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DOMINATION GAME: EFFECT OF EDGE CONTRACTION AND EDGE SUBDIVISION

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Abstract

In this paper the behavior of the game domination number $\gamma_g(G)$ and the Staller start game domination number $\gamma_g'(G)$ by the contraction of an edge and the subdivision of an edge are investigated. Here we prove that contracting an edge can decrease $\gamma_g(G)$ and $\gamma_g'(G)$ by at most two, whereas subdividing an edge can increase these parameters by at most two. In the case of no-minus graphs it is proved that subdividing an edge can increase both these parameters by at most one but on the other hand contracting an edge can decrease these by two. All possible values of these parameters are also analysed here.

Keywords: domination game, edge contraction, edge subdivision.

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1. Introduction

The domination game was introduced in [5] by Brešar, Klavžar and Rall. A vertex dominates itself and its neighbors. In the domination game, two players, Dominator and Staller, alternate turns choosing a vertex in a finite, undirected graph G, and adding it to a set of vertices S. The rule of the game is that whenever a player chooses a vertex to add to S, the vertex must dominate at least one vertex, which is not yet dominated by the vertices of S. The game ends when S is a dominating set of the graph and the cardinality of S is called the *score* of the game. The two players have conflicting goals — Dominator tries to minimize the final score while Staller tries to maximize it.

Two graph parameters associated with this game were introduced in [5]. Assuming both players play optimally, the game domination number $\gamma_g(G)$ is the score of the game on G when Dominator starts (D game), and the Staller start game domination number $\gamma_g'(G)$ is the score when Staller starts (S game). Both parameters are studied in parallel since many results hold for both of them.

In terms of the order of the graph, Kinnersley, West and Zamani [14] conjectured that a general upper bound for $\gamma_g(G)$ is $\frac{3}{5}$ of its order if G is an isolate free graph. Bujtás [8, 9, 10] developed an innovative discharging-like method to attack this conjecture. The conjecture was confirmed by Henning and Kinnersley on the class of graphs with minimum degree at least two [12]. Schmidt [16] determined the largest known class of trees for which the conjecture holds.

It is known that the difference between $\gamma_g(G)$ and $\gamma_g'(G)$ is at most one, and that it can occur in both directions. The consequences of an edge removal in a graph were considered in [7] and it is proved that $\gamma_g(G)$ and $\gamma_g'(G)$ can either increase or decrease by at most two.

This motivated us to study any operation which gives a monotone behavior of these parameters in the sense that these parameters may either increase or decrease. We succeeded in showing that edge contraction as well as edge subdivision preserves the monotone behavior for both these parameters. The order of a graph G will be denoted by $n(G)$. The open neighborhood $N_G(x) = \{y : \ xy \in E(G)\}$ and the closed neighborhood $N_G[x] = N_G(x) \cup \{x\}$ will be abbreviated as $N(x)$ and $N[x]$ when G will be clear from the context. Two vertices u and v are true twins if $N[u] = N[v]$.

The sequence of moves in a D game will be denoted with $d_1, s_1, d_2, s_2, \ldots$, and the sequence of moves in an S game with $s_1', d_1', s_2', d_2', \ldots$. A partially dominated graph is a graph together with a declaration that some vertices are already dominated. Let $G|S$ denote the partially dominated graph where the vertices of S are considered already dominated. We get the residual graph from $G|S$ by removing all edges between dominated vertices, and all vertices v with $N[v] \subseteq S$. We have the following result.

Theorem 1 (The Continuation Principle [14])**.** *Let G be a graph and $A, B \subseteq V(G)$. If $B \subseteq A$ then $\gamma_g(G|A) \le \gamma_g(G|B)$ and $\gamma_g'(G|A) \le \gamma_g'(G|B)$.*

This result together with earlier observations [5] on the problem allowed one to deduce that the game domination number and the Staller start game domination number may differ by at most one.

Theorem 2 [5, 14]**.** *For any graph G and a subset S of vertices,* $|\gamma_g(G|S) - \gamma'_g(G|S)| \leq 1$.

The Staller-pass game [5]**.** The Staller-pass game is the variant of the domination game in which at some point in the game, instead of picking a vertex, Staller may decide to pass, and it is Dominator's turn again. Denote the size of the final dominating set in the Staller-pass game on G, under optimal play, by $\gamma_g^{sp}(G)$.

Theorem 3 [15]**.** *If* $n \geq 1$*, then*

$$\gamma_g(P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil - 1; & n \equiv 3 \pmod 4, \\ \left\lceil \frac{n}{2} \right\rceil; & \textit{otherwise} \end{cases}$$

and

$$\gamma'_g(P_n) = \left\lceil \frac{n}{2} \right\rceil .$$

A leaf of a graph G is a vertex of degree one and a support vertex of G is a vertex adjacent to a leaf. A vertex of G is a universal vertex if it is adjacent to all vertices of G other than itself. An edge contraction in a graph G is an operation which removes an edge e from G while simultaneously merging the two vertices that it previously joined and is denoted by $G.e$. Subdividing an edge e in a graph G is an operation which consists of removing e and adding a new vertex adjacent to both extremities of e, forming a path of length two connecting the previously adjacent vertices and is denoted by $G \odot e$.

2. Edge Contraction and Edge Subdivision

We first prove the following bounds on the game domination number of the graph obtained by edge contraction.

Theorem 4. *Let G be a graph and $e \in E(G)$. If $G.e$ is the graph obtained from G by contracting the edge e, then*

$$\gamma_g(G) - 2 \leq \gamma_g(G.e) \leq \gamma_g(G),$$
$$\gamma'_g(G) - 2 \leq \gamma'_g(G.e) \leq \gamma'_g(G).$$

Proof. Let G be a graph, and $e = uv$ be an edge in G. In $G.e$, we denote by w the new vertex obtained by the identification of u and v.

We first prove the upper bounds by describing a strategy for Dominator. We use the imagination strategy, as used in [5]. During the course of the game on $G.e$, Dominator imagines another game played on G. Every time in his turn to play, he plays an optimal move in the imagined game and copies this move to the real game. Then he copies Staller's answer in the real game to his imagined game. In addition, in the imagined game, Dominator may consider some extra vertices dominated during the course of the game, and adapt his strategy. Note that by the Continuation Principle, considering more vertices dominated in the imagined game cannot make the imagined game last longer. If Staller plays the vertex w, then Dominator will consider that she played the vertex u in G and also adds the neighbourhood of v to the set of dominated vertices. Similarly if Dominator is supposed to copy to the real game a move in the imagined game on the vertex u or v, then Dominator plays to w and adds both the neighbourhoods of u and v to the set of dominated vertices in the imagined game. Finally, if any player moves on a neighbour of w in $G.e$, Dominator will assume that both u and v get dominated in the imagined game. We know that Dominator, and possibly not Staller, is playing optimally in the imagined game. This guarantees that the imagined game in G should last no longer than $\gamma_g(G)$ for the D game or than $\gamma_g'(G)$ for the S game. Moreover, at each stage of the game, if S is the set of dominated vertices in the real game, either S does not contain w or the set of dominated vertices in the imagined game is precisely $(S \setminus \{w\}) \cup \{u, v\}$. Eventually, when the imagined game is over, the real game is also finished and Dominator ensured that the number of moves in the real game was no more than the number of moves in the imagined game. Staller, and possibly not Dominator, is playing optimally in the real game on $G.e$. This implies that the total number of moves made in the real game is at least $\gamma_g(G)$ for the D game and at least $\gamma_g'(G)$ for the S game. Dominator, and possibly not Staller is playing optimally in the imagined game on G. This implies that the total number of moves made in the imagined game is at most $\gamma_g(G.e)$ for the D game and at most $\gamma_g'(G.e)$ for the S game, thus proving the upper bound.

Now we prove that $\gamma_g(G) - 2 \leq \gamma_g(G.e)$. Consider a real Dominator start game played on G and Dominator imagines another Dominator start game played on $G.e$. Again, Dominator copies every move of Staller in the real game except u and v to the imagined game and copies back his optimal response in the imagined game except w to the real game on G. Every move of Dominator in the imagined game except w is a legal move in the real game. Suppose at some stage Dominator chooses w in the imagined game, then he chooses either u or v in the real game instead of copying w. Clearly one of u or v is a legal move in G. If Staller plays either u or v in the real game then Dominator plays w for Staller in the imagined game when it is a legal move. Assume first that every move of Staller in the real game is also a legal move in the imagined game. There may be vertices remaining undominated in the real game when the imagined game is finished. If neither Dominator nor Staller played u or v, then the only undominated vertices must be one among $\{u, v\}$. Otherwise, all the undominated vertices must be included either in $N(v)$ or in $N(u)$. In both cases the real game can be finished by playing either u or v depending on the game. Thus the game finishes in at most two more moves.

Assume now that the k^{th} move of Staller is not a legal move in the imagined game. Again, the only vertices that may be dominated in the imagined game but not in the real game are vertices from $N[u] \cup N[v]$. More precisely, if any of u or v was played in the real game then these vertices are contained in $N(v)$ or in $N(u)$ respectively, otherwise only u or v may be such a vertex. In any case Dominator plays any legal move x in the real game. Let S be the set of vertices dominated in the real game on G after the $(k+2)^{th}$ move and let S' be the set of vertices dominated in the imagined game after the k^{th} move. Defining $S'' = S' \cup N[x]$ by adding the newly dominated vertices in $N[x]$ to the set S' of dominated vertices in the imagined game after the k^{th} move. By the Continuation Principle, we get that $\gamma_g'(G.e|S'') \leq \gamma_g'(G.e|S')$. The residual graph from $G|S$ and the residual graph from $G.e|S''$ are isomorphic. Staller, and possibly not Dominator, is playing optimally in the real game on G. We then have

$$\gamma_g(G) \leq k + 2 + \gamma_g'(G|S) = k + 2 + \gamma_g'(G.e|S'') \leq k + 2 + \gamma_g'(G.e|S').$$

Also Dominator, and possibly not Staller, is playing optimally in the imagined game on $G.e$. Thus we get

$$k + \gamma_g'(G.e|S') \leq \gamma_g(G.e).$$

Therefore $\gamma_g(G) \leq \gamma_g(G.e) + 2$.

The same argument also holds for Staller start game domination number and hence the bounds proposed for the S game can be proved similarly. ∎

Now we consider the case of edge subdivision. Since in $G \odot e$, for any edge e' incident to the added degree 2 vertex, $(G \odot e).e'$ is the initial graph G, we get the following as a corollary of Theorem 4.

Corollary 5. *Let G be a graph and $e \in E(G)$. The graph $G \odot e$ obtained from G by subdividing the edge e satisfies*

$$\begin{aligned} \gamma_g(G) &\leq \gamma_g(G \odot e) \leq \gamma_g(G) + 2, \\ \gamma_g'(G) &\leq \gamma_g'(G \odot e) \leq \gamma_g'(G) + 2. \end{aligned}$$

3. No-Minus Graphs

In [11], a special family of graphs was introduced, called *no-minus* graphs. A graph G is no-minus if for any subset of vertices $S \subseteq V$, $\gamma_g(G|S) \leq \gamma_g'(G|S)$. In a no-minus graph it is of no advantage for either player to pass a move. It is known already that forests [14], tri-split graphs and dually chordal graphs [11] are no-minus graphs. We have just proved that $0 \leq \gamma_g(G) - \gamma_g(G.e) \leq 2$ and $0 \leq \gamma'_g(G) - \gamma'_g(G.e) \leq 2$. We shall now describe no-minus graphs, especially trees, attaining all possible values for these differences.

3.1. Edge contraction

Proposition 6. *For any $l \geq 3$ there exists a connected no-minus graph G with an edge e such that $\gamma_g(G) = l$ and $\gamma_g(G.e) = l - 2$.*

Proof. For $l \geq 3$, we construct the following family of connected no-minus graphs denoted by $G_l, l \geq 0$. Let G_0 be the graph constructed in the following way. Take two copies of $K_{1,2}$ and label their centre vertices as x and y. Join x and y by the edge e. For $l \geq 1$, the graph G_l is obtained from G_0 by identifying the end vertices of l copies of P_3 with x. See Figure 1. We claim that $\gamma_g(G_l) = l + 3$ and $\gamma_g(G_l.e) = l + 1$. Note that if Dominator plays his first move on x, then only $l + 2$ vertices remain undominated which yields $\gamma_g(G_l) \leq l + 3$.

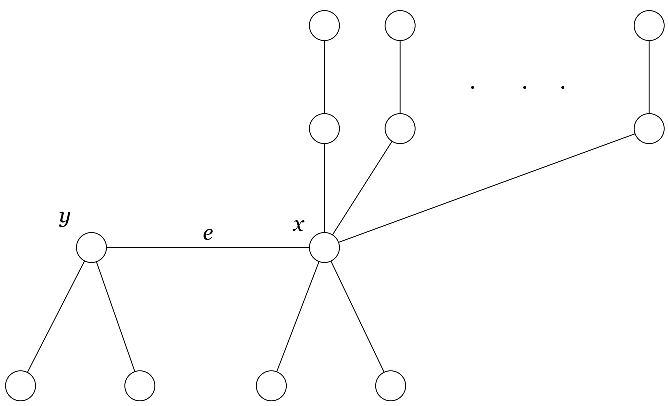

Figure 1. The graph G_l.

Now we present a strategy for Staller which ensures that at least $l + 3$ moves are needed to finish the game on G_l. If Dominator starts by playing on x, then Staller selects a leaf adjacent to y. The resulting residual graph at this stage is a partially dominated graph consisting of $l + 1$ copies of K_2 and hence at least $l + 1$ more moves needed to finish the game. Therefore a total of $l + 3$ moves will be played. Otherwise, if Dominator does not start by playing on x then Staller responds by playing on a leaf adjacent to x. Note that all vertices of G_l are either leaves or support vertices and the number of support vertices in G_l is $l + 2$. By the Continuation Principle, Dominator prefers to select support vertices to leaves. The number of support vertices of the residual graph G'_l after the first two moves of G_l is $l+1$. Clearly $\gamma_g(G_l) \geq 2 + \gamma_g(G'_l)$ and $\gamma_g(G'_l) \geq \gamma(G'_l) \geq l+1$ (using the fact that the domination number of a graph is at least the number of support vertices of that graph). Thus $\gamma_g(G_l) \geq l+3$, and we get that $\gamma_g(G_l) = l+3$.

Let $G_l.e$ be the graph obtained from G_l by contracting the edge e in G_l. Here x and y are identified by a new vertex, say w. If Dominator selects the vertex w then only l vertices remain undominated which yields $\gamma_g(G_l.e) \leq l+1$. There are $l+1$ support vertices in $G_l.e$ and hence $\gamma_g(G_l.e) \geq \gamma(G_l.e) \geq l+1$. Thus we get that $\gamma_g(G_l.e) = l+1$. ■

Proposition 7. *For any $l \geq 2$ there exists a connected no-minus graph G with an edge e such that $\gamma_g(G) = l$ and $\gamma_g(G.e) = l-1$.*

Proof. For $l \geq 2$, we construct the graph H_l from a star $K_{1,l}$ by subdividing each edge except one. We claim that $\gamma_g(H_l) = l$. Note that if Dominator plays his first move on the centre vertex, then only $l-1$ vertices remain undominated which yields $\gamma_g(H_l) \leq l$. On the other hand the number of support vertices in H_l is l and $\gamma_g(H_l) \geq \gamma(H_l) \geq l$. Consider the graph $H_l.e$ where e is an edge not incident to the centre of H_l. We claim that $\gamma_g(H_l.e) \leq l-1$. If Dominator plays his first move on the centre vertex of $H_l.e$, then only $l-2$ vertices remain undominated which yields $\gamma_g(H_l.e) \leq l-1$.

It is clear that $H_l.e$ has $l-1$ support vertices. Hence $\gamma_g(H_l.e) \geq \gamma(H_l.e) \geq l-1$. Thus $\gamma_g(H_l.e) = l-1$. ■

Proposition 8. *For any $l \geq 1$ there exists a connected no-minus graph G with an edge e such that $\gamma_g(G) = l$ and $\gamma_g(G.e) = l$.*

Proof. For $l \geq 1$, construct the graph F_l from a star $K_{1,l+1}$ by subdividing each edge except two. Clearly F_l has $(l+1)-2+1 = l$ support vertices. Hence $\gamma_g(F_l) \geