $P_4 \times K_4$ satisfying the necessary condition does not separate all vertices of $P_4 \times K_4$. So, $\gamma^{ID}(P_4 \times K_4) \geq 7$. The set $D = \{(0, v_0), (0, v_1), (1, v_2), (1, v_3), (2, v_1), (2, v_2), (2, v_3)\}$ is an identifying code of $P_4 \times K_4$ of cardinality 7 (see Figure 20). ■

Figures 20–29 illustrate identifying codes of $P_n \times K_4$ for different values of n.

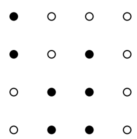

Figure 20. An identifying code of $P_4 \times K_4$.

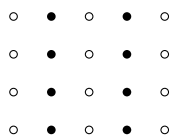

Figure 21. An identifying code of $P_5 \times K_4$.

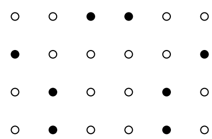

Figure 22. An identifying code of $P_6 \times K_4$.

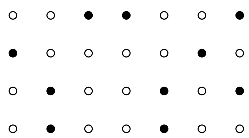

Figure 23. An identifying code of $P_7 \times K_4$.

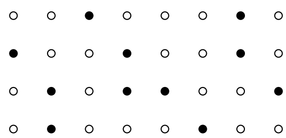

Figure 24. An identifying code of $P_8 \times K_4$.

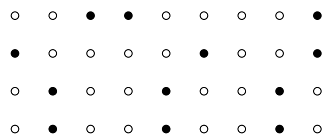

Figure 25. An identifying code of $P_9 \times K_4$.

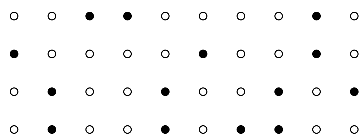

Figure 26. An identifying code of $P_{10} \times K_4$.

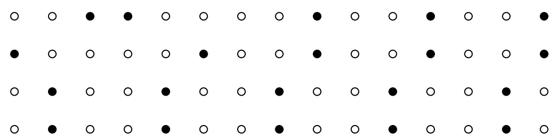

Figure 27. An identifying code of $P_{15} \times K_4$.

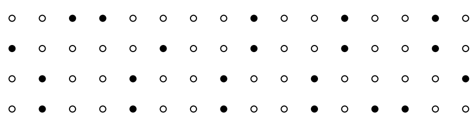

Figure 28. An identifying code of $P_{16} \times K_4$.

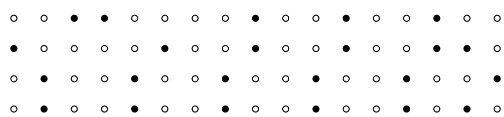

Figure 29. An identifying code of $P_{17} \times K_4$.

Theorem 16. *For $n \geq 5$,*

$$\gamma^{ID}(P_n \times K_4) \leq \begin{cases} 4t & \text{if } n = 3t,\ t \geq 2, \\ 4t+2 & \text{if } n = 3t+1,\ t = 2,3, \\ 4t+1 & \text{if } n = 3t+1,\ t \geq 4, \\ 4t+4 & \text{if } n = 3t+2,\ t = 1, \\ 4t+3 & \text{if } n = 3t+2,\ t \geq 2. \end{cases}$$

Proof. It is easy to check that following codes are identifying of $P_n \times K_4$.

Case 1. For $n = 3t$, with $n \geq 6$, $\{(0, v_1), (1, v_2), (1, v_3), (2, v_0), (3, v_0), (4, v_2), (4, v_3), (5, v_1), (7, v_2), (7, v_3), (8, v_0), (8, v_1), (10, v_2), (10, v_3), (11, v_0), (11, v_1), \ldots, (3t-2, v_2), (3t-2, v_3), (3t-1, v_0), (3t-1, v_1)\}$ is an identifying code of cardinality $4t$ (see Figure 22, 25, and 27).

Case 2. For $n = 7$, $\{(0, v_1), (1, v_2), (1, v_3), (2, v_0), (3, v_0), (4, v_2), (4, v_3), (5, v_1), (6, v_0), (6, v_2)\}$ is an identifying code of cardinality 10 (see Figure 23).

