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SCAFFOLD FOR THE POLYHEDRAL EMBEDDING OF CUBIC GRAPHS

Flor Aguilar-Campos

Gabriela Araujo-Pardo

Instituto de Matemáticas
Universidad Nacional Autónoma de México, México

e-mail: flor@matem.unam.mx
garaujo@matem.unam.mx

and

Natalia García-Colín

Computer Science Department
ULB (scientific colaborator)

e-mail: natalia.garcia.colin@ulb.ac.be

Abstract

Let G be a cubic graph and Π be a polyhedral embedding of this graph. The *extended graph*, G^e, of Π is the graph whose set of vertices is $V(G^e) = V(G)$ and whose set of edges $E(G^e)$ is equal to $E(G) \cup \mathcal{S}$, where $\mathcal{S}$ is constructed as follows: given two vertices t_0 and t_3 in $V(G^e)$ we say $[t_0t_3] \in \mathcal{S}$, if there is a 3-path, $(t_0t_1t_2t_3) \in G$ that is a Π-facial subwalk of the embedding. We prove that there is a one to one correspondence between the set of possible extended graphs of G and polyhedral embeddings of G.

Keywords: cubic graph, polyhedral embedding.

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1. Introduction

Our motivation for studying polyhedral embeddings of cubic graphs is twofold. On one hand, it is interesting as the natural alternative point of view on combinatorial characterisations of triangulations of surfaces [1], given that the dual structure of a triangulation is precisely a polyhedral embedding of a 3-regular graph in a surface. On the other hand, graph embeddings of 3-regular graphs are interesting in their own right as a plethora of papers on the subject prove, particularly as related to Grünbaum's conjectured generalization of the Four Color Theorem. *If a cubic graph admits a polyhedral embedding in an orientable surface, then it is* 3*-edge colorable* [5].

More precisely, in [1] they prove that the information on the size of the pairwise intersection of triangles in a triangulation suffices in order to determine its whole combinatorial structure. However, it is known that if we only have information on the pairs of triangles that intersect edge to edge (that is, the dual graph of the triangulation) then we cannot uniquely determine the whole incidence structure of the triangulation. For example, in [5] they prove that some connected cubic graphs can be embedded into more than one surface.

In like manner, *in this paper we prove that some additional combinatorial information suffices to uniquely determine a polyhedral embedding of a* 3*-regular graph in a surface with no boundary.* For a more precise statement of our main result (Theorem 3) we need to introduce some terminology.

A topological map of a graph into a surface is called a *graph embedding.* If we consider the graph G together with its embedding Π, we say that G is Π-*embedded.* Each disjoint region of the complement of the image of an embedded graph is called a *face of the embedding.* The closed walk in the underlying graph G that corresponds to the boundary of a face is called a Π-*facial walk.* The embedding Π determines a *set of* Π-*facial walks.* Each edge is either contained in two Π-facial walks or it appears twice in the same facial walk. If a Π-facial walk is a cycle, it is also called a Π-*facial cycle.* Two embeddings of G are *equivalent* if they have the same set of facial walks, up to automorphisms of G.

Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be distinct Π-facial walks. We say that $\mathcal{P}_1$ and $\mathcal{P}_2$ *meet properly* if the intersection of $\mathcal{P}_1$ and $\mathcal{P}_2$ is either empty, a single vertex, or an edge. In this paper we will consider only embeddings of cubic simple graphs whose facial walks meet properly, this latter characteristic defines a polyhedral embedding.

Definition 1. Π is said to be a *polyhedral embbeding* of a graph G, if every Π-facial walk is a cycle and any two Π-facial cycles meet properly.

The set of all the Π-facial cycles is called Π-*facial cycle system.*

An indication that there is some non-triviality in determining polyhedral embeddings is that a significant part of Ringel and Youngs' Map Color Theorem [6] was to determine which complete graphs have such embeddings, even though for a complete graph K_n (with $n \geq 5$) a polyhedral embedding is necessarily a triangulation. Furthermore, in [2] it is proven that the decision problem about the existence of polyhedral embeddings of a graph is NP-complete. The problem remains NP-complete even if it is restricted to the case of embeddings in orientable surfaces and it is required that the graph is 6-connected.

Concerning the uniqueness of the embedding, recall that the *face-width* is defined as the minimum integer r such that G has r facial walks whose union contains a cycle which is noncontractible on the surface. Whitney [9] proved that every 3-connected planar graph has an essentially unique embedding in the plane. Robertson and Vitray [7] extended the previous result to an arbitrary surface of genus g by assuming that the *face-width* is at least $2g+3$. Seymour and Thomas [8] and Mohar [3] improved the bound on the face width to $O\left(\frac{\log g}{\log\log g}\right)$. Moreover, Robertson and Vitray [7] proved the following result.

Proposition 2. *An embedding of a graph G is polyhedral if and only if G is 3-connected and the embedding has face-width at least 3.*

It is also known that for each surface S, there is a constant $\zeta = \zeta(S)$ such that every 3-connected graph admits at most ζ embeddings of face width greater than three [4], then it can be concluded that a graph may have many different polyhedral embedings in the same or different surfaces.

1.1. The extended graph of an embedding

The *extended graph* of the polyhedral embedding Π, $G^e(\Pi)$, is the graph whose set of vertices is $V(G^e(\Pi)) = V(G)$ and whose set of edges $E(G^e(\Pi))$ is equal to $E(G) \cup \mathcal{S}$. We will call $\mathcal{S}$ the set of *scaffold edges* and construct it as follows: given two vertices t_0 and t_3 in $V(G)$, $[t_0t_3] \in \mathcal{S}$, if there are vertices t_1 and t_2, different from t_0 and t_3, such that $(t_0t_1t_2t_3)$ is a Π-facial subwalk. In such case, we say that the 3-*path of G corresponding to* $[t_0t_3]$ is $(t_0t_1t_2t_3)$. Notice that the scaffold edges may be double (but not triple or more). That is, if $[t_0t_3] \in \mathcal{S}$ and its corresponding path is $(t_0t_1t_2t_3)$, there may be a second 3-path between t_0 and t_3, internally disjoint from $(t_0t_1t_2t_3)$, say $(t_0t'_1t'_2t_3)$, that *also* corresponds to $[t_0t_3]$. In this case we say $[t_0t_3]$ is a *double scaffold edge*, and we denote this by a double bracket $[[t_0t_3]]$. It will become obvious later, in Proposition 9, that this only happens when $[[t_0t_3]]$ appears as a *chord* of a 6-cycle which is a Π-facial cycle of the embedding, or when two 4-cicles intersect in one edges. Notice that $E(G) \subset E(G^e(\Pi))$. As such, we will refer to the edges in $E(G)$ as simply *edges.*

Given two different polyhedral embeddings Π and Π' is not obvious that their corresponding extended graphs $G^e(\Pi)$ and $G^e(\Pi')$ are combinatorically different. Figuring this out is precisely the aim of this paper.

Theorem 3. *Let G be a finite cubic graph. Then there is a one to one correspondence between the set of embeddings of G, $\mathfrak{P}(G) = \{\Pi \,|\, \Pi$ is an embedding of $G\}$, and the set of extended graphs $\{G^e(\Pi) \,|\, \Pi \in \mathfrak{P}(G)\}$.*

There is an example of a planar graph, its facial cycles and its corresponding extended graph.

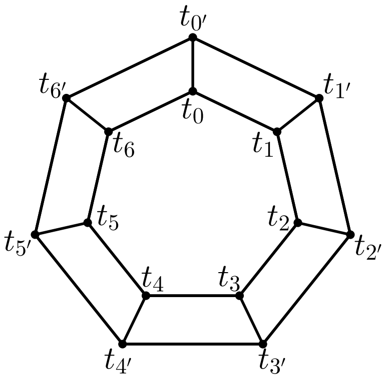

Figure 1. Planar graph G.

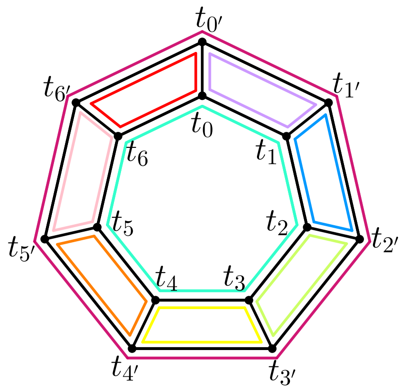

Figure 2. Facial cycles of G.

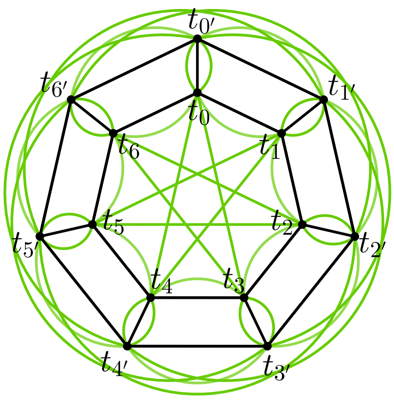

Figure 3. Extended graph of G.

2. Preliminaries

Firstly, note that as G is a cubic graph and Π is an embedding of G then every path of length two is in a Π-facial cycle. This follows as there are three faces incident to every vertex of the embedding, thus any path of length two is contained in one of the faces incident to the vertex in the center of the path.

Proposition 4. *Let $n \geq 5$ and $\mathcal{C} = (t_0 t_1 \cdots t_{n-1} t_0)$ be a cycle in G corresponding to a Π-facial cycle. Then there is no edge $(t_i t_j)$ in $E(G)$, with $i \neq j$, $|i - j| \geq 2$ and this difference taken modulo n.*

Proof. We will proceed by contradiction. Suppose that $(t_i t_j) \in E(G)$ where $|i - j| \geq 2$ and let $\mathcal{C}_{ij}$ be a facial cycle that passes through the edge $(t_i t_j)$. Since the graph has degree three, $\mathcal{C}_{ij}$ contains either the edge (t_it_{i+1}) or $(t_{i-1}t_i)$, and one of (t_jt_{j+1}) or $(t_{j-1}t_j)$. This contradicts the definition of polyhedrality, since the Π_{ij}-facial cycle would intersect $\mathcal{C}$ in two edges. ■

Proposition 5. *Let $n \geq 5$ and $\mathcal{C} = (t_0t_1 \cdots t_{n-1}t_0)$ be a cycle in G corresponding to a Π-facial cycle, then for all t_i, t_j, there is no $t_k \in V(G)$ such that $(t_it_kt_j)$ is a path of G, where $|i-j| \geq 2$ and this difference is taken modulo n.*

Proof. Notice that every path of length two belongs to a Π-facial cycle, and then the proof follows by using similar arguments to those in the proof of Proposition 4. ■

Proposition 6. *Let G be a cubic graph. If $\mathcal{C}$ is a 3-cycle of G then $\mathcal{C}$ is a facial cycle in every polyhedral embedding of G.*

Proof. Let $(t_0t_1t_2t_0)$ be a cycle in G. Since every path of length two belongs to a Π-facial cycle, say $(t_0t_1t_2)$ is in a facial cycle $\mathcal{C}$. If $(t_0t_2) \notin \mathcal{C}$ then we would contradict Proposition 4, and the statement follows. ■

Proposition 7. *Let G be a cubic graph. If $\mathcal{C}$ is a 4-cycle of G and $G \neq K_4$ then $\mathcal{C}$ is a facial cycle in every polyhedral embedding of G.*

Proof. If G has a polyhedral embedding and $G \neq K_4$, then every 4-cycle of G is induced, since G is 3-connected by Proposition 2. Let $(t_0t_1t_2t_3t_0)$ be a cycle in G. Since every path of length two belongs to a Π-facial cycle, say $(t_0t_1t_2)$ is in a facial cycle $\mathcal{C}$. If the path $(t_2t_3t_0) \notin \mathcal{C}$ then we would contradict Proposition 5, and the statement follows. ■

Proposition 8. *Given two paths, $(t_0t_1t_2t_3), (t_0t_1t_2t_{3'})$, of G then either $(t_0t_1t_2t_3)$ or $(t_0t_1t_2t_{3'})$, but not both, is a Π-facial subwalk.*

Proof. Given that every path of length two is in a Π-facial cycle then either $(t_0t_1t_2t_3)$ or $(t_0t_1t_2t_{3'})$ is a Π-facial subwalk. Suppose that both $(t_0t_1t_2t_3)$ and $(t_0t_1t_2t_{3'})$ are Π-facial subwalks, if each of these paths belongs to a different facial cycle, then they would intersect improperly, contradicting Definition 1. If they belong to the same facial cycle, then such facial cycle self intersects, contradicting Definition 1. ■

Proposition 9. *Let $\mathcal{P} = (q_0 = t_0, t_1, \ldots, t_n = q_m)$ and $\mathcal{Q} = (q_0 = t_0, q_1, \ldots, q_m = t_n)$ be two internally disjoint Π-facial subwalks, such that $(t_0t_n) \notin E(G)$. Then $\mathcal{P} \cup \mathcal{Q}$ must be a Π-facial cycle.*

Proof. We will proceed by contradiction. Let $\mathcal{C}_{\mathcal{P}}$ be the Π-facial cycle associated to $\mathcal{P}$ and $\mathcal{C}_{\mathcal{Q}}$ be the Π-facial cycle associated to $\mathcal{Q}$, where $\mathcal{C}_{\mathcal{P}} \neq \mathcal{C}_{\mathcal{Q}}$. Since G is a cubic graph, let u and v be the remaining vertices adjacent to $t_0 = q_0$ and

$t_n = q_m$, respectively. Since the edge $(t_0 t_n) \notin E(G)$, observe that $v \neq t_0$ and $u \neq t_n$. This implies that $(t_0 u)$ and $(t_n v)$ are different.

Notice both $\mathcal{C}_{\mathcal{P}}$ and $\mathcal{C}_{\mathcal{Q}}$ have to contain two edges incident to $t_0 = q_0$, but $t_0 = q_0$ has degree three; thus $\mathcal{C}_{\mathcal{P}}$ intersects $\mathcal{C}_{\mathcal{Q}}$ in at least one edge incident to $t_0 = q_0$. The same holds for $t_n = q_m$, hence $\mathcal{C}_{\mathcal{P}}$ and $\mathcal{C}_{\mathcal{Q}}$ would have to intersect in at least two edges, contradicting Definition 1. ∎

3. Proof of the Main Theorem

In this section we will denote as G^e any extended graph in $\{G^e(\Pi) \,|\, \Pi \in \mathfrak{P}(G)\}$, where we emphasize that we do not claim knowledge of what embedding G^e corresponds to.

The proof of the main result of the paper will be split in to two subsections: the simple case and the difficult case.

3.1. The simple case

We present the simple case first as within its proof it becomes obvious why the second part requires much more detail.

Theorem 10. *Let G^e be an extended graph of a finite cubic graph, G, such that for all edges $[t, t'] \in \mathcal{S}$ there is a unique path of length three in G whose ends are t and t'. Then G^e uniquely determines the Π-facial cycle system of an embedding.*

Proof. From this information we will construct the Π-facial cycle system. Let $[t_0 t_3] \in \mathcal{S}$, by hypothesis we may say that $(t_0 t_1 t_2 t_3)$ is the unique path of length three between t_0 and t_3, and there is a Π-facial cycle that contains the facial subwalk $(t_0 t_1 t_2 t_3)$. Let t_4 and $t_{4'}$ be the remaining vertices adjacent to t_3. Then by Proposition 8 either $(t_1 t_2 t_3 t_4)$ or $(t_1 t_2 t_3 t_{4'})$ is a facial subwalk, but not both. Additionally, by hypothesis, only one of $[t_1 t_4]$ or $[t_1 t_{4'}]$ is in $\mathcal{S}$. Hence we know with certainty if the facial cycle that contains $(t_0 t_1 t_2 t_3)$ continues on to t_4 or $t_{4'}$. We can continue with this procedure until we obtain the unique Π-facial cycle that contains $(t_0 t_1 t_2 t_3)$.

Now consider the edge $(t_1 t_2) \in E(G)$, this edge must belong to another Π-facial cycle. Let $t_{1'}$ and $t_{2'}$ be the remaining vertices adjacent to t_1 and t_2, respectively. Then, as the embedding is polyhedral, the other facial cycle containing $(t_1 t_2)$, necessarily contains $(t_{1'} t_1 t_2 t_{2'})$, thus $[t_{1'} t_{2'}] \in \mathcal{S}$. Now we may use the same procedure as before to find the rest of the edges in this facial cycle.

We can find every Π-facial cycle in this manner, by selecting at every step an edge of the union of the preceding facial cycles that hasn't appeared in two facial cycles yet. This procedure is finite, since the graph G is finite. ∎

As a consequence of Theorem 10 we have the following.

Corollary 11. *Let G^e be an extended graph of a finite cubic graph, G, with no 6-cycles, then G^e uniquely determines the Π-facial cycle system of an embedding of G.*

3.2. The difficult case

Clearly, the difficulty arises when we have the possibility that the hypothesis of Theorem 10 does not hold for an extended graph, G^e. Namely, **not** for all scaffold edges $[t,t'] \in \mathcal{S}$ there is a unique path of length three in G whose ends are t and t'. This motivates the following definitions.

Definition 12 (Fork)**.** Let $Y \subset G^e$ be a subgraph with set of vertices $V(Y) = \{t_1, t_2, t_3, t_4, t_{4'}\}$ and set of edges $E(Y) = \{(t_1t_2), (t_2t_3), (t_3t_4), (t_3t_{4'})\} \cup \{[t_1t_4], [t_1t_{4'}]\}$. We will call such graph a *fork.*

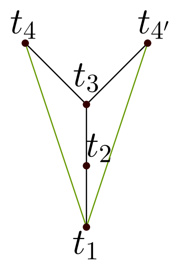

Figure 4. Fork.

Here, if we tried to reproduce the reconstruction procedure presented in the proof of Theorem 10, when arriving at a fork we would have a *disjunction* consisting of whether $(t_1t_2t_3t_4)$ is the 3-path corresponding to $[t_1t_4]$ or $(t_1t_2t_3t_{4'})$ is the 3-path corresponding to $[t_1t_{4'}]$. A great part of this section consists of proving that, in most cases, this *disjuntion can be solved* by looking at the rest of the structure of the graph. We will say that the *disjuntion cannot be solved* whenever the rest of the structure of the extended graph does not force either $(t_1t_2t_3t_4)$ to be the 3-path corresponding to $[t_1t_4]$ or $(t_1t_2t_3t_{4'})$ to be the 3-path corresponding to $[t_1t_{4'}]$.

We will now prove some more technical lemmas which will help us discover the very particular structure of graphs for which disjunctions cannot be solved.

Proposition 13. *Let $\mathcal{P}_1 = (ut_0t_1v)$, $\mathcal{P}_2 = (ut_2t_3v)$ and $\mathcal{P}_3 = (ut_4t_5v)$ be three internally disjoint 3-paths in G such that $[uv] \in G^e$. Then $[[uv]]$ is a double scaffold edge. Furthermore, $\mathcal{P}_i \cup \mathcal{P}_j$ is a Π-facial cycle for a pair $i, j \in \{1, 2, 3\}$.*

Proof. Without loss of generality, suppose that $[uv]$ corresponds to the path $\mathcal{P}_1$, which means that (ut_0t_1v) is Π-facial subwalk. By Proposition 8, it must satisfied that either $(ut_0t_1vt_3)$ or $(ut_0t_1vt_5)$ is a Π-facial subwalk. Suppose, with out loss of generality, that $(ut_0t_1vt_3)$ is a Π-facial subwalk, then by Proposition 5 $(ut_0t_1vt_3t_2u)$ is Π-facial cycle, hence $[[uv]]$ is a double scaffold edge. See Figure 5. ■

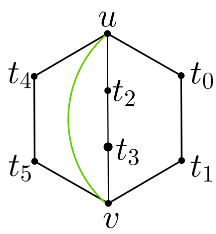

Figure 5.

Corollary 14. *Let $\mathcal{P}_1 = (ut_0t_1v)$, $\mathcal{P}_2 = (ut_2t_3v)$ and $\mathcal{P}_3 = (ut_4t_5v)$ be three internally disjoint 3-paths in G such that $[uv] \in G^e$ and $(t_1t_0ut_4)$ is a Π-facial subwalk. Then (vt_5t_4u) is a Π-facial subwalk.*

Proof. Since $[uv] \in G^e$, Proposition 13 implies $[[uv]]$ is a double scaffold edge and either $(ut_0t_1vt_5t_4u)$, $(ut_0t_1vt_3t_2u)$ or $(ut_2t_3vt_5t_4u)$ is a Π-facial cycle. Since $(t_1t_0ut_4)$ is a Π-facial subwalk, by Proposition 8 $(t_1t_0ut_2)$ is not a Π-facial subwalk. This implies that $(ut_0t_1vt_3t_2u)$ cannot be a Π-facial cycle and, necessarily, either $(ut_0t_1vt_5t_4u)$ or $(ut_2t_3vt_5t_4u)$ is a Π-facial cycle. Notice that the path (vt_5y_4u) appears in both cases, hence, this path is a Π-facial subwalk. ■

Corollary 15. *Let (ut_0t_1v) and (ut_2t_3v) be two internally disjoint 3-paths in G such that $[uv] \in G^e$ and (ut_0t_1v) is not a Π-facial subwalk. Then (ut_2t_3v) is a Π-facial subwalk.*

Proof. We proceed by contradiction. If (ut_2t_3v) is not a Π-facial subwalk, then there is a third 3-path between u and v, (ut_4t_5v), internally disjoint to (ut_0t_1v) and(ut_2t_3v) which corresponds to the scaffold edge $[uv]$. But, by Proposition 13 the edge $[[uv]]$ is double. Since (ut_0t_1v) is not a Π-facial subwalk, necessarily (ut_2t_3v) and (ut_4t_5v) are Π-facial subwalks, which is a contradiction. ■

Definition 16 (Butterfly one and two)**.** Let $B \subset G^e$ be a subgraph of G with set of vertices $V(B) = \{t_0, t_{0'}, t_1, t_2, t_3, t_4, t_{4'}, t_5, t_{5'}\}$, and set of edges equal to the union of the two cycles $E(Y) = (t_0t_1t_2t_3t_4t_5t_0) \cup (t_{0'}t_1t_2t_3t_{4'}t_{5'}t_{0'})$, we will call this graph a *butterfly*. A *butterfly one*, B_1, is a butterfly with the additional scaffold edges $[t_1t_4], [t_1t_{4'}], [t_0t_3], [t_{0'}t_3]$. A *butterfly two*, B_2, is a butterfly with the additional scaffold edges $[t_1t_4], [t_1t_{4'}], [t_0t_3]$.

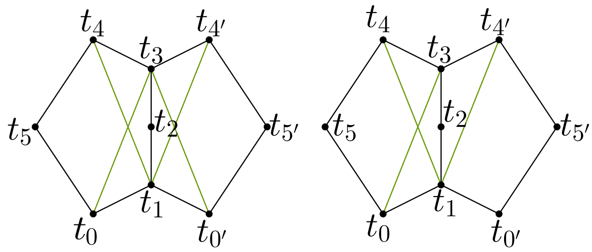

Figure 6. The butterfly one is shown on the left and the butterfly two is shown on the right.

We are ready to state and proof the main theorems of this section. The spirit is the following: if a fork Y appears as a subgraph of G^e in such a way that the disjunction cannot be solved, then G^e will contain either butterfly one or butterfly two as subgraphs and, in either case, we will be able to reconstruct the full extended graph uniquely.

Lemma 17. *Let Y be a fork labelled as in Definition 12 then if either $(t_1t_2t_3t_4)$ is the only 3-path between t_1 and t_4 or $(t_1t_2t_3t_{4'})$ is the only 3-path between t_1 and $t_{4'}$ then the disjunction can be solved.*

Proof. By Definition, every scaffold edge corresponds to a 3-path. Thus, when there is only one 3-path between the end vertices of a scaffold edge necessarily such path is its corresponding 3-path. ■

Next we will prove different theorems. Along with the proofs we state different claims and the procedure that we use is giving a claim followed immediately by its proof. The figures that shows the following results appears in the Apendix at the end of the paper.

Lemma 18. *Let Y be a fork labelled as in Definition 12 such that the disjunction cannot be solved. Then G^e contains a subgraph isomorphic to butterfly one, B_1, or butterfly two, B_2.*

Proof. Since G is a cubic graph, let t_0 and $t_{0'}$ be the remaining vertices adjacent to t_1, different from t_2.

Claim 1. *The vertices t_0 and $t_{0'}$ are different from each of $t_3, t_4, t_{4'}$.*

Proof. Note that the proofs for t_0 and $t_{0'}$ must be analogous. We only detail the proof for t_0. As G is a 3-regular graph, $t_0 \neq t_3$ (otherwise t_3 would have degree four). If $t_0 = t_4$, then by Proposition 7, $(t_1t_2t_3t_4t_1)$ is Π-facial cycle, and $(t_1t_2t_3t_4)$ is forced to be the 3-path corresponding to $[t_1t_4]$ thus the disjunction can be solved. Hence $t_0 \neq t_4$. The case $t_0 \neq t_{4'}$ is analogous to the case $t_0 \neq t_4$. □

Applying Lemma 17 we obtaing the following.

Claim 2. *For both edges $[t_1t_4]$ and $[t_1t_{4'}]$ there exist at least two internally disjoint 3-paths between its end vertices.*

Proof. Without loss of generality we may say that $(t_1t_0t_5t_4)$ is the additional 3-path with ends $[t_1t_4]$, internally disjoint from $(t_1t_2t_3t_4)$. Note that then $t_5 \neq t_0, t_1, t_2, t_3, t_4$. □

Claim 3. *The vertex t_5 is different from the vertices $t_{0'}, t_{4'}$.*

Proof. If $t_5 = t_{4'}$, Proposition 6 implies that $(t_3t_4t_{4'}t_3)$ is a Π-facial cycle; by Proposition 8 $(t_2t_3t_{4'}t_0)$ is a Π-facial subwalk; Proposition 5 implies that $(t_0t_1t_2t_3t_{4'}t_0)$ is a Π-facial cycle, so $(t_1t_2t_3t_{4'})$ is a 3-path corresponding to $[t_1t_{4'}]$; hence, the disjunction could be solved, contradicting the hypothesis. Therefore $t_5 \neq t_{4'}$.

If $t_5 = t_{0'}$, using a similar arguments we would arrive to the conclusion that $(t_1t_2t_3t_4)$ is a 3-path corresponding to $[t_1t_4]$ contradicting our hypothesis. Therefore $t_5 \neq t_{0'}$. □

Now, we will give the additional 3-path between t_1 and $t_{4'}$ internally disjoint to $(t_1t_2t_3t_{4'})$. We have three cases: (Case A) when the 3-path is $(t_1t_{0'}t_{5'}t_{4'})$, (Case B) when the 3-path is $(t_1t_0x_0t_{4'})$ and (Case C) when the 3-path is $(t_1t_0t_5t_{4'})$. We will deal with each case separately.

Case A. Assume the 3-path is $(t_1t_{0'}t_{5'}t_{4'})$. Since G is a cubic graph and all the previous vertices already have degree two, $t_{5'}$ must be a new vertex of G, otherwise we would have a vertex with degree four. By Proposition 8, given that the 3-paths, $(t_0t_1t_2t_3)$ and $(t_{0'}t_1t_2t_3)$ are in G, one of these paths is a Π-facial subwalk. So we have two cases: either $[t_0t_3]$ and $[t_{0'}t_3]$ are in $\mathcal{S}$ which implies that $B_1 \subset G^e$, or only of them ($[t_0t_3]$ or $[t_{0'}t_3]$) is in $\mathcal{S}$, which implies that $B_2 \subset G^e$. Either way, the assertion of the lemma follows. This ends the proof for Case A.

Case B. Assume the 3-path is $(t_1t_0x_0t_{4'})$. First, observe that Claim 3 implies that x_0 is a new vertex. Moreover, the 3-path $(t_0t_1t_2t_3)$ is in G and it is possible that $[t_0t_3] \in \mathcal{S}$ or $[t_0t_3] \notin \mathcal{S}$. We will analyse both cases.

Case B1. $[t_0t_3] \in \mathcal{S}$.

Claim B1.1. *$[[t_0t_3]]$ is a double scaffold edge.*

Proof. Since there are three internally disjoint 3-paths between t_0 and t_3: $(t_0t_1t_2t_3)$, $(t_0t_5t_4t_3)$ and $(t_0x_0t_{4'}t_3)$, then $[[t_0t_3]]$ is a double scaffold edge, by Corollary 13. □

Claim B1.2. *$[t_1t_4]$ or $[t_1t_{4'}]$ is a double scaffold edge.*

Proof. Note that if either $[t_1t_4]$ or $[t_1t_{4'}]$ is a double scaffold edge then we necessarily have three 3-paths internally disjoint between the pair t_1, t_4 in the first instance and $t_1, t_{4'}$ in the second instance, which would confirm our claim.

Thus, we may now suppose that neither $[t_1t_4]$ nor $[t_1t_{4'}]$ are double scaffold edges, then $(t_0t_5t_4t_3t_{4'}x_0t_0)$ is a Π-facial cycle, by Corollary 13. The latter and Corollary 15 imply that $(t_1t_2t_3t_4)$ and $(t_1t_2t_3t_{4'})$ are the Π-facial subwalks corresponding to $[t_1t_4]$ and $[t_1t_{4'}]$ respectively, which contradicts Definition 1. Hence, at least one of $[t_1t_4]$ or $[t_1t_{4'}]$ has to be a double scaffold edge. □

Claim B1.3. *B_1 or B_2 is contained in G^e.*

Proof. Without loss of generality, suppose that $[[t_1t_{4'}]]$ is a double scaffold edge. Observe that $(t_1t_{4'}) \notin E(G)$, otherwise it would contradict the 3-regularity of G. Since there are already two internally disjoint 3-paths between t_1 and $t_{4'}$: $(t_1t_2t_3t_{4'})$ and $(t_1t_0x_0t_{4'})$, we have to have a third one, else $(t_1t_2t_3t_{4'}x_0t_0t_1)$ would be a Π-facial cycle, by Proposition 9, which implies that $(t_1t_2t_3t_{4'})$ is a Π-facial subwalk. That is, the disjunction can be solved, contradicting our hypothesis.

Let $(t_1t_{0'}t_{5'}t_{4'})$ be the third path. Notice that $t_{5'}$ is a new vertex, otherwise it would contradict the hypothesis that G is a cubic graph. Then, if $[t_{0'}t_3] \in \mathcal{S}$ then $B_1 \subset G^e$ or if $[t_{0'}t_3] \notin \mathcal{S}$ then $B_2 \subset G^e$.

If we assume instead that $[[t_1t_4]]$ is a double scaffold edge, the proof follows in a similar way. □

This completes the proof for *Case* B1.

Case B2. $[t_0t_3] \notin \mathcal{S}$. This implies that $(t_0t_1t_2t_3)$ is not a Π-facial subwalk and, by Proposition 8, $(t_{0'}t_1t_2t_3)$ is a Π-facial subwalk, hence $[t_{0'}t_3] \in \mathcal{S}$.

Claim B2.1. *If $(t_{0'}t_1t_2t_3t_4)$ is a Π-facial subwalk then $(t_5t_0t_1t_2)$ is a Π-facial subwalk. If $(t_{0'}t_1t_2t_3t_{4'})$ then $(t_{0'}t_1t_0t_5t_4)$ is a Π-facial subwalk.*

Proof. Suppose that $(t_{0'}t_1t_2t_3t_4)$ is a Π-facial subwalk. The later implies that $(t_1t_2t_3t_{4'})$ is not a Π-facial subwalk. Since $[t_1t_{4'}] \in \mathcal{S}$ and using Corollary 15, $(t_1t_0x_0t_{4'})$ is a Π-facial subwalk. This Π-facial subwalk cannot contain the edge (t_1t_2) otherwise, by Proposition 5, $(t_1t_0x_0t_{4'}t_3t_2t_1)$ would be a Π-facial cycle intersecting the Π-facial subwalk $(t_{0'}t_1t_2t_3)$ in two edges, which contradicts the definition of polyhedral embedding. Then, $(t_{0'}t_1t_0x_0t_{4'})$ is a Π-facial subwalk. We know that two different Π-facial cycles pass through each edge of G; this fact together with Definition 1 and Proposition 8 imply that $(t_5t_0t_1t_2)$ is a Π-facial subwalk.

Now, suppose that $(t_{0'}t_1t_2t_3t_{4'})$ is a Π-facial subwalk. This implies that $(t_1t_2t_3t_4)$ is not the Π-facial subwalk corresponding to the scaffold edge $[t_1t_4]$. Hence, using Corollary 15 we know that $(t_1t_0t_5t_4)$ is the Π-facial subwalk corresponding to the scaffold edge $[t_1t_4]$. This Π-facial subwalk cannot contain the edge (t_1t_2), otherwise $(t_2t_1t_0t_5t_4t_3t_2)$ would be a Π-facial cycle, by Proposition 5, and it would intersect the Π-facial subwalk $(t_{0'}t_1t_2t_3)$ in two edges, contradicting the definition of polyhedral embedding. Then, $(t_{0'}t_1t_0t_5t_4)$ is a Π-facial subwalk. $\square$

Claim B2.2. *The fork Y' given by the set of vertices $\{t_5, t_0, t_1, t_{0'}, t_2\}$, and the edges, $\{(t_5t_0), (t_0t_1), (t_1t_2), (t_1t_{0'}), [t_5t_2], [t_5t_{0'}]\}$ is contained in G^e.*

Proof. If the statement were false, i.e., if one of the scaffold edges $[t_5t_2]$ or $[t_5t_{0'}]$ is not in $\mathcal{S}$, then the disjunction can be solved, contradicting our hypothesis. Then, the statement follows. $\square$

In a way similar to our treatment of the cases for fork Y, we will analyse the fork Y'. We need to have two internally disjoint 3-paths for each pair of vertices $\{t_5, t_2\}$ and $\{t_5, t_{0'}\}$. Notice that for the pair $\{t_5, t_2\}$ we already have two internally disjoint 3-paths: $(t_5t_4t_3t_2)$ and $(t_5t_0t_1t_2)$. Therefore, we only need to find a second 3-path between t_5 and $t_{0'}$. We have three cases.

Case B2.A. $(t_5t_4t_3t_{0'})$ cannot be the second 3-path between t_5 and $t_{0'}$.

Suppose the second 3-path is $(t_5t_4t_3t_{0'})$. This would imply that t_3 has degree four, contradicting that G is a cubic graph.

Case B2.B. If $(t_5t_4x_4t_{0'})$ is the second 3-path between t_5 and $t_{0'}$ then B_2 is contained in G^e.

Suppose the second 3-path is $(t_5t_4x_4t_{0'})$, where x_4 is the remaining vertex adjacent to t_4. Observe that x_4 is a new vertex, since x_4 is different to t_3, t_4, t_5, and if x_4 is equal to one of $t_0, t_1, t_2, t_{4'}, x_0$, that would contradict the fact that G is a cubic graph. Finally, $x_4 \neq t_{0'}$, otherwise, by Proposition 5, $(t_{0'}t_1t_2t_3t_4t_{0'})$ would be a Π-facial cycle, implying that $(t_1t_2t_3t_4)$ is a Π-facial subwalk, contradicting our hypothesis.

Hence, we would have to B_2 contained in G^e, given by the set of vertices: $\{t_0, t_{0'}, t_1, t_2, t_3, t_4, t_{4'}, x_0, x_4\}$ and by the set of edges of the two 6-cycles: $(t_1t_2t_3 t_{4'}x_0t_0t_1) \cup (t_1t_2t_3t_4x_4t_{0'}t_1)$ and the scaffold edges $\{[t_1t_4], [t_1t_{4'}], [t_{0'}t_3]\}$.

Case B2.C. $(t_5x_5x_{0'}t_{0'})$ cannot be the second 3-path.

Suppose the second 3-path is $(t_5x_5x_{0'}t_{0'})$, where x_5 and $x_{0'}$ are the remaining vertices adjacent to t_5 and $t_{0'}$ respectively.

Observe that $x_{0'}$ and x_5 are different from the vertices $t_0, t_{0'}, t_1, t_2, t_3, t_4, t_5$ by a similar proof to the case A. Also $x_{0'}$ and x_5 are different from the vertices $t_{4'}$ and x_0, otherwise contradicts the fact that G is a cubic graph. So there is a butterfly contained in G given by the union of the two 6-cycles $(t_0, t_1, t_2, t_3, t_4, t_5t_0) \cup (t_0t_1t_{0'}x_{0'}x_5t_5t_0)$ and the set of scaffold edges $\{[t_5t_2], [t_5t_{0'}], [t_1t_4], [t_1x_5]\}$ if is a butterfly one, or the set of scaffold edges $\{[t_5t_2], [t_5t_{0'}], [t_1t_4]\}$ if is a butterfly two.

This completes the proof for Case B2 and Case B.

Case C. Assume that the 3-path is $(t_1t_0t_5t_{4'})$. Notice that $(t_3t_4t_5t_{4'}t_3)$ is a Π-facial cycle of G.

Suppose that $[t_0t_3] \in \mathcal{S}$. Since $(t_3t_4t_5t_{4'}t_3)$ is a Π-facial cycle, $(t_0t_5t_4t_3)$ cannot be a Π-facial subwalk, by Propositon 8. Using Corollary 15, $(t_0t_1t_2t_3)$ is the Π-facial subwalk corresponding to $[t_0t_3]$. The Π-facial subwalk $(t_0t_1t_2t_3)$ cannot continue trough the vertex t_4, otherwise by Proposition 5 implies that $(t_0t_1t_2t_3t_4t_5t_0)$ is a Π-facial cycle, and it would intersect the Π-facial cycle $(t_3t_4t_5t_{4'}t_3)$ in two edges, contradicting the definition of polyhedral embedding. So $(t_0t_1t_2t_3t_{4'})$ is a Π-facial subwalk, i.e., the disjunction can be solved, contradicting out hypothesis.

Then, suppose that $[t_0t_3] \notin \mathcal{S}$. Since $(t_0t_1t_2t_3)$ is not a Π-facial subwalk, using Proposition 8 implies that $(t_{0'}t_1t_2t_3)$ is a Π-facial subwalk. Thus either $(t_{0'}t_1t_2t_3t_4)$ or $(t_{0'}t_1t_2t_3t_{4'})$ is a Π-facial subwalk.

Claim C1. *The remaining vertex adjacent to t_2, x_2, is a new vertex of G.*

Proof. We are going to proceed by contradiction.

1. $(x_2 \neq t_5)$. If $x_2 = t_5$ it would contradict the 3-regularity of G.
2. $(x_2 \neq t_4)$. If $x_2 = t_4$ then $(t_2t_3t_4t_2)$ is a Π-facial cycle, by Proposition 6. The latter implies that $(t_{0'}t_1t_2t_3t_4)$ cannot be a Π-facial subwalk, since it would intersect the Π-facial cycle $(t_2t_3t_4t_2)$ in two edges, contradicting the polyhedrality of the embedding. Then, by Proposition 8, $(t_{0'}t_1t_2t_3t_{4'})$ is a Π-facial subwalk, i.e., $(t_1t_2t_3t_{4'})$ is the Π-facial subwalk corresponding to the edge $[t_1t_{4'}]$, and the disjunction can be solved, contradicting out hypothesis.
3. $(x_2 \neq t_{4'})$ This proof follows in a manner similar to the previous one.
4. $(x_2 \neq t_{0'})$. If $x_2 = t_{0'}$ it would contradict the definition of polyhedral embedding, since $(t_{0'}t_1t_2t_3)$ is a Π-facial subwalk.
5. $(x_2 \neq t_0)$. If $x_2 = t_0$ then $(t_0t_1t_2t_0)$ would be a Π-facial cycle, by Proposition 6. Since there are two different Π-facial cycles passing through every edge, then Proposition 8 implies that $(t_3t_2t_0t_5)$ is a Π-facial subwalk. Thus, $(t_3t_2t_0t_5t_4t_3)$ is a Π-facial cycle, by Proposition 5. But notice that this would contradict the polyhedrality of the embedding.

Hence, x_2 is a new vertex. □

Claim C2. *If $(t_{0'}t_1t_2t_3t_4)$ is a Π-facial subwalk then $(x_2t_2t_3t_{4'})$ is a Π-facial subwalk.*

Proof. Notice that $(t_1t_2t_3t_4)$ is the Π-facial subwalk corresponding to $[t_1t_4]$. Given that two different Π-facial cycles pass through every edge and using Proposition 8 we can deduce that $(x_2t_2t_3t_{4'})$ is a Π-facial subwalk. □

Claim C3. *If $(t_{0'}t_1t_2t_3t_{4'})$ is a Π-facial subwalk then $(t_1t_2t_3t_{4'})$ is a Π-facial subwalk.*

Proof. This statement follows immediately from the hypothesis, since $(t_1t_2t_3t_{4'})$ is contained in the Π-facial subwalk $(t_{0'}t_1t_2t_3t_{4'})$. □

Claim C4. *The fork Y' given by the set of vertices $\{t_{4'}, t_3, t_2, t_1, x_2\}$ and the set of edges $\{(t_3t_{4'}), (t_2t_3), (t_1t_2), (t_2x_2), [t_1t_{4'}], [t_{4'}x_2]\}$ is contained in G^e.*

Proof. Notice that only remains to prove that $[t_{4'}x_2]$ is in $\mathcal{S}$. If this scaffold edge is not in $\mathcal{S}$, then $(x_2t_2t_3t_{4'})$ is not a Π-facial subwalk. The latter and Proposition 8 imply that $(t_{0'}t_1t_2t_3t_{4'})$ is a Π-facial subwalk. Thus there is no disjunction contradicting our hypothesis. □

Let $x_{4'}$ be the remaining vertex adjacent to $t_{4'}$.

Claim C5. $(t_2t_3t_{4'}x_{4'})$ *is a* Π*-facial subwalk.*

Proof. Since every path of length two is a Π-facial subwalk, then $(t_2t_3t_{4'})$ is a Π-facial subwalk. But this Π-facial subwalk cannot continue to t_5, otherwise it would intersect the Π-facial cycle $(t_3t_4t_5t_{4'}t_3)$ in two edges, contradicting Definition 1. Hence, we have to continue towards the remaining neighbour of $t_{4'}$, $x_{4'}$, and the statement follows. □

Claim C6. *There are two internally disjoint* 3*-paths for each pair of vertices* $\{t_1, t_{4'}\}$ *and* $\{t_{4'}, x_2\}$.

Proof. It follows immediately by Lemma 17. □

The two internally disjoint 3-paths between t_1 and $t_{4'}$ are: $(t_1t_2t_3t_{4'})$ and $(t_1t_0t_5t_{4'})$. The first of the two internally disjoint 3-paths between $t_{4'}$ and x_2 is $(t_{4'}t_3t_2x_2)$; as for the second there are three options: (Case C1) $(t_{4'}t_5t_0x_2)$, (Case C2) $(t_{4'}t_5t_4x_2)$ and (Case C3) $(t_{4'}x_{4'}x_{2'}x_2)$. We will deal with each instance separately.

Case C1. Assume that the second internally disjoint 3-path between $t_{4'}$ and x_2 is $(t_{4'}t_5t_0x_2)$.

Claim C1.1. *The remaining vertex adjacent to $t_{4'}$, $x_{4'}$, is a new vertex.*

Proof. We are going to proceed by contradiction.

1. ($x_{4'} \neq t_0$, $x_{4'} \neq t_1$, $x_{4'} \neq t_2$, $x_{4'} \neq t_3$, $x_{4'} \neq t_5$). If any of the inequalities was an equality instead we would have a contradiction on the fact that G is a cubic graph.

2. ($x_{4'} \neq t_{0'}$). If $x_{4'} = t_{0'}$ then by Proposition 5, $(t_{0'}t_1t_2t_3t_{4'}t_{0'})$ is a Π-facial cycle because $(t_{0'}t_1t_2t_3)$ is a Π-facial subwalk. This implies that $(t_1t_2t_3t_{4'})$ is the Π-facial subwalk corresponding to the edge $[t_1t_{4'}]$, i.e., the disjunction can be solved, contradicting our hypothesis. Hence $x_{4'} \neq t_{0'}$.

3. ($x_{4'} \neq t_4$). If $x_{4'} = t_4$, since $(t_3t_4t_5t_{4'}t_3)$ is a Π-facial cycle this contradicts Proposition 4. Hence $x_{4'} \neq x_4$.

4. ($x_{4'} \neq x_2$). If $x_{4'} = x_2$ then the Π-facial subwalk $(t_{0'}t_1t_2t_3)$ cannot continue through $t_{4'}$, since there is a 2-path $(t_2x_2t_{4'})$, which would contradict Proposition 5. Then $(t_{0'}t_1t_2t_3)$ continues through t_4, implying that $(t_1t_2t_3t_4)$ is the Π-facial

subwalk corresponding to $[t_1t_4]$, i.e., the disjunction can be solved, contradicting our hypothesis. Hence $x_{4'} \neq x_2$.

Thus, $x_{4'}$ is a new vertex. □

Claim C1.2. *The remaining vertex adjacent to t_4, x_4, is a new vertex.*

Proof. We are going to proceed by contradiction.

1. ($x_4 \neq t_0$, $x_4 \neq t_1$, $x_4 \neq t_2$, $x_4 \neq t_3$, $x_4 \neq t_5$, $x_4 \neq t_{4'}$). If any of the inequalities was an equality instead we would have a contradiction with the fact that G is a cubic graph.

2. ($x_4 \neq x_2$). If $x_4 = x_2$ then by Proposition 7, $(t_2t_3t_4x_2t_2)$ is a Π-facial cycle. Thus the Π-facial subwalk $(t_{0'}t_1t_2t_3)$ cannot continue through t_4, otherwise it would contradict the polyhedrality of G. Then $(t_{0'}t_1t_2t_3t_{4'}x_{4'})$ is a Π-facial subwalk. This implies that $(t_1t_2t_3t_{4'})$ is the Π-facial subwalk corresponding to $[t_1t_{4'}]$, therefore the disjunction can be solved, contradicting our hypothesis. Hence $x_4 \neq x_2$.

3. ($x_4 \neq t_{0'}$). If $x_4 = t_{0'}$ then using Proposition 5 it can be said that $(t_{0'}t_1t_2t_3t_4t_{0'})$ is a Π-facial cycle. The latter implies that $(t_1t_2t_3t_4)$ is the Π-facial subwalk corresponding to $[t_1t_4]$, which contradicts our hypothesis. Hence $x_4 \neq t_{0'}$.

4. ($x_4 \neq x_{4'}$). If $x_4 = x_{4'}$, since $(t_3t_4t_5t_{4'}t_3)$ is a Π-facial cycle, it would contradict Proposition 5.

Hence, x_4 is a new vertex. □

Claim C1.3. *The remaining vertex adjacent to x_2, $x_{2'}$, is a new vertex.*

Proof. We are going to proceed by contradiction.

1. ($x_{2'} \neq t_0, x_{2'} \neq t_1, x_{2'} \neq t_2, x_{2'} \neq t_3, x_{2'} \neq t_4, x_{2'} \neq t_{4'}, x_{2'} \neq t_5$). If any of the inequalities was an equality instead we would have a contradiction with the fact that G is a cubic graph.

2. ($x_{2'} \neq t_{0'}$). If $x_{2'} = t_{0'}$ since $(t_0t_1t_2x_2t_0)$ is a Π-facial cycle, it would contradict Proposition 5. Hence $x_{2'} \neq t_{0'}$.

3. ($x_{2'} \neq x_4$). If $x_{2'} = x_4$ then $(x_4x_2t_2t_3)$ is a Π-facial subwalk, since $(t_0t_1t_2x_2t_0)$ is a Π-facial cycle. Proposition 5 and the latter imply that $(x_4x_2t_2t_3t_4x_4)$ is a Π-facial cycle. Therefore the Π-facial subwalk $(t_{0'}t_1t_2t_3)$ cannot continue though t_4, otherwise it would contradict the polyhedrality of the embedding. Then $(t_{0'}t_1t_2t_3t_{4'})$ is a Π-facial subwalk, which implies that $(t_1t_2t_3t_{4'})$ is the Π-facial subwalk corresponding to $[t_1t_{4'}]$ and the disjunction can be solved, contradicting our hypothesis. Hence $x_{2'} \neq x_4$.

4. ($x_{2'} \neq x_{4'}$). This proof follows in a similar way that the previous one.

Thus $x_{2'}$ is a new vertex of G. □

Claim C1.4. *Either $(t_1t_2t_3t_{4'})$ or $(t_1t_0t_5t_{4'})$ is a Π-facial subwalk.*

Proof. We know that $[t_1t_4]$ and $[t_1t_{4'}]$ are in $\mathcal{S}$, also Proposition 8 implies that either $(t_1t_2t_3t_4)$ or $(t_1t_2t_3t_{4'})$ is a Π-facial subwalk.

If $(t_1t_2t_3t_{4'})$ is a Π-facial subwalk the statement follows. If, on the other hand, $(t_1t_2t_3t_4)$ is not a Π-facial subwalk, Corollary 15 implies that $(t_1t_0t_5t_{4'})$ is the Π-facial subwalk corresponding to $[t_1t_{4'}]$. □

Claim C1.5. *For either choice between $(t_1t_2t_3t_{4'})$ or $(t_1t_0t_5t_{4'})$ being a Π-facial subwalk, there is a Π-facial subwalk $\mathcal{P} = (t_1t_{0'} \cdots x_{4'}t_{4'})$, such that either in $\mathcal{P} \cup (t_1t_2t_3t_{4'})$ is a Π-facial cycle in the first instance or $\mathcal{P} \cup (t_1t_0t_5t_{4'})$ is a Π-facial cycle in the second instance.*

Proof. First, suppose that $(t_1t_2t_3t_{4'})$ is a Π-facial subwalk. This walk cannot be extended as $(t_0t_1t_2t_3t_{4'}t_5)$, otherwise it would intersect the Π-facial cycles $(t_0t_1t_2x_2t_0)$ and $(t_3t_4t_5t_{4'}t_3)$ in two edges, contradicting the definition of polyhedral embedding. Hence the only possible way to extend this walk is $(t_{0'}t_1t_2t_3t_{4'}x_{4'})$ and the claim follows.

Suppose $(t_1t_0t_5t_{4'})$ is a Π-facial subwalk. Similarly, this walk cannot be extended as $(t_2t_1t_0t_5t_{4'}t_3)$, otherwise it would intersect the Π-facial cycles $(t_0t_1t_2x_2t_0)$ and $(t_3t_4t_5t_{4'}t_3)$ in two edges, contradicting the definition of polyhedral embedding. Hence, the only possible way to extend this walk is $(t_{0'}t_1t_0t_5t_{4'}x_{4'})$ and the claim follows.

In both cases, a path $\mathcal{P} = (t_1t_{0'} \cdots x_{4'}t_{4'})$ completes either Π-facial subwalk in to a Π-facial cycle. □

Claim C1.6. *The Π-facial subwalk $\mathcal{P}$ in the previous claim is $(t_1t_{0'}x_{4'}t_{4'})$.*

Proof. We proceed by contradiction. Suppose that there is at least one vertex $t_p \in V(G)$ such that $\mathcal{P} = (t_1t_{0'}t_p \cdots x_{4'}t_{4'})$. We will now prove that we can deduce further structure of G^e from this assumption and arrive to a contradiction for either $(t_1t_2t_3t_{4'})$ or $(t_1t_0t_5t_{4'})$ being a Π-facial subwalk. □

Claim C1.6.1. *The fork given by the set of vertices $\{t_p, t_{0'}, t_1, t_0, t_2\}$ and the set of edges $\{(t_pt_{0'}), (t_{0'}t_1), (t_1t_0), (t_1t_2), [t_pt_0], [t_pt_2]\}$ has to be contained in G^e.*

In this instance, by Proposition 8, either $(t_pt_{0'}t_1t_0)$ or $(t_pt_{0'}t_1t_2)$ is a Π-facial subwalk. Therefore, if one of the scaffold edges $[t_pt_0]$ or $[t_pt_2]$ is not in $\mathcal{S}$ we can deduce whether $\mathcal{P} \cup (t_1t_2t_3t_{4'})$ or $\mathcal{P} \cup (t_1t_0t_5t_{4'})$ is a Π-facial cycle, by Proposition 5, and the disjunction can be solved, contradicting the hypothesis and the claim follows. □

Claim C1.6.2. *The paths $(t_px_{2'}x_2t_0)$ and $(t_px_{2'}x_2t_2)$ are in G.*

Proof. As we have argued in other claims, if there is to be a disjunction, by Lemma 17, there are a pair of 3-paths: one joining the vertices $\{t_p, t_0\}$ and another one joining the vertices $\{t_p, t_2\}$, internally disjoint to $(t_pt_{0'}t_1t_0)$ and $(t_pt_{0'}t_1t_2)$, respectively. Looking at the degree of the vertices involved, it is easy to see that the only possibility is that t_p is adjacent to $x_{2'}$ and the additional 3-paths are $(t_p x_{2'} x_2 t_0)$ and $(t_p x_{2'} x_2 t_2)$. □

Claim C1.6.3. *$\mathcal{P}$ does not continue through $x_{2'}$ after t_p.*

Proof. We will proceed by contradiction. Suppose $\mathcal{P} = (t_1 t_{0'} t_p x_{2'} \cdots x_{4'} t_{4'})$, then $\mathcal{P}$ cannot continue to the vertex t_2 after t_1, otherwise there is a 2-path $(t_2 x_2 x_{2'})$ contradicting Proposition 5. Then, $\mathcal{P}$ continue to t_0 after t_1 and using Proposition 5 we conclude that $\mathcal{P} \cup (t_1 t_0 t_5 t_{4'}) = (t_1 t_{0'} t_p x_{2'} \cdots x_{4'} t_{4'} t_5 t_0 t_1)$ is a Π-facial cycle. This implies that undoubtedly $(t_1 t_2 t_3 t_4)$ is the Π-facial subwalk corresponding to the scaffold edge $[t_1 t_4]$. Ergo, the disjunction can be solved, contradicting our hypothesis. Hence, $x_{2'}$ is not in $\mathcal{P}$. □

Finally, we will now see what happens in each of the cases: either $\mathcal{P} \cup (t_1 t_2 t_3 t_{4'})$ or $\mathcal{P} \cup (t_1 t_0 t_5 t_{4'})$ is a Π-facial cycle. Inasmuch as both cases are symmetric, we will only prove the first one in detail. Suppose that $\mathcal{P} \cup (t_1 t_2 t_3 t_{4'})$ is a Π-facial cycle. This implies that the 3-path $(t_1 t_2 t_3 t_{4'})$ corresponds to the edge $[t_1 t_{4'}] \in \mathcal{S}$ then the 3-path that corresponds to $[t_1 t_4]$ is $(t_1 t_0 t_5 t_4)$, given that the embedding is polyhedral, we can conclude that $(t_{0'} t_1 t_0 t_5 t_4 x_4)$ is a Π-facial subwalk.

Analyzing the existing facial subwalks and using extensive use of the fact that the embedding is polyhedral is easy to see that $(x_4 t_4 t_3 t_2 x_2 x_{2'})$ is a Π-facial subwalk. Notice that this walk cannot continue through the vertex t_p after $x_{2'}$, otherwise it would intersect the Π-facial cycle $\mathcal{P} \cup (t_1 t_2 t_3 t_{4'})$ in two edges.

Similarly, we can conclude that $(x_{4'} t_{4'} t_5 t_0 x_2 x_{2'})$ is a Π-facial subwalk, because it cannot continue through the vertex t_p after $x_{2'}$, otherwise it would intersect the Π-facial cycle $\mathcal{P} \cup (t_1 t_2 t_3 t_{4'})$ in two edges.

We have concluded that both Π-facial subwalks $(x_4 t_4 t_3 t_2 x_2 x_{2'})$ and $(x_{4'} t_{4'} t_5 t_0 x_2 x_{2'})$ continue after $x_{2'}$ through the same vertex (not equal to t_p). This is, they will intersect in two edges, contradicting the definition of polyhedral embedding. As we said the proof of the second case is analagous, then in both cases we get to a contradiction, and we can conclude that there is no additional vertex in the path $\mathcal{P} = (t_1 t_{0'} x_{4'} t_{4'})$, and obviously the edge $(t_{0'} x_{4'})$ is in G. Thus, there is a butterfly, B_2, contained in G: it is given by the union of the two 6-cycles $(t_0 t_1 t_2 t_3 t_4 t_5 t_0) \cup (t_{0'} t_1 t_2 t_3 t_{4'} x_{4'} t_{0'})$ and the set of scaffold edges $\{[t_1 t_4], [t_1 t_{4'}], [t_{0'} t_3]\}$.

This ends the proof of Case C1.

Case C2. Assume that the second internally disjoint 3-path between $t_{4'}$ and x_2 is $(t_{4'} t_5 t_4 x_2)$. Notice that $(t_2 t_3 t_4 x_2 t_2)$ is a Π-facial cycle by Proposition 7. Using Claim 1 and Definition 1 we can deduce that $(t_{0'} t_1 t_2 t_3 t_{4'})$ is a Π-facial subwalk, i.e., the disjunction can be solved, contradicting our hypothesis. Hence, this case can never occur. This ends the proof of Case C2.

Case C3. Assume that the second internally disjoint 3-path between $t_{4'}$ and x_2 is $(t_{4'} x_{4'} x_{2'} x_2)$. Remember that Claim C5 asserts that $(t_2 t_3 t_{4'} x_{4'})$ is a Π-facial subwalk. So, there is a butterfly two contained in G^e, B_2, whose set of vertices is $\{t_0, t_1, t_2, t_3, t_{4'}, t_5, x_{4'}, x_2, x_{2'}\}$ and whose set of edges is given by the union of edges in the two 6-cycles, $(t_0t_1t_2t_3t_{4'}t_5t_0) \cup (t_2t_3t_{4'}x_{4'}x_{2'}x_2t_2)$, and whose set of scaffold edges is $\{[t_{4'}t_1], [t_{4'}x_2], [t_2x_{4'}]\}$. This ends the proof for Case C3. ■

Theorem 19. *Let G^e be such that $B_1 \subset G^e$ (labelled as in Definition* 16*, Figure* 6*) and such that the disjunctions that arise from the two fork subgraphs of B_1 induced by the vertices $\{t_1, t_2, t_3, t_4, t_{4'}\}$ and $\{t_0, t_{0'}, t_1, t_2, t_3\}$ cannot be solved. Then G is the Petersen's graph, P, and G^e is the union of P and the set of (single) scaffold edges $\mathcal{S} = E(K_{10}) \setminus E(P)$.*

Proof. As $B_1 \subset G^e$ (labelled as in Definition 16) then G contains the two 6-cycles $(t_0t_1t_2t_3t_4t_5t_0)$ and $(t_{0'}t_1t_2t_3t_{4'}t_{5'}t_{0'})$. Also, $[t_0t_3]$, $[t_{0'}t_3]$, $[t_1t_4]$, $[t_1t_{4'}] \in \mathcal{S}$.

Claim 1. *Either $(t_0t_1t_2t_3t_{4'})$ or $(t_{0'}t_1t_2t_3t_4)$ (but not both) is a Π-facial subwalk.*

Proof. Suppose, without loss of generality, that $(t_0t_1t_2t_3)$ is a Π-facial subwalk. If $(t_1t_2t_3t_4)$ is the 3-path corresponding to $[t_1t_4]$, by Definition 1, $(t_0t_1t_2t_3t_4)$ is a Π-facial subwalk and by Proposition 5, $(t_0t_1t_2t_3t_4t_5t_0)$ is a Π-facial cycle. Furthermore, by Proposition 8 and Corollary 15 imply that $(t_1t_{0'}t_{5'}t_{4'})$ is the Π-facial subwalk corresponding to $[t_1t_{4'}]$ and $(t_{0'}t_{5'}t_{4'}t_3)$ is the Π-facial subwalk corresponding to $[t_{0'}t_3]$. It follows that, by Definition 1, $(t_1t_{0'}t_{5'}t_{4'}t_3)$ is a Π-facial subwalk; and by Proposition 5, $(t_{0'}t_1t_2t_3t_{4'}t_{5'}t_{0'})$ is a Π-facial cycle. But this last statement contradicts Definition 1, as two different facial walks would intersect in two edges. Hence $(t_1t_2t_3t_4)$ is not the 3-path corresponding to $[t_1t_4]$, so $(t_0t_1t_2t_3t_{4'})$ is a Π-facial subwalk and then by Proposition 8, the statement follows.

The case when we suppose $(t_{0'}t_1t_2t_3)$ as a Π-facial subwalk is resolved similarly. □

Claim 2. *Neither $(t_5t_0t_1t_2t_3t_{4'})$ nor $(t_0t_1t_2t_3t_{4'}t_{5'})$ can be a Π-facial subwalk.*

Proof. Suppose $(t_5t_0t_1t_2t_3t_{4'})$ is a Π-facial subwalk, by Proposition 5, $(t_0t_1t_2t_3 t_4t_5t_0)$ is a Π-facial cycle, and has to be the same Π-facial walk that contains $(t_5t_0t_1t_2t_3t_{4'})$, this contradicts the definition of polyhedral embedding (Definition 1).

The case for $(t_0t_1t_2t_3t_{4'}t_{5'})$ follows similarly. □

Claim 3. *Neither $(t_{5'}t_{0'}t_1t_2t_3t_4)$ nor $(t_{0'}t_1t_2t_3t_4t_5)$ can be a Π-facial subwalk.*

Proof. The proof of this statement follows in the same way as that of Claim 2. □

Claim 4. *Either $(t_0t_1t_2t_3t_{4'})$ is a Π-facial subwalk, implying that $(t_{0'}t_1t_0t_5t_4)$ and $(t_{0'}t_{5'}t_{4'}t_3t_4)$ are Π-facial subwalks, or $(t_{0'}t_1t_2t_3t_4)$ is a Π-facial subwalk, implying that $(t_0t_1t_{0'}t_{5'}t_{4'})$ and $(t_0t_5t_4t_3t_{4'})$ are Π-facial subwalks.*

Proof. By Claim 1, we know that $(t_0t_1t_2t_3t_{4'})$ and $(t_{0'}t_1t_2t_3t_4)$ cannot be Π-facial subwalks simultaneously.

Assume that $(t_0t_1t_2t_3t_{4'})$ is a Π-facial subwalk, then by Corollary 15, $(t_1t_0t_5t_4)$ is the Π-facial subwalk of $[t_1t_4]$. Since t_1 has degree three, necessarily $(t_2t_1t_0t_5t_4)$ or $(t_{0'}t_1t_0t_5t_4)$ is a Π-facial subwalk. However, if $(t_2t_1t_0t_5t_4)$ is a Π-facial subwalk then it intersects the Π-facial subwalk $(t_0t_1t_2t_3t_{4'})$ in two edges, contradicting Definition 1. Hence $(t_{0'}t_1t_0t_5t_4)$ is a Π-facial subwalk. Since $(t_0t_1t_2t_3)$ is a Π-facial subwalk, by Proposition 8, $(t_{0'}t_1t_2t_3)$ is not a Π-facial subwalk. Thus, by Corollary 15, the 3-path corresponding to $[t_{0'}t_3]$ is $(t_{0'}t_{5'}t_{4'}t_3)$. Since t_3 has degree three, necessarily $(t_{0'}t_{5'}t_{4'}t_3t_2)$ or $(t_{0'}t_{5'}t_{4'}t_3t_4)$ is a Π-facial subwalk. However, if $(t_{0'}t_{5'}t_{4'}t_3t_2)$ was a Π-facial subwalk then it would intersect the Π-facial subwalk $(t_0t_1t_2t_3t_{4'})$ in two edges, contradicting Definition 1. Hence $(t_{0'}t_{5'}t_{4'}t_3t_4)$ is a Π-facial subwalk.

The other case follows similarly. □

Claim 5. $(t_0t_1t_2t_3t_{4'}t_0)$ *and* $(t_{0'}t_1t_2t_3t_4t_{0'})$ *are* 5*-cycles of* G. (*Not necessarily facial cycles of the embedding.*)

Proof. By Claim 1, we know that either $(t_0t_1t_2t_3t_{4'})$ or $(t_{0'}t_1t_2t_3t_4)$ is a Π-facial subwalk. Assume that $(t_0t_1t_2t_3t_{4'})$ is a Π-facial subwalk. Then, by Claim 4, $(t_{0'}t_1t_0t_5t_4)$ and $(t_{0'}t_{5'}t_{4'}t_3t_4)$ are Π-facial subwalks and observe that they intersect in their start and end vertices. By Proposition 9, either $(t_{0'}t_{5'}t_{4'}t_3t_4t_5t_0t_1t_{0'})$ is a Π-facial cycle, or the edge $(t_{0'}t_4)$ is in $E(G)$. However, $(t_{0'}t_{5'}t_{4'}t_3t_4t_5t_0t_1t_{0'})$ cannot be a Π-facial cycle, since it would intersect the Π-facial subwalk $(t_0t_1t_2t_3t_{4'})$ in two edges and this would contradict the assumption that the embedding is polyhedral. Then, if $(t_0t_1t_2t_3t_{4'})$ is a Π-facial subwalk, the edge $(t_{0'}t_4) \in E(G)$, and $(t_{0'}t_1t_2t_3t_4t_{0'})$ is a 5 -cycle of G.

If we assume that $(t_{0'}t_1t_2t_3t_4)$ is a Π-facial subwalk, the proof follows in a similar way and we can deduce that $(t_0t_{4'}) \in E(G)$ and $(t_0t_1t_2t_3t_{4'})$ is a 5-cycle of G.

Notice that, as a consequence of the previous two arguments, it cannot happen that both of $(t_{0'}t_4)$ and $(t_0t_{4'})$ are not in $E(G)$ simultaneously. Otherwise, this would contradict Claim 1. We will now argue that both $(t_{0'}t_4)$ and $(t_0t_{4'})$ are in $E(G)$.

Suppose that $(t_0t_{4'}) \notin E(G)$, then $(t_{0'}t_4) \in E(G)$. If $(t_{0'}t_1t_2t_3t_4)$ is a Π-facial subwalk then, by Claim 4, $(t_0t_1t_{0'}t_{5'}t_{4'})$ and $(t_0t_5t_4t_3t_{4'})$ are Π-facial subwalks. It follows that, by Proposition 9, $(t_0t_1t_{0'}t_{5'}t_{4'}t_3t_4t_5t_0)$ is a Π-facial cycle, but it intersects the Π-facial subwalk $(t_{0'}t_1t_2t_3t_4)$ in two edges, which contradicts the definition of polyhedral embedding. Hence $(t_{0'}t_1t_2t_3t_4)$ is not a Π-facial subwalk, and, by Claim 1, $(t_0t_1t_2t_3t_{4'})$ has to be a Π-facial subwalk, implying that the disjunction can be solved. Therefore $(t_0t_{4'}) \in E(G)$.

The case where we begin by assuming that $(t_{0'}t_4) \notin E(G)$ follows analogously, thus $(t_{0'}t_4) \in E(G)$.

These arguments prove that $(t_0t_{4'})$ and $(t_{0'}t_4)$ are in $E(G)$, and $(t_0t_1t_2t_3t_{4'}t_0)$ and $(t_{0'}t_1t_2t_3t_4t_{0'})$ are 5-cycles of G. □

Claim 6. *The scaffold edges* $[t_0t_2]$, $[t_1t_3]$, $[t_2t_{0'}]$, $[t_2t_{4'}]$, $[t_2t_4]$ *are in* $\mathcal{S}$.

Proof. By Claims 1 and 5, and Proposition 4 either $(t_0t_1t_2t_3t_{4'}t_0)$ or $(t_{0'}t_1t_2t_3t_4t_{0'})$ is a Π-facial cycle. So, if any of edges $[t_0t_2]$, $[t_1t_3]$, $[t_2t_{0'}]$, $[t_2t_{4'}]$, $[t_2t_4]$ were not in $\mathcal{S}$, we could easily discard the possibility that one of the two cycles is a Π-facial cycle, thus, we could solve the disjunction and the claim follows. □

Claim 7. *There is a new vertex* t_x *adjacent to* t_2, t_5 *and* $t_{5'}$.

Proof. By Claim 6, the edges $[t_0t_2]$, $[t_2t_{0'}]$, $[t_2t_{4'}]$, $[t_2t_4]$ are in $\mathcal{S}$. If for any of this edges there was only one 3-path that could correspond to it, then, by Proposition 5, we could know whether $(t_0t_1t_2t_3t_{4'}t_0)$ or $(t_{0'}t_1t_2t_3t_4t_{0'})$ is a Π-facial cycle, thus we could solve the disjunction. Hence there have to be additional 3-paths that can correspond to $[t_0t_2]$, $[t_2t_{0'}]$, $[t_2t_{4'}]$, $[t_2t_4]$.

In order to find a second path between t_2 and $t_{4'}$ of length three, which is internally disjoint to $(t_2t_3t_{4'})$ and $(t_2t_1t_0t_{4'})$, let t_x be the remaining vertex adjacent to t_2. Observe that $t_x \neq t_5$, otherwise by Proposition 7 $(t_0t_1t_2t_5t_0)$ and $(t_2t_3t_4t_5t_2)$ are Π-facial cycles and using Claim 1, we can deduce easily if $(t_0t_1t_2t_3t_{4'})$ or $(t_0t_1t_2t_3t_{4'})$ is a Π-facial subwalk, contradicting our hypothesis. Analogously $t_x \neq t_{5'}$. Thus t_x is a new vertex. Notice that $[t_xt_0], [t_xt_4], [t_xt_{0'}], [t_xt_{4'}]$ are in $\mathcal{S}$, otherwise, by Proposition 8, we can deduce the sub-paths in the Π-facial subwalks that correspond to $[t_1t_4]$ and $[t_1t_{4'}]$, which implies that we can solve the disjunction. Then, the second 3-path between t_2 and $t_{4'}$ contains t_x and $t_{5'}$. This implies that $(t_xt_{5'})$ is an edge of G.

Similarly, for $[t_0t_2]$ there are already two disjoint paths $(t_2t_1t_0)$ and $(t_2t_3t_{4'}t_0)$ of length two and three respectively. Then, the third path necessarily contains t_x and t_5, which implies (as before) that (t_xt_5) is an edge of G. □

Note that we now know the complete underlying cubic graph G, namely G is the Petersen's graph.

Claim 8. $[t_0t_{0'}]$, $[t_0t_{5'}]$, $[t_0t_4]$, $[t_1t_5]$, $[t_1x]$, $[t_1t_{5'}]$, $[t_2t_{5'}]$, $[t_2t_5]$, $[t_3x]$, $[t_3t_{5'}]$, $[t_3t_5]$, $[t_{4'}t_5]$, $[t_4t_{4'}]$, $[t_0t_{4'}]$, $[t_5t_{0'}]$, $[t_4t_{5'}]$ *and* $[t_5t_{5'}]$ *are in* $\mathcal{S}$.

Proof. We will use identical arguments to those in Claim 6. For example if $[t_0t_{0'}]$ is not in $\mathcal{S}$, then any 3-path whith start t_0 and end $t_{0'}$ cannot be a facial subwalk. This implies that by process of elimination and using Proposition 8 and Proposition 5, we can deduce the Π-facial cycles, and the disjunction can be solved. □

This finalizes the proof. Therefore, G is Petersen Graph and $\mathcal{S} = E(K_{10} \setminus E(P))$. ■

Theorem 20. *Let G^e be such that $B_2 \subset G^e$ (labelled as in Definition* 16, *Figure* 6) *and such that the disjunction that arises from the fork subgraph of B_2 induced by the vertices $\{t_1, t_2, t_3, t_4, t_{4'}\}$ cannot be solved. Then G is the Franklin graph, F, and G^e is the union of F and the set $\mathcal{S}$ given by* simple *scaffold edges* $[t_0t_5]$, $[t_5x_5]$, $[x_5x_0]$, $[x_0t_0]$, $[t_1t_2]$, $[t_2x_2]$, $[x_2t_{0'}]$, $[t_{0'}t_1]$, $[t_3t_{4'}]$, $[t_{4'}\ t_{5'}]$, $[t_{5'}t_4]$, $[t_4t_3]$, *and the* double *scaffold edges* $[[t_0t_3]], [[x_0t_2]], [[t_1t_{4'}]], [[t_{0'}t_5]], [[t_4x_2]], [[x_5t_{5'}]]$, $[[t_0t_{5'}]]$, $[[t_1t_4]]$, $[[t_2t_5]], [[t_3x_5]], [[t_{4'}x_2]], [[x_0t_{0'}]]$.

Proof. Observe that in B_2, $(t_0t_1t_2t_3t_4t_5t_0)$ and $(t_{0'}t_1t_2t_3t_{4'}t_{5'}t_{0'})$ are two 6-cycles such that $[t_1t_4]$, $[t_1t_{4'}]$, $[t_0t_3] \in G^e$ and $[t_{0'}t_3] \notin G^e$, which implies by Proposition 8, that $(t_0t_1t_2t_3)$ is Π-facial subwalk of $[t_0t_3]$.

Claim 1. $[[t_0t_3]]$, $[[t_1t_4]]$ *and* $[[t_2t_5]]$ *are double scaffold edges.*

Proof. If one of $[[t_0t_3]]$, $[[t_1t_4]]$ or $[[t_2t_5]]$ is not a double scaffold edge, then $(t_1t_2t_3t_4)$ cannot be a Π-facial subwalk, otherwise by Definition 1 and Proposition 5, this would imply that $(t_0t_1t_2t_3t_4t_5t_0)$ is a Π-facial cycle, but this is impossible as we started by assuming that at least one of $[[t_0t_3]]$, $[[t_1t_4]]$ or $[[t_2t_5]]$ is not a double scaffold edge.

It follows by Proposition 8 that $(t_1t_2t_3t_{4'})$ is the Π-facial subwalk that corresponds to the scaffold edge $[t_1t_{4'}]$, and then, by Proposition 8 and Corollary 15, the Π-facial subwalk corresponding to the edge $[t_1t_4]$ should be $(t_1t_0t_5t_4)$, which implies that the disjunction could be solved, contradicting our hypothesis. Hence the claim holds. □

Claim 2. *For each pair of vertices $\{t_0, t_3\}$, $\{t_1, t_4\}$ and $\{t_2, t_5\}$, there is a third* 3*-path between them whose vertices are disjoint to the cycle $(t_0t_1t_2t_3t_4t_5t_0)$.*

Proof. Suppose that for at least one of the pairs of vertices mentioned above, there are only two 3-paths, by Proposition 9, $(t_0t_1t_2t_3t_4t_5t_0)$ is Π-facial cycle and then $(t_1t_2t_3t_4)$ is the Π-facial subwalk corresponding to $[t_1t_4]$, i.e., we can solve the disjunction, contradicting our hypothesis. Hence the claim holds. □

Since G is a cubic graph, let x_i be the remaining vertex adjacent to t_i for $i \in \{0, 2, 5\}$, then the following holds.

Claim 3. *The vertex x_0 is different with the vertices t_i for all $i \in \{0, 1, 2, 3, 4, 5, 0', 4', 5'\}$ and adjacent to $t_{4'}$.*

Proof. First, in order to prove that all vertices are distinct we proceed by contradiction.

1. ($x_0 \neq t_{0'}$.) If $x_0 = t_{0'}$, then by Proposition 6, $(t_0t_{0'}t_1t_0)$ is a Π-facial cycle, this implies, by Proposition 8, that $(t_5t_0t_1t_2)$ is a Π-facial subwalk. Since $(t_0t_1t_2t_3)$ is Π-facial subwalk, by Definition 1, necessarily $(t_5t_0t_1t_2t_3)$ is a Π-facial

subwalk, and using Proposition 5, $(t_0t_1t_2t_3t_4t_5t_0)$ is a Π-facial cycle. The latter implies that $(t_1t_2t_3t_4)$ is the Π-facial subwalk corresponding to $[t_1t_4]$ and thus we can solve the disjunction, contradicting our hypothesis. Hence $x_0 \neq t_{0'}$.

2. ($x_0 \neq t_2$.) If $x_0 = t_2$, by Proposition 6, $(t_0t_1t_2t_0)$ is a Π-facial cycle, but this contradicts, by Definition 1, the fact that $(t_0t_1t_2t_3)$ is a Π-facial subwalk. Hence $x_0 \neq t_2$.

3. ($x_0 \neq t_4$.) If $x_0 = t_4$, since $(t_0t_1t_2t_3)$ is a Π-facial subwalk, Proposition 5 implies that $(t_0t_1t_2t_3t_4t_0)$ is a Π-facial cycle, and $(t_1t_2t_3t_4)$ is the Π-facial subwalk corresponding to $[t_1t_4]$. Thus, the disjunction can be solved, contradicting our hypothesis. Hence, $x_0 \neq t_4$.

4. ($x_0 \neq t_3$.) If $x_0 = t_3$, then t_3 would have degree four and G would not be a cubic graph.

5. ($x_0 \neq t_{4'}$.) If $x_0 = t_{4'}$, since $(t_0t_1t_2t_3)$ is a Π-facial subwalk, Definition 1 and Proposition 5 imply that $(t_0t_1t_2t_3t_{4'}t_0)$ is a Π-facial cycle and that $(t_1t_2t_3t_{4'})$ is the Π-facial subwalk corresponding to $[t_1t_{4'}]$. Thus we can solve the disjunction, contradicting our hypothesis. Hence $x_0 \neq t_{4'}$.

6. ($x_0 \neq t_{5'}$.) If $x_0 = t_{5'}$, then by Proposition 7, $(t_0t_1t_{0'}t_{5'}t_0)$ is a Π-facial cycle; the latter and Proposition 8 imply that $(t_2t_1t_0t_5)$ is a Π-facial subwalk, and by Definition 1, since $(t_0t_1t_2t_3)$ is a Π-facial subwalk, $(t_5t_0t_1t_2t_3)$ is a Π-facial subwalk. By Proposition 5, it follows that $(t_0t_1t_2t_3t_4t_5t_0)$ is a Π-facial cycle and $(t_1t_2t_3t_4)$ is the Π-facial subwalk that corresponds to $[t_1t_4]$. The previous statements imply that we can solve the disjunction, contradicting our hypothesis. Hence $x_0 \neq t_{5'}$.

Finally, Claim 2 implies x_0 and $t_{4'}$ are adjacent. □

Claim 4. *The vertex x_2 is different from the vertices t_i for all $i \in \{0,1,2,3,4,5,0',4',5'\}$ and x_0.*

Proof. We proceed by contradiction.

1. ($x_2 \neq t_0$.) If $x_2 = t_0$ then t_0 would have degree four and contradict the fact that G is a cubic graph. Hence $x_2 \neq t_0$.

2. ($x_2 \neq t_{0'}$.) If $x_2 = t_{0'}$, by Proposition 6, $(t_{0'}t_1t_2t_{0'})$ is a Π-facial cycle. Thus, by the previous statement and Definition 1, $(t_3t_2t_{0'}t_{5'})$ is a Π-facial subwalk. By the previous statement and Proposition 5, $(t_3t_2t_{0'}t_{5'}t_{4'}t_3)$ is a Π-facial cycle. The latter and Definition 1, imply that $(t_1t_2t_3t_{4'})$ is not a Π-facial subwalk. This, in turn, implies that $(t_1t_2t_3t_4)$ is a Π-facial subwalk, and the disjunction can be solved, contradicting our hypothesis. Hence $x_2 \neq t_{0'}$.

3. ($x_2 \neq t_4$.) If $x_2 = t_4$, by Proposition 6, $(t_2t_3t_4t_2)$ is a Π-facial cycle. The latter together with Definition 1 imply $(t_1t_2t_3t_4)$ is not a Π-facial subwalk, thus $(t_1t_2t_3t_{4'})$ is a Π-facial subwalk and the disjunction can be solved, contradicting our hypothesis. Hence $x_2 \neq t_4$.

4. ($x_2 \neq t_{4'}$.) The proof follows identical arguments to those used in the previous case.

5. ($x_2 \neq t_5$.) If $x_2 = t_5$, by Proposition 7, $(t_2t_3t_4t_5t_2)$ is a Π-facial cycle. The latter and Definition 1, imply that $(t_1t_2t_3t_4)$ is not a Π-facial subwalk, thus $(t_1t_2t_3t_{4'})$ is a Π-facial subwalk and the disjunction can be solved, contradicting our hypothesis. Hence $x_2 \neq t_5$.

6. ($x_2 \neq t_{5'}$.) The proof is similar to the previous case.

7. ($x_2 \neq x_0$.) If $x_2 = x_0$, then by Proposition 7, $(t_0t_1t_2x_0t_0)$ is a Π-facial cycle. Thus the Π-facial cycle $(t_0t_1t_2x_0)$ intersects to the Π-facial subwalk $(t_0t_1t_2t_3)$ in three vertices contradicting that we have a polyhedral embedding. □

Claim 5. *The vertex x_5 is different from the vertices t_i for all $i \in \{0, 1, 2, 3, 4, 5, 0', 4', 5'\}$ and x_0, x_2, and adjacent to x_2.*

Proof. Once more, we proceed by contradiction to prove that the vertices are distinct.

1. ($x_5 \neq t_{0'}$.) If $x_5 = t_{0'}$, then Proposition 7 implies that $(t_5t_0t_1t_{0'}t_5)$ is a Π-facial cycle. The latter and Definition 1 imply that $(t_1t_0t_5t_4)$ is not a Π-facial subwalk. Then, by Corollary 15, it follows that $(t_1t_2t_3t_4)$ is the Π-facial subwalk corresponding to the edge $[t_1t_4]$. However, using Proposition 5 we can deduce that $(t_0t_1t_2t_3t_4t_5t_0)$ is a Π-facial cycle which intersects the Π-facial cycle $(t_5t_0t_1t_{0'}t_5)$ in two edges, contradicting the polyhedrality of the embedding. In conclusion, this case can never occur.

2. ($x_5 \neq t_2$, $x_5 \neq t_1$, $x_5 \neq t_3$, $x_5 \neq t_{4'}$.) If any of the inequalities was an equality instead we would have a contradiction on the fact that G is a cubic graph.

3. ($x_5 \neq x_0$.) If $x_5 = x_0$, then it is only possible to find two internally disjoint 3-paths between t_2 and t_5. Since $[[t_2t_5]]$ is a double scaffold edge and $(t_2t_5) \notin E(G)$ (otherwise contradicts that G is a cubic graph), Proposition 9 implies that $(t_0t_1t_2t_3t_4t_5t_0)$ is a Π-facial cycle. In turn, the latter implies that $(t_1t_2t_3t_4)$ is the Π-facial subwalk corresponding to $[t_1t_4]$ and that $(t_1t_{0'}t_{5'}t_{4'})$ is a Π-facial subwalk, thus we can solve the disjunction, contradicting our hypothesis. Hence $x_5 \neq x_0$.

4. ($x_5 \neq x_2$.) If $x_5 = x_2$, Proposition 5 implies that $(t_0t_1t_2t_3t_4t_5t_0)$ is not a Π-facial cycle. Since $(t_0t_1t_2t_3)$ is a Π-facial subwalk, using Proposition 13 we can deduce that $(t_0t_1t_2t_3t_{4'}x_0t_0)$ is a Π-facial cycle; i.e., $(t_1t_2t_3t_{4'})$ is the Π-facial subwalk that corresponds to $[t_1t_{4'}]$. The latter and Corollary 15 imply that $(t_1t_0t_5t_4)$ is the Π-facial subwalk corresponding to $[t_1t_4]$, and then we can solve the disjunction, contradicting our hypothesis. Hence $x_5 \neq x_2$.

5. ($x_5 \neq t_{5'}$.) If $x_5 = t_{5'}$, it is only possible to find two 3-paths internally disjoint between them t_2 and t_5. Since $[[t_2t_5]]$ is double and $(t_2t_5) \notin E(G)$ (otherwise contradicts that G is a cubic graph), Proposition 9 implies that $(t_0t_1t_2t_3t_4t_5t_0)$ is a Π-facial cycle. Hence, $(t_1t_2t_3t_4)$ is the Π-facial subwalk corresponding to

$[t_1t_4]$. Using the latter and Corollary 15 is easy to see that $(t_1t_{0'}t_{5'}t_{4'})$ is the Π-facial subwalk corresponding to $[t_1t_{4'}]$. That is we can solve the disjunction, contradicting our hypothesis. Hence $x_5 \neq x_{5'}$.

Finally, x_2 and x_5 are adjacent by Claim 2. □

Claim 6. *The vertex x_4 is different from the vertices t_i, for all $i \in \{0, 1, 2, 3, 4, 5, 0', 4'\}$ and x_0, x_2, x_5.*

Proof. We proceed by contradiction.

1. $(x_4 \neq t_0,\ x_4 \neq t_1,\ x_4 \neq t_2,\ x_4 \neq t_{4'})$. If any of the inequalities was an equality instead we would have a contradiction on the fact that G is a cubic graph.

2. $(x_4 \neq t_{0'}.)$ If $x_4 = t_{0'}$ then there is a 2-path $(t_1t_{0'}t_4)$ and two 3-paths, $(t_1t_2t_3t_4)$ and $(t_1t_0t_5t_4)$ between t_1 and t_4. Since $[[t_1t_4]]$ is a double scaffold edge, and there is no edge (t_1t_4), by Proposition 9 $(t_0t_1t_2t_3t_4t_5t_0)$is a Π-facial cycle. But this contradicts Proposition 5 since there is a 2-path $(t_1t_{0'}t_4)$ between t_1 and t_4. Hence $x_4 \neq x_{0'}$.

3. $(x_4 \neq x_0.)$ If $x_4 = x_0$, Proposition 7 implies that $(t_0t_5t_4x_0t_0)$ is a Π-facial cycle. The latter, Proposition 8 and Corollary 15 imply that $(t_1t_2t_3t_4)$ is the Π-facial subwalk corresponding to $[t_1t_4]$ and we can solve the disjunction, contradicting our hypothesis. Hence $x_4 \neq x_0$.

4. $(x_4 \neq x_2.)$ If $x_4 = x_2$, then Proposition 7 implies that $(t_2t_3t_4x_2t_2)$ is a Π-facial cycle. The latter and Proposition 8 imply that $(t_1t_2t_3t_{4'})$ is the Π-facial subwalk corresponding to $[t_1t_{4'}]$. Subsequently, using Corollary 15 it is possible to deduce that $(t_1t_0t_5t_4)$ is the Π-facial subwalk corresponding to $[t_1t_4]$, that is, we can solve the disjunction, contradicting our hypothesis. Hence $x_4 \neq x_2$.

5. $(x_4 \neq x_5.)$ If $x_4 = x_5$, using Proposition 6 we deduce that $(t_4t_5x_5t_4)$ is a Π-facial cycle. In addition to this, using Proposition 8 we can prove that $(t_0t_5t_4t_3)$ is a Π-facial subwalk, ussing that $(t_0t_3) \notin E(G)$ and applying Proposition 9, we have $(t_0t_1t_2t_3t_4t_5t_0)$ is a Π-facial cycle. The latter implies that $(t_1t_2t_3t_4)$ is the Π-facial subwalk corresponding to $[t_1t_4]$. Using that $(t_1t_2t_3t_4)$ is the Π-facial subwalk corresponding to $[t_1t_4]$ and Corollary 15 we arrive to the conclusion that $(t_1t_{0'}t_{5'}t_{4'})$ is the Π-facial subwalk corresponding to $[t_1t_{4'}]$. Thus, we can solve the disjuntion, contradicting our hypothesis. Hence $x_4 \neq x_5$. □

Claim 7. $[[t_1t_{4'}]]$ *is a double scaffold edge.*

Proof. Since $[t_1t_{4'}] \in \mathcal{S}$ and between t_1 and $t_{4'}$ there are three internally disjoint 3-paths, $(t_1t_2t_3t_{4'})$, $(t_1t_{0'}t_{5'}t_{4'})$ and $(t_1t_0x_0t_{4'})$, by Proposition 13, necessarily $[[t_1t_{4'}]]$ is double. □

Claim 8. $(t_1t_0x_0t_{4'})$ *is a* Π-*facial subwalk.*

Proof. Since $(t_0t_1t_2t_3)$ is a Π-facial subwalk and $[[t_1t_{4'}]]$ is a double scaffold edge (by Claim 7), then Corollary 14 implies that $(t_1t_0x_0t_{4'})$ is a Π-facial subwalk. □

Claim 9. *$[[t_0t_{5'}]]$, $[[x_0t_{0'}]]$, $[[t_0t_3]]$ and $[[x_0t_2]]$ are double scaffold edges and there exist three internally disjoint paths between each pair of end vertices.*

Proof. If some edge, $[t_0t_{5'}]$, $[x_0t_{0'}]$, $[t_0t_3]$ or $[x_0t_2]$ does not belong to $\mathcal{S}$, or it is not double or they are all double but for one of them there exist only two 3-paths between its end vertices, then by Proposition 9 we know if $(t_1t_2t_3t_{4'})$ is Π-facial subwalk or not. This would implies that the disjunction can be solved, contradicting the hypothesis. □

Let $x_{5'}$ be the remaining vertex adjacent to $t_{5'}$.

Claim 10. *$(t_0t_1t_{0'}t_{5'})$ is a Π-facial subwalk.*

Proof. Notice that for the double scaffold edge $[[t_0t_{5'}]]$ there exist three internally disjoint 3-paths, $(t_0x_0t_{4'}t_{5'})$, $(t_0t_5x_{5'}t_{5'})$ and $(t_0t_1t_{0'}t_{5'})$. Since $(t_1t_0x_0t_{4'})$ is a Π-facial subwalk, then by Corollary 14, necessarily $(t_0x_0t_{4'}t_{5'}t_{0'}t_1t_0)$ or $(t_0t_5x_5t_{5'}t_{0'}t_1t_0)$ is a Π-facial cycle, i.e., $(t_0t_1t_{0'}t_{5'})$ is a Π-facial subwalk. □

Claim 11. *$(t_1t_0t_5t_4)$ is a Π-facial subwalk.*

Proof. Since $[[t_1t_4]]$ is a double scaffold edge, there exist three internally disjoint 3-paths: $(t_1t_0t_5t_4)$, $(t_1t_{0'}x_4t_4)$ and $(t_1t_2t_3t_4)$. Since $(t_0t_1t_2t_3)$ is a Π-facial subwalk, then by Corollary 14, $(t_1t_0t_5t_4)$ is a Π-facial subwalk. □

Claim 12. $x_4 = t_{5'}$.

Proof. If $x_4 \neq t_{5'}$, then $(t_1t_0t_5t_4x_4t_{0'}t_1)$ is not a Π-facial cycle because $(t_0t_1t_{0'}t_{5'})$ is a Π-facial subwalk, and this would contradict Definition 1. In addition, Claim 11 implies that $(t_1t_0t_5t_4t_3t_2t_1)$ is a Π-facial cycle and $(t_1t_{0'}t_{5'}t_{4'})$ is the Π-facial subwalk of $[t_1t_{4'}]$. Thus the disjunction could be solved, contradicting our hypothesis. □

Claim 13. *$(t_2t_3t_4t_5)$ is a Π-facial subwalk.*

Proof. Notice that for the double scaffold edge $[[t_2t_5]]$ there are three internally disjoint 3-paths $(t_2t_1t_0t_5)$, $(t_2x_2x_5t_5)$ and $(t_2t_3t_4t_5)$. Since $(t_0t_1t_2t_3)$ is a Π-facial subwalk, by Corollary 14, $(t_2t_3t_4t_5)$ is a Π-facial subwalk. □

Claim 14. *$[[t_3x_5]]$, $[[t_4x_2]]$, $[[t_0t_3]]$ and $[[t_1t_4]]$ are double scaffold edges in $\mathcal{S}$ and there exist three internally disjoint paths between its end vertices.*

Proof. If one of $[t_3x_5]$, $[t_4x_2]$, $[t_0t_3]$ or $[t_1t_4]$ does not belong in $\mathcal{S}$, or it is not a double scaffold edge, or they are all double but for one of them there exist only two 3-paths between its end vertices, then by Proposition 9, we know if $(t_1t_2t_3t_4)$ is a Π-facial subwalk or not. Consequently, using Proposition 13, we could conclude if the Π-facial subwalk corresponding to the edge $[t_1t_4]$ is either $(t_1t_2t_3t_4)$ or $(t_1t_{0'}t_{5'}t_4)$ thus we could solve the disjunction, contradicting our hypothesis. □

Observe that for the edges $[[t_0t_3]]$ and $[[t_1t_4]]$ there already exist three internally disjoint paths between their end vertices. For $[[t_3x_5]]$ and $[[t_4x_2]]$ we need to find additional edges of G to complete their corresponding 3-paths.

Claim 15. $(x_0x_5) \in E(G)$.

Proof. Note that for $[[t_3x_5]]$ there are already two 3-paths, $(t_3t_2x_2x_5)$ and $(t_3t_4t_5x_5)$. Since the only adjacent vertex to t_3 that is not in either of the paths already mentioned is $t_{4'}$, then $t_{4'}$ must belong to the third 3-path. The third 3-path can continue through either $t_{5'}$ or x_0. However, the vertex $t_{5'}$ is adjacent to $t_4, t_{4'}$ and $t_{0'}$, then the third 3-path necessarily has to continue to x_0. Since x_0 has degree two thus far, the only option left now is that it is adjacent to x_5. □

Claim 16. $(x_2x_{0'}) \in E(G)$.

Proof. The proof of this statement follows in a manner similar to the proof of Claim 15. □

Claim 17. *$[[x_2t_{4'}]]$, $[[t_{5'}x_5]]$ and $[[t_5t_{0'}]]$ are double scaffold edges.*

Proof. Notice that if $[x_2t_{4'}] \notin \mathcal{S}$, by Proposition 8, $(t_1t_2t_3t_{4'})$ is the Π-facial subwalk corresponding to $[t_1t_{4'}]$, and then the disjunction can be solved, contradicting the hypothesis. Hence $[x_2t_{4'}] \in \mathcal{S}$. Since there are three internally disjoint 3-paths between its end vertices, by Proposition 13, $[[x_2t_{4'}]]$has to be a double scaffold edge. The proof follows similarly for $[[t_{5'}x_5]]$ and $[[t_5t_{0'}]]$. □

Claim 18. *The edges $[t_1t_{0'}]$, $[t_{0'}x_2]$, $[x_2t_2]$, $[t_1t_2]$, $[t_3t_{4'}]$, $[t_{4'}t_{5'}]$, $[x_4t_4]$, $[t_3t_4]$, $[x_0t_0]$, $[t_0t_5]$, $[t_5x_5]$ and $[x_5x_0]$ are in $\mathcal{S}$.*

Proof. Since $(t_1t_2x_2t_{0'}t_1)$, $(t_0t_5x_5x_0t_0)$ and $(t_3t_4t_{5'}t_{4'}t_3)$ are Π-facial cycles (by Proposition 7), the edges $[t_1t_{0'}]$, $[t_{0'}x_2]$, $[x_2t_2]$, $[t_1t_2]$, $[t_3t_{4'}]$, $[t_{4'}t_{5'}]$, $[x_4t_4]$, $[t_3t_4]$, $[x_0t_0]$, $[t_0t_5]$, $[t_5x_5]$ and $[x_5x_0]$ are in $\mathcal{S}$.

Claim 18 completes the proof of the theorem. ■

4. Conclusion

The main aim of the project, that this paper initiates, is to develop an algorithmic procedure for constructing all the extended graphs of a given cubic graph.

Thus, the next step is characterizing when a set of scaffold edges build on top of a cubic graph is indeed an extended graph.

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APPENDIX

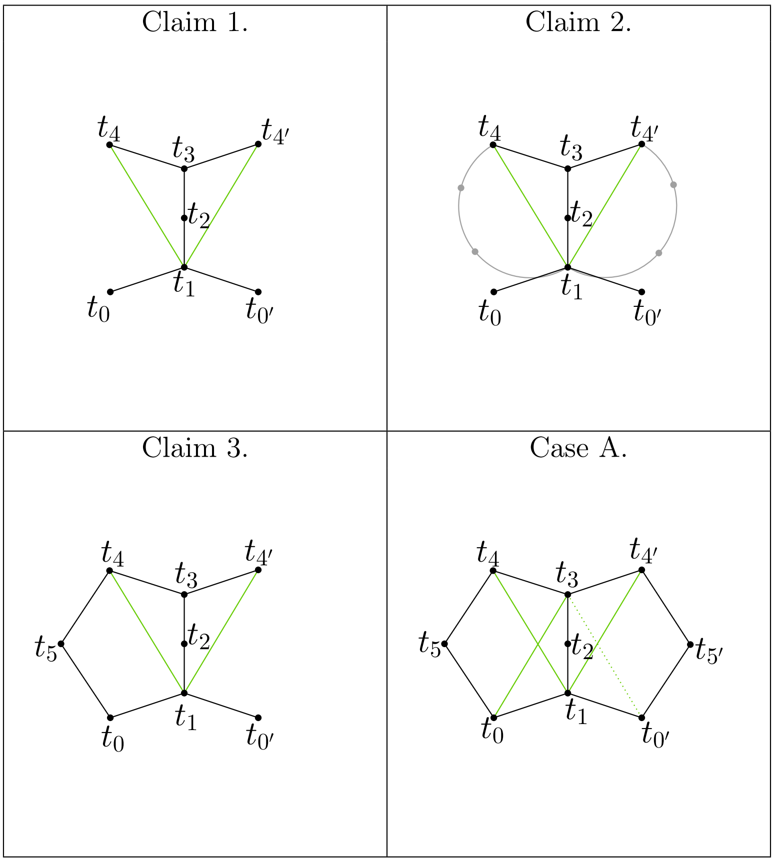


Table 1. Theorem 15. Case A

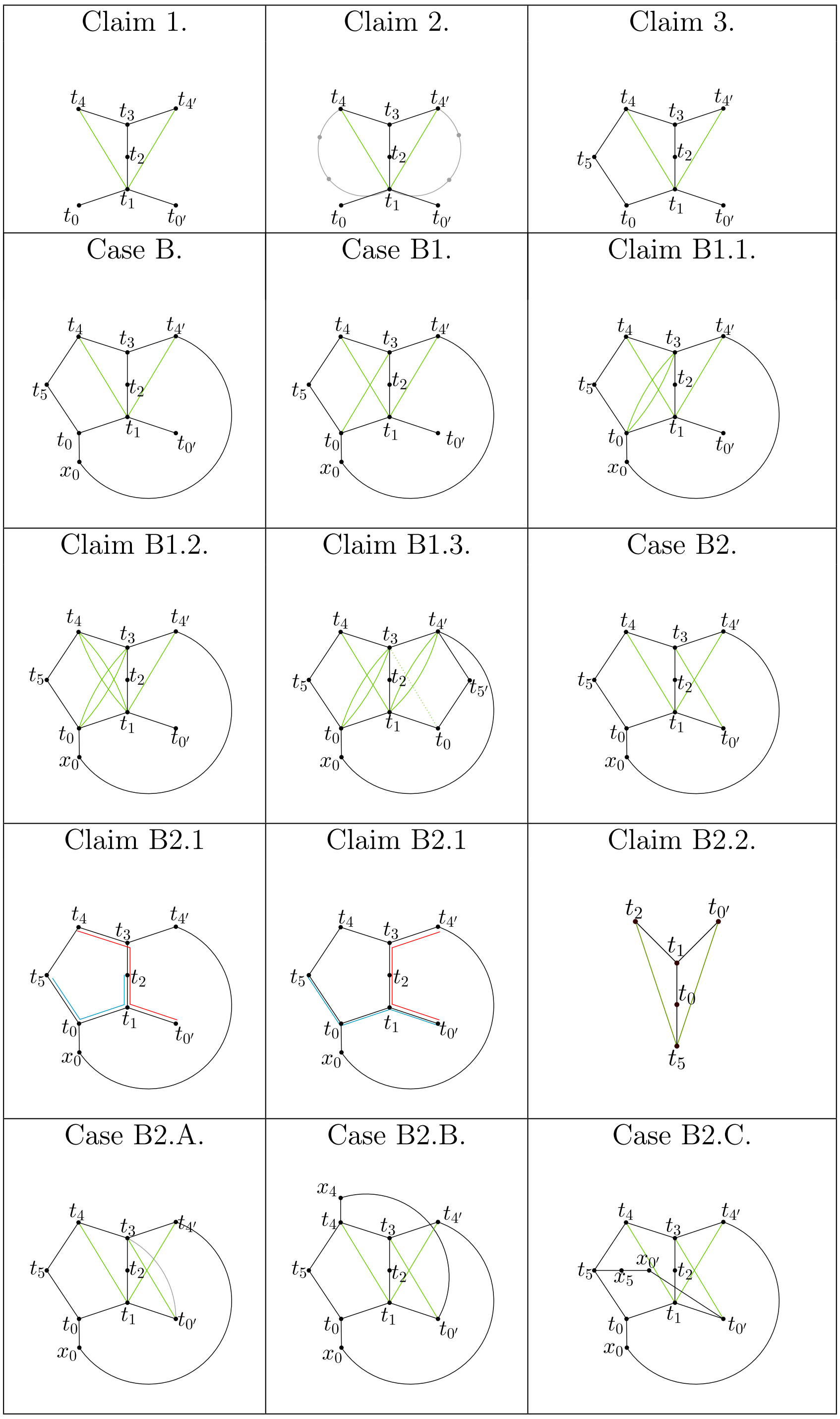


Table 2. Theorem 15. Case B

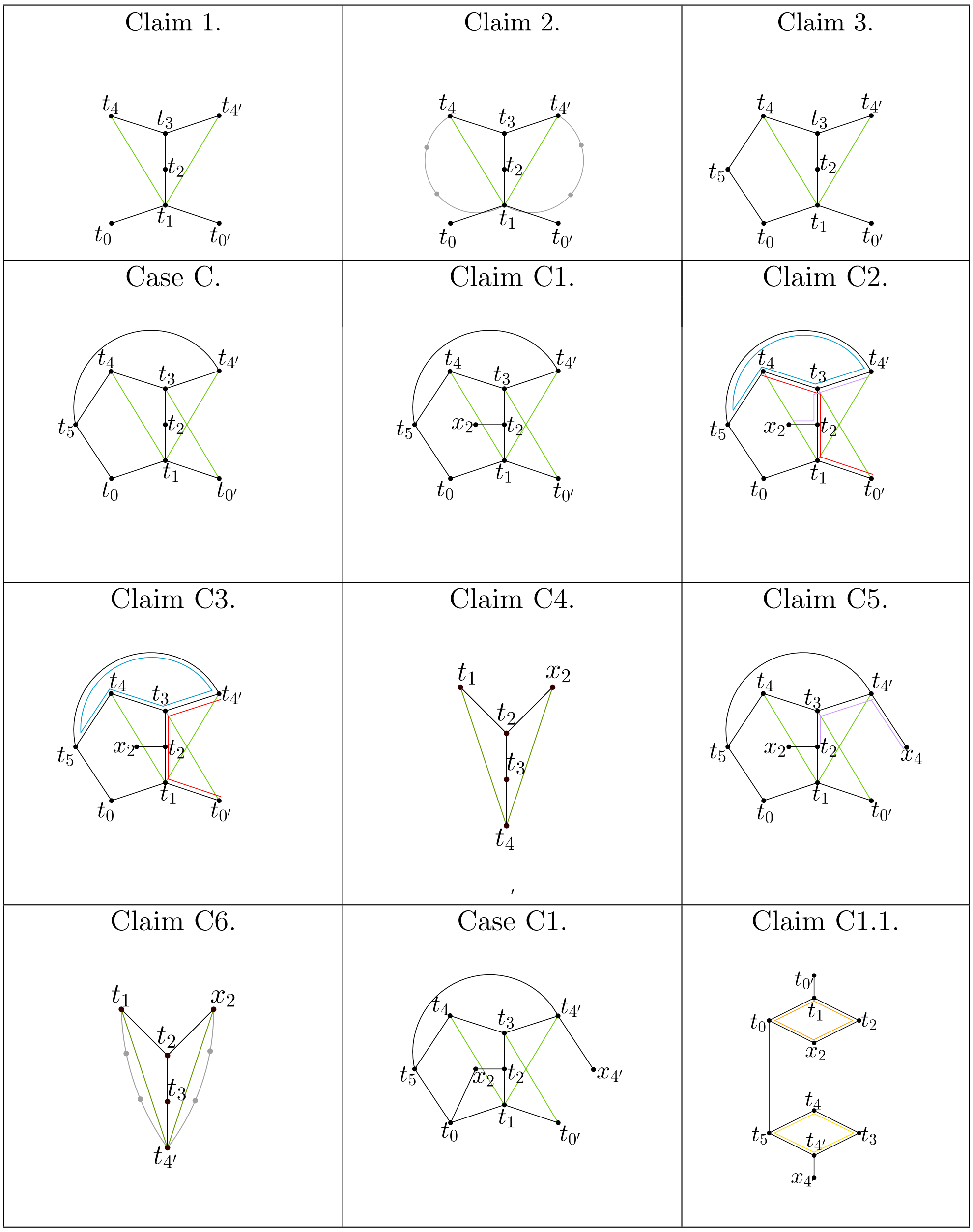
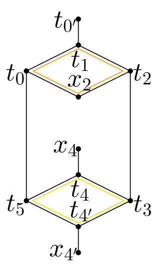
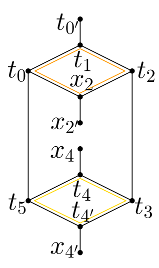


Table 3. Theorem 15. Case C

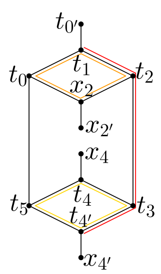
Claim C1.2



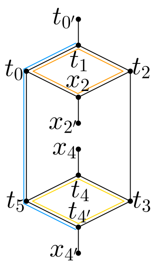
Claim C1.3



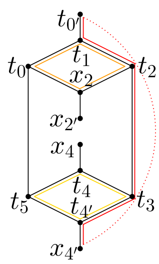
Claim C1.4



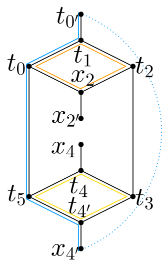
Claim C1.4



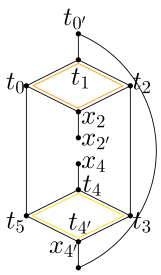
Claim C1.5



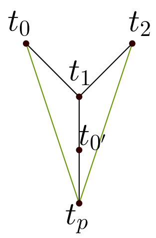
Claim C1.5



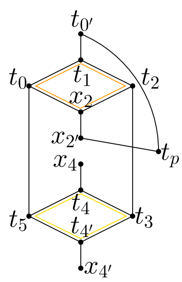
Claim C1.6



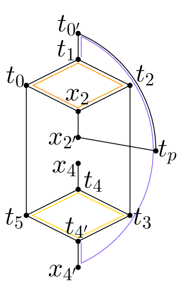
Claim C1.6.1



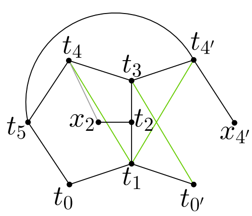
Claim C1.6.2



Claim C1.6.3



Case C2



Case C3.

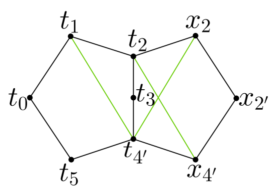


Table 4. Theorem 15. Case C

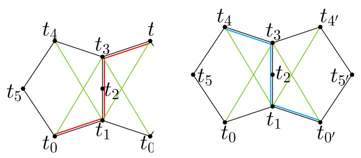
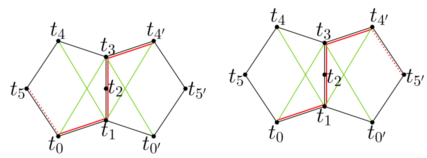
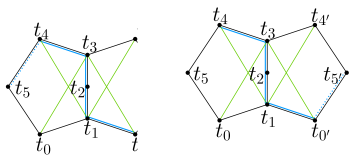
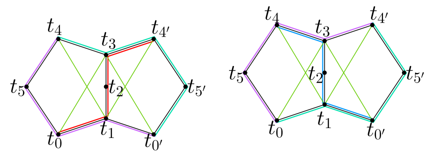
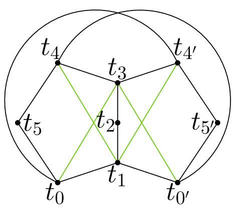
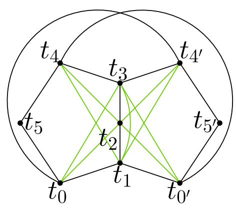
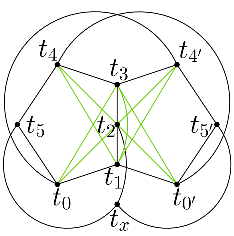
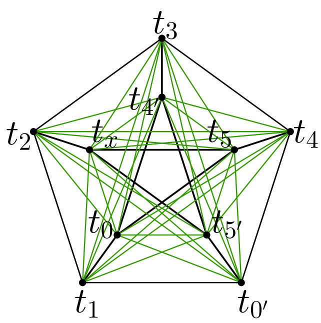
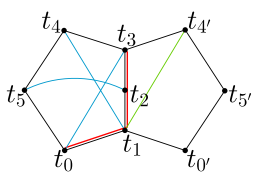
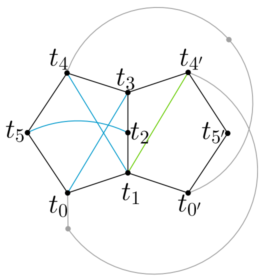
Claim 1. 	Claim 2. 
Claim 3. 	Claim 4. 
Claim 5. 	Claim 6. 
Claim 7. 	Claim 8. 

Table 5. Theorem 16

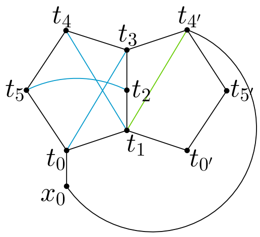
Claim 1.



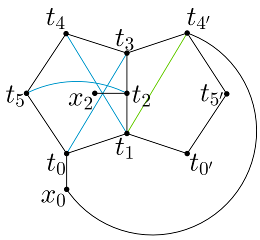
Claim 2.



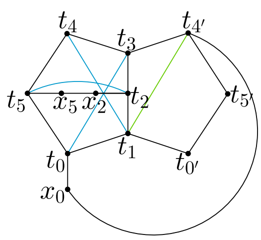
Claim 3.



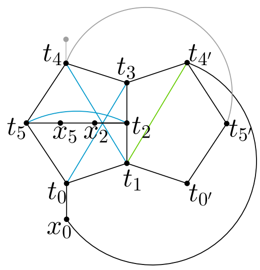
Claim 4.



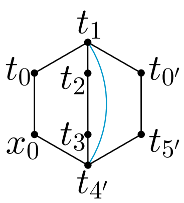
Claim 5.



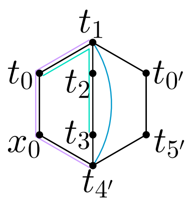
Claim 6.



Claim 7.



Claim 8.



Claim 9.

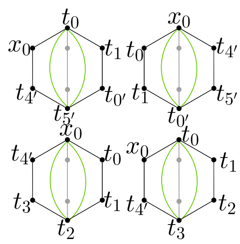


Table 6. Theorem 17

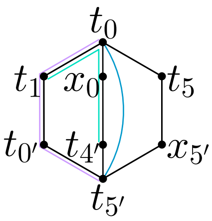
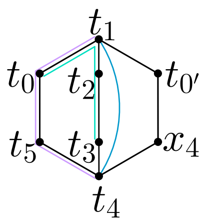
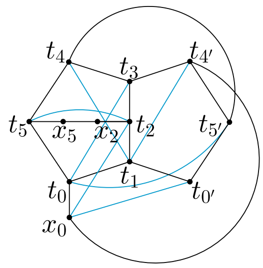
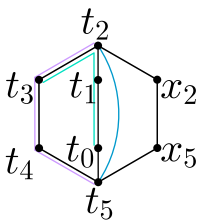
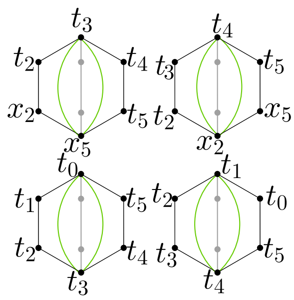
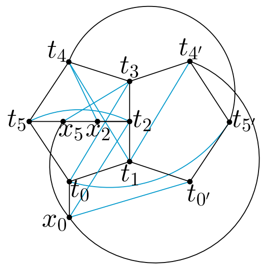
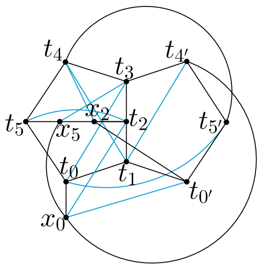
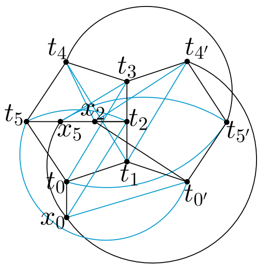
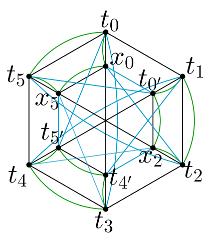
Claim 10.	Claim 11.	Claim 12.
		
Claim 13.	Claim 14.	Claim 15.
		
Claim 16.	Claim 17.	Claim 18.
		

Table 7. Theorem 17