For $n = 10$, $\{(0, v_1), (1, v_2), (1, v_3), (2, v_0), (3, v_0), (4, v_2), (4, v_3), (5, v_1), (6, v_3), (7, v_2), (7, v_3), (8, v_0), (8, v_1), (9, v_2)\}$ is an identifying code of cardinality 14 (see Figure 26).

For $n = 3t + 1$, with $n \geq 13$, $\{(0, v_1), (1, v_2), (1, v_3), (2, v_0), (3, v_0), (4, v_2), (4, v_3), (5, v_1), (7, v_2), (7, v_3), (8, v_0), (8, v_1), \ldots, (3t-5, v_2), (3t-5, v_3), (3t-4, v_0), (3t-4, v_1), (3t-3, v_3), (3t-2, v_3), (3t-1, v_0), (3t-1, v_1), (3t, v_2)\}$ is an identifying code of cardinality $4t + 1$ (see Figure 28).

Case 3. For $n = 5$, $\{(1, v_0), (1, v_1), (1, v_2), (1, v_3), (3, v_0), (3, v_1), (3, v_2), (3, v_3)\}$ is an identifying code of cardinality 8 (see Figure 21).

For $n = 8$, $\{(0, v_1), (1, v_2), (1, v_3), (2, v_0), (3, v_1), (3, v_2), (4, v_2), (5, v_3), (6, v_0), (6, v_1), (7, v_2)\}$ is an identifying code of cardinality 11 (see Figure 24).

For $n = 3t + 2$, with $n \geq 11$, $\{(0, v_1), (1, v_2), (1, v_3), (2, v_0), (3, v_0), (4, v_2), (4, v_3), (5, v_1), (7, v_2), (7, v_3), (8, v_0), (8, v_1), \ldots, (3t-2, v_2), (3t-2, v_3), (3t-1, v_0), (3t-1, v_1), (3t, v_1), (3t, v_3), (3t+1, v_2)\}$ is an identifying code of cardinality $4t+3$ (see Figure 29). ■

Theorem 17. *If a subset D of $V(P_n \times K_4)$, for $n \geq 5$, is a minimum identifying code, then $|r_i(D)| \geq 3$ for $0 \leq i \leq n-1$ and $|D_{3q} \cup D_{3q+1} \cup D_{3q+2}| \geq 4$ for all $0 \leq q < \left\lfloor \frac{n}{3} \right\rfloor$.*

Proof. If $n = 5, 6, 7$, and 8, it is easy to check that the result holds (see Figure 21–24). Therefore, in this proof, we assume that $n \geq 9$.

The proof of $|r_i(D)| \geq 3$ follows from Proposition 3.1. Since $|r_i(D)| \geq 3$ (for $0 \leq i \leq n-1$), $|D_{3q} \cup D_{3q+1} \cup D_{3q+2}| \geq 3$ for all $0 \leq q < \left\lfloor \frac{n}{3} \right\rfloor$. In Theorem 16, we constructed an identifying code B in $P_n \times K_4$ such that $|B_{3q} \cup B_{3q+1} \cup B_{3q+2}| = 4$

for all $0 \leq q < \lfloor \frac{n}{3} \rfloor$. Therefore, to prove that for an identifying code D in $P_n \times K_4$, $|D_{3q} \cup D_{3q+1} \cup D_{3q+2}| \geq 4$ for all $0 \leq q < \lfloor \frac{n}{3} \rfloor$, it is enough to prove that if there exists one q such that $|D_{3q} \cup D_{3q+1} \cup D_{3q+2}| = 3$, then either D is not identifying, or D is identifying with cardinality more than that of Theorem 16. Thus, assume that there exists one q such that $0 \leq q < \lfloor \frac{n}{3} \rfloor$ and $|D_{3q} \cup D_{3q+1} \cup D_{3q+2}| = 3$. If $q = 0$ or $q = \lfloor \frac{n}{3} \rfloor - 1$, then D is not even dominating. So, assume that $1 \leq q \leq \lfloor \frac{n}{3} \rfloor - 2$.

In each of the following cases, $k_1 + k_2 + k_3 = 3$, where either $|D_{3q}| = k_1$, $|D_{3q+1}| = k_2$, and $|D_{3q+2}| = k_3$ or $|D_{3q}| = k_3$, $|D_{3q+1}| = k_2$, and $|D_{3q+2}| = k_1$. We apply the same technique for the construction of D in both cases. Here, we discuss only one of them. Moreover, while placing vertices of $D_{3q} \cup D_{3q+1} \cup D_{3q+2}$, $|r_{3q+1}(D)| \geq 3$ is considered in every case.

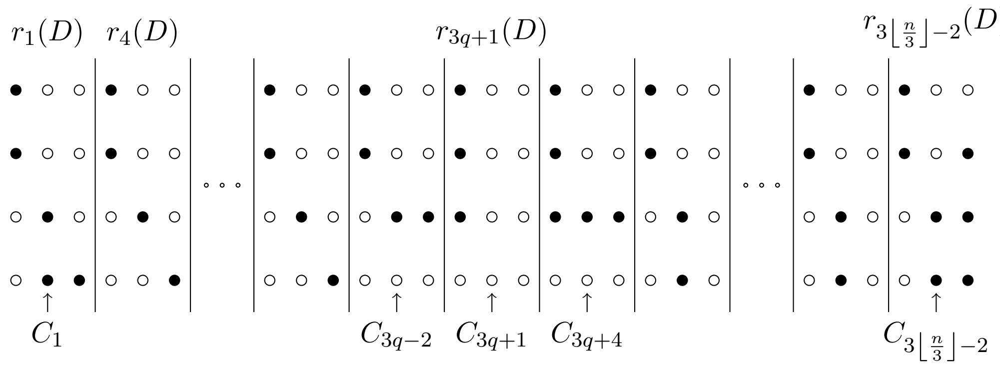

Figure 30. An identifying code of $P_{3t} \times K_4$.

Case 1. $|D_{3q}| = 3$, $|D_{3q+1}| = 0$, and $|D_{3q+2}| = 0$, say $(3q, v_0), (3q, v_1), (3q, v_2) \in D$ (see Figure 30). In this case, to dominate $(3q, v_3)$ at least one of $\{(3q-1, v_i) : 0 \leq i \leq 2\}$ must lie in D, say $(3q-1, v_2) \in D$. To separate columns C_{3q-1} and C_{3q+1}, either $(3q-2, v_2) \in D$ or $|D_{3q-2}| \geq 2$. If $(3q-2, v_2) \in D$, then $|D_{3q-3}| \geq 2$ since $|r_{3q-2}(D)| \geq 3$, which implies that $|D_{3q-3} \cup D_{3q-2} \cup D_{3q-1}| \geq 4$. If $(3q-2, v_2) \notin D$, then to separate C_{3q-1} and C_{3q+1}, $|D_{3q-2}| \geq 2$, say $\{(3q-2, v_1), (3q-2, v_3)\} \subseteq D$. To dominate $(3q-2, v_2)$, $|D_{3q-3}| \geq 1$, which implies that $|D_{3q-3} \cup D_{3q-2} \cup D_{3q-1}| \geq 4$. By continuing in this manner, we get that $|D_{3s-3} \cup D_{3s-2} \cup D_{3s-1}| \geq 4$ for all $1 \leq s \leq q-1$ and $|D_0 \cup D_1 \cup D_2| \geq 5$. Similarly, since $|r_{3q+2}(D)| \geq 3$, $|D_{3q+3}| \geq 3$, say $\{(3q+3, v_i) : 0 \leq i \leq 2\} \subseteq D$. Then, to dominate $(3q+3, v_3)$, $|D_{3q+4}| \geq 1$, say $(3q+4, v_2) \in D$. To separate columns C_{3q+2} and C_{3q+4}, $|D_{3q+5}| \geq 1$, which implies that $|D_{3q+3} \cup D_{3q+4} \cup D_{3q+5}| \geq 5$. By continuing in this manner, we get that $|D_{3s} \cup D_{3s+1} \cup D_{3s+2}| \geq 4$ for all $q+2 \leq s \leq \lfloor \frac{n}{3} \rfloor - 2$. If $n = 3t$, for $n \geq 12$, then $\left|D_{\lfloor \frac{n}{3} \rfloor-3} \cup D_{\lfloor \frac{n}{3} \rfloor-2} \cup D_{\lfloor \frac{n}{3} \rfloor-1}\right| \geq 7$ and, therefore, $|D| \geq 4t + 4$. If $n = 9$, $|D| \geq 15$. If $n = 3t + 1$, for $n \geq 10$, then $|D_{\lfloor \frac{n}{3} \rfloor-3} \cup D_{\lfloor \frac{n}{3} \rfloor-2} \cup D_{\lfloor \frac{n}{3} \rfloor-1}| \geq 7$, and $|D| \geq 4t+4$. If $n = 3t+2$, for $n \geq 11$, then

$\left|D_{\lfloor \frac{n}{3} \rfloor -3} \cup D_{\lfloor \frac{n}{3} \rfloor -2} \cup D_{\lfloor \frac{n}{3} \rfloor -1}\right| \geq 4$, $|r_{n-1}(D)| \geq 4$, in which case $|D| \geq 4t+5$.

Case 2. $|D_{3q}| = 0$, $|D_{3q+1}| = 3$, and $|D_{3q+2}| = 0$, say $(3q+1, v_0), (3q+1, v_1), (3q+1, v_2) \in D$. In this case $(3q+1, v_3)$ is not dominated by D.

Case 3. $|D_{3q}| = 1$, $|D_{3q+1}| = 2$, and $|D_{3q+2}| = 0$, say $(3q, v_0), (3q+1, v_1), (3q+1, v_2) \in D$. In this case $(3q+1, v_0)$ is not dominated by D.

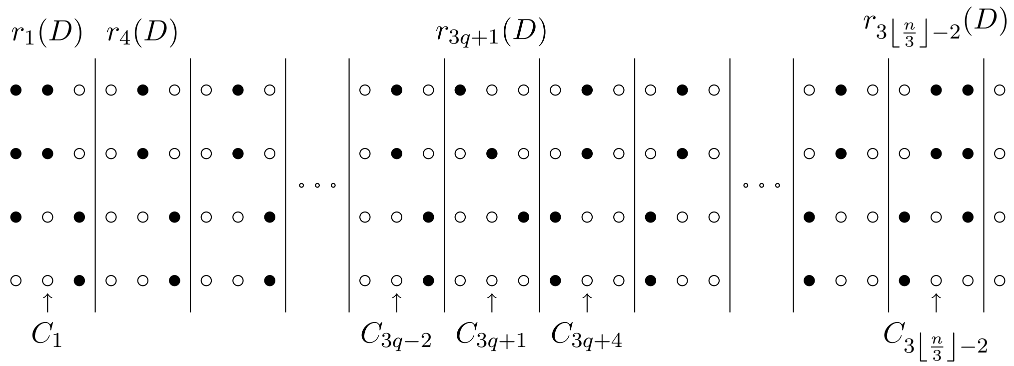

Figure 31. An identifying code of $P_{3t+1} \times K_4$.

Case 4. $|D_{3q}| = 1$, $|D_{3q+1}| = 1$, and $|D_{3q+2}| = 1$, say $(3q, v_0), (3q+1, v_1), (3q+2, v_2) \in D$. By following the procedure used in Case 1 (see Figure 31), $|D_0 \cup D_1 \cup D_2| \geq 7$, $|D_{3s} \cup D_{3s+1} \cup D_{3s+2}| \geq 4$ for $1 \leq s \leq q-1$ and $q+1 \leq s \leq \lfloor \frac{n}{3} \rfloor - 2$. If $n = 3t$, for $n \geq 9$, then $\left|D_{\lfloor \frac{n}{3} \rfloor -3} \cup D_{\lfloor \frac{n}{3} \rfloor -2} \cup D_{\lfloor \frac{n}{3} \rfloor -1}\right| \geq 7$, and $|D| \geq 4t+5$. If $n = 3t+1$, for $n \geq 10$, then $\left|D_{\lfloor \frac{n}{3} \rfloor -3} \cup D_{\lfloor \frac{n}{3} \rfloor -2} \cup D_{\lfloor \frac{n}{3} \rfloor -1}\right| \geq 7$, and $|D| \geq 4t+5$. If $n = 3t+2$, for $n \geq 11$, then $\left|D_{\lfloor \frac{n}{3} \rfloor -3} \cup D_{\lfloor \frac{n}{3} \rfloor -2} \cup D_{\lfloor \frac{n}{3} \rfloor -1}\right| \geq 4$, and $|r_{n-1}(D)| \geq 4$, in which case $|D| \geq 4t+6$.

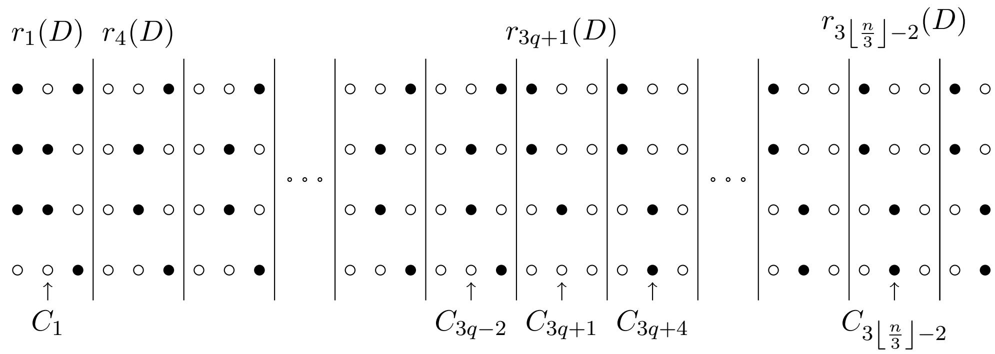

Figure 32. An identifying code of $P_{3t+2} \times K_4$.

Case 5. $|D_{3q}| = 2$, $|D_{3q+1}| = 1$, and $|D_{3q+2}| = 0$, say $(3q, v_0), (3q, v_1), (3q + 1, v_2) \in D$. By following the procedure used in Case 1 (see Figure 32), $|D_0 \cup D_1 \cup D_2| \geq 7$, $|D_{3s} \cup D_{3s+1} \cup D_{3s+2}| \geq 4$ for $1 \leq s \leq q-1$ and $q+1 \leq s \leq \lfloor \frac{n}{3} \rfloor - 2$. If $n = 3t$, for $n \geq 9$, then $\left| D_{\lfloor \frac{n}{3} \rfloor - 3} \cup D_{\lfloor \frac{n}{3} \rfloor - 2} \cup D_{\lfloor \frac{n}{3} \rfloor - 1} \right| \geq 7$, and $|D| \geq 4t + 5$. If $n = 3t + 1$, for $n \geq 10$, then $\left| D_{\lfloor \frac{n}{3} \rfloor - 3} \cup D_{\lfloor \frac{n}{3} \rfloor - 2} \cup D_{\lfloor \frac{n}{3} \rfloor - 1} \right| \geq 7$, and $|D| \geq 4t + 5$. If $n = 3t + 2$, for $n \geq 11$, then $\left| D_{\lfloor \frac{n}{3} \rfloor - 3} \cup D_{\lfloor \frac{n}{3} \rfloor - 2} \cup D_{\lfloor \frac{n}{3} \rfloor - 1} \right| \geq 4$, and $|r_{n-1}(D)| \geq 4$, in which case $|D| \geq 4t + 6$.

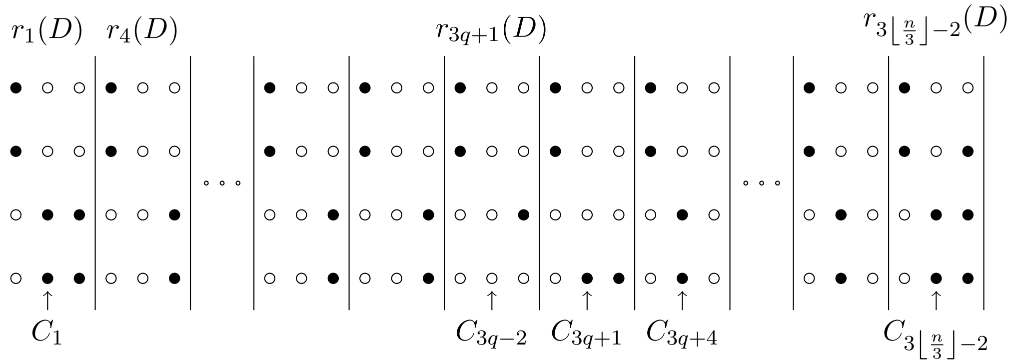

Figure 33. An identifying code of $P_{3t} \times K_4$.

Case 6. $|D_{3q}| = 2$, $|D_{3q+1}| = 0$, and $|D_{3q+2}| = 1$, say $(3q, v_0), (3q, v_1), (3q + 2, v_2) \in D$. By following the procedure used in Case 1 (see Figure 33), $|D_0 \cup D_1 \cup D_2| \geq 6$, $|D_{3s} \cup D_{3s+1} \cup D_{3s+2}| \geq 4$ for $1 \leq s \leq q-1$ and $q+1 \leq s \leq \lfloor \frac{n}{3} \rfloor - 2$. If $n = 3t$, for $n \geq 9$, then $\left| D_{\lfloor \frac{n}{3} \rfloor - 3} \cup D_{\lfloor \frac{n}{3} \rfloor - 2} \cup D_{\lfloor \frac{n}{3} \rfloor - 1} \right| \geq 7$, and $|D| \geq 4t + 4$. If $n = 3t + 1$, for $n \geq 10$, then $\left| D_{\lfloor \frac{n}{3} \rfloor - 3} \cup D_{\lfloor \frac{n}{3} \rfloor - 2} \cup D_{\lfloor \frac{n}{3} \rfloor - 1} \right| \geq 7$, and $|D| \geq 4t + 4$. If $n = 3t + 2$, for $n \geq 11$, then $\left| D_{\lfloor \frac{n}{3} \rfloor - 3} \cup D_{\lfloor \frac{n}{3} \rfloor - 2} \cup D_{\lfloor \frac{n}{3} \rfloor - 1} \right| \geq 4$, and $|r_{n-1}(D)| \geq 4$, in which case $|D| \geq 4t + 5$. ∎

Theorem 18. *For $n \geq 5$,*

$$\gamma^{ID}(P_n \times K_4) \geq \begin{cases} 4t & \text{if } n = 3t,\ t \geq 2, \\ 4t+2 & \text{if } n = 3t+1,\ t = 2, 3, \\ 4t+1 & \text{if } n = 3t+1,\ t \geq 4, \\ 4t+4 & \text{if } n = 3t+2,\ t = 1, \\ 4t+3 & \text{if } n = 3t+2,\ t \geq 2. \end{cases}$$

Proof. Assume that D is a minimum identifying code of $P_n \times K_4$. Then, by Theorem 17, $|r_i(D)| \geq 3$ for $0 \leq i \leq n-1$ and $\left| \bigcup_{j=3q}^{3q+2} D_j \right| \geq 4$ for all $0 \leq q < \lfloor \frac{n}{3} \rfloor$.

Case 1. If $n = 3t$, for $n \geq 6$, then $|D| \geq 4t$.

Case 2. If $n = 3t + 1$, for $n \geq 7$. In this case, if $0 \leq |D_{n-2}| \leq 2$, then $|D_{n-1}| \geq 1$ since $|r_{n-1}(D)| \geq 3$. Thus, $|D| \geq 4t + 1$. It is easy to check that for $t = 2$, and 3, $|D| \geq 4t + 2$.

Case 2.1. If $|D_{n-2}| = 3$, then the necessary condition $r_{n-1}(D)$ is satisfied. So, D_{n-1} may remain empty.

Case 2.1.1. Assume that $(n-2, v_0), (n-2, v_1), (n-2, v_2), (n-3, v_3) \in D$. To separate vertices of columns C_{n-1} and C_{n-3}, $|D_{n-4}| \geq 1$. To cover $(n-2, v_3)$, either $|D_{n-1}| \geq 1$ or $|D_{n-3}| \geq 2$. Thus, $|D| \geq 4t + 2$.

Case 2.1.2. Assume that $(n-2, v_0), (n-2, v_1), (n-2, v_2), (n-3, v_2) \in D$. To separate vertices of columns C_{n-1} and C_{n-3}, $(n-4, v_2) \in D$. Thus, $|D_{n-4} \cup D_{n-3} \cup D_{n-2}| \geq 5$. Similarly, $|D_{3q} \cup D_{3q+1} \cup D_{3q+2}| \geq 4$ for all $1 \leq q \leq \lfloor \frac{n}{3} \rfloor - 1$. Similarly, $|D_0 \cup D_1 \cup D_2| \geq 6$. So, $|D| \geq 4t + 3$.

Case 2.1.3. Assume that $(n-2, v_0), (n-2, v_1), (n-2, v_2), (n-4, v_2) \in D$. To cover $(n-2, v_3)$, $|D_{n-1}| \geq 1$, say $(n-1, v_2)$. Thus, $|D_{3q} \cup D_{3q+1} \cup D_{3q+2}| \geq 4$ for all $1 \leq q \leq \lfloor \frac{n}{3} \rfloor$. Similarly, $|D_0 \cup D_1 \cup D_2| \geq 6$. So, $|D| \geq 4t + 3$.

Case 2.2. $|D_{n-2}| = 4$. Then, $|D_{n-1}| \geq 0$. To separate vertices of columns C_{n-1} and C_{n-3}, $|D_{n-4}| \geq 2$. Thus, $|D_{n-4} \cup D_{n-3} \cup D_{n-2}| \geq 6$. So, $|D| \geq 4t + 2$.

Case 3. If $n = 3t + 2$, for $n \geq 5$, then $|D| \geq 4t + |r_{n-1}(D)| \geq 4t + 3$. When $t = 1$, it can be easily checked that a set of any seven vertices of $P_5 \times K_4$ satisfying the necessary condition (Theorem 17) is not separating all vertices of $P_5 \times K_4$. Therefore, when $t = 1$, $|D| \geq 8$. ∎

Now, we will discuss the case of $P_n \times K_3$. Figures 34–39 illustrate identifying codes of $P_n \times K_3$ for different values of n.

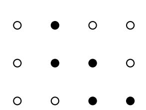

Figure 34. An identifying code of $P_4 \times K_3$.

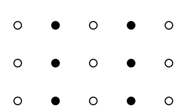

Figure 35. An identifying code of $P_5 \times K_3$.

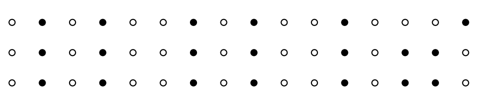

Figure 36. An identifying code of $P_{16} \times K_3$.

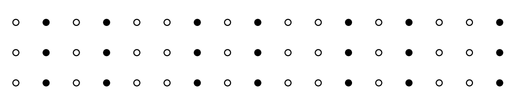

Figure 37. An identifying code of $P_{17} \times K_3$.

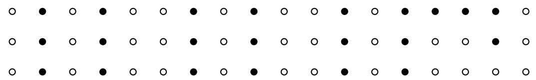

Figure 38. An identifying code of $P_{18} \times K_3$.

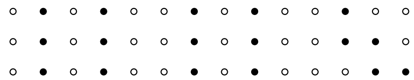

Figure 39. An identifying code of $P_{14} \times K_3$.

By using the idea applied in Theorem 17, we state the following result without proof.

Theorem 19. *If a subset D of $V(P_n \times K_3)$, for $n \geq 4$, is a minimum identifying code, then $|r_i(D)| \geq 2$ for $0 \leq i \leq n-1$ and $|D_q \cup D_{q+1} \cup D_{q+2} \cup D_{q+3} \cup D_{q+4}| \geq 6$ for all $0 \leq q \leq n-5$.*

By Theorem 19 and by using the idea applied in Theorem 18, we state the following result without proof.

Theorem 20. *For $n \geq 4$,*

$$\gamma^{ID}(P_n \times K_3) \geq \begin{cases} 5 & \text{if } n = 4, \\ 6t & \text{if } n = 5t, \ t \geq 1, \\ 6t + i + 1 & \text{if } n = 5t + i, \ t \geq 1, \ 1 \leq i \leq 4. \end{cases}$$

Theorem 21. *For $n \geq 4$,*

$$\gamma^{ID}(P_n \times K_3) \leq \begin{cases} 5 & \text{if } n = 4, \\ 6t & \text{if } n = 5t, \ t \geq 1, \\ 6t + i + 1 & \text{if } n = 5t + i, \ t \geq 1, \ 1 \leq i \leq 4. \end{cases}$$

Proof. It is easy to check that the given codes are identifying in $P_n \times K_3$.

Case 1. If $n = 5t$, then $D = \bigcup_{j=0}^{t-1}\{(5j+1, v_0), (5j+1, v_1), (5j+1, v_2), (5j+3, v_0), (5j+3, v_1), (5j+3, v_2)\}$ is an identifying code of cardinality $6t$ (see Figure 35).

Case 2. If $n = 5t+1$, then $D = \bigcup_{j=0}^{t-2}\{(5j+1, v_0), (5j+1, v_1), (5j+1, v_2), (5j+3, v_0), (5j+3, v_1), (5j+3, v_2)\} \cup \{(5t-4, v_0), (5t-4, v_1), (5t-4, v_2), (5t-2, v_1), (5t-2, v_2), (5t-1, v_1), (5t-1, v_2), (5t, v_0)\}$ is an identifying code of cardinality $6t+2$ (see Figure 36).

Case 3. If $n = 5t+2$, then $D = \bigcup_{j=0}^{t-1}\{(5j+1, v_0), (5j+1, v_1), (5j+1, v_2), (5j+3, v_0), (5j+3, v_1), (5j+3, v_2)\} \cup \{(5t+1, v_0), (5t+1, v_1), (5t+1, v_2)\}$ is an identifying code of cardinality $6t+3$ (see Figure 37).

Case 4. If $n = 5t+3$, then $D = \bigcup_{j=0}^{t-1}\{(5j+1, v_0), (5j+1, v_1), (5j+1, v_2), (5j+3, v_0), (5j+3, v_1), (5j+3, v_2)\} \cup \{(5t-1, v_0), (5t, v_0), (5t+1, v_0), (5t+1, v_1)\}$ is an identifying code of cardinality $6t+4$ (see Figure 38).

Case 5. If $n = 5t+4$, then $D = \bigcup_{j=0}^{t-1}\{(5j+1, v_0), (5j+1, v_1), (5j+1, v_2), (5j+3, v_0), (5j+3, v_1), (5j+3, v_2)\} \cup \{(5t+1, v_0), (5t+1, v_1), (5t+2, v_1), (5t+2, v_2), (5t+3, v_2)\}$ is an identifying code of cardinality $6t + 5$ (see Figure 34, and 39). ∎

We summarize our results in the following table.

n	m	$\gamma^{ID}(P_n \times K_m)$
$n = 3$	$m \geq 3$	$2m - 1^{[\text{Theorem 6.1, 6.3}]}$
$n = 4$	$m \geq 5$	$2m - 2^{[\text{Theorem 6.4}]}$
$n = 6$	$m \geq 5$	$2m^{[\text{Theorem 4.1, 5.2}]}$
$n = 3t,\ n \geq 9$	$m \geq 5$	$\frac{n}{3}(m-1) + 3^{[\text{Theorem 4.1, 5.2}]}$
$n = 3t+1,\ n \geq 7$	$m \geq 5$	$(\lfloor \frac{n}{3} \rfloor + 1)(m-1)^{[\text{Theorem 4.1, 5.1}]}$
$n = 3t+2,\ n \geq 5$	$m \geq 5$	$(\lfloor \frac{n}{3} \rfloor + 1)(m-1)^{[\text{Theorem 4.1, 5.1}]}$
$n = 4$	$m = 4$	$7^{[\text{Theorem 7.1}]}$
$n = 5$	$m = 4$	$8^{[\text{Theorem 7.2, 7.4}]}$
$n = 7$	$m = 4$	$10^{[\text{Theorem 7.2, 7.4}]}$
$n = 10$	$m = 4$	$14^{[\text{Theorem 7.2, 7.4}]}$
$n = 3t,\ n \geq 6$	$m = 4$	$\frac{4n}{3}^{[\text{Theorem 7.2, 7.4}]}$
$n = 3t+1,\ n \geq 13$	$m = 4$	$4\lfloor \frac{n}{3} \rfloor + 1^{[\text{Theorem 7.2, 7.4}]}$
$n = 3t+2,\ n \geq 8$	$m = 4$	$4\lfloor \frac{n}{3} \rfloor + 3^{[\text{Theorem 7.2, 7.4}]}$
$n = 5t,\ n \geq 5$	$m = 3$	$\frac{6n}{5}^{[\text{Theorem 7.6, 7.7}]}$
$n = 5t+i,\ n \geq 4,\ 1 \leq i \leq 4$	$m = 3$	$6\lfloor \frac{n}{5} \rfloor + i + 1^{[\text{Theorem 7.6, 7.7}]}$

Concluding remarks. In this work, we studied identifying codes in $P_n \times K_m$. If one goes to infinity (in n and m), the density of a minimum identifying code is 1/3 in $P_n \times K_m$, 2/3 in $P_3 \times K_m$, 1/2 in $P_4 \times K_m$, 1/3 in $P_n \times K_4$, and 2/5 in $P_n \times K_3$. It is interesting to observe that, Lu *et al.* [23] found the density of a minimum identifying code in $C_n \times K_m$ for $n \geq 5$ and $m \geq 6$, where C_n is a cycle of length n, and it is 1/3 if one goes to infinity (in n and m).

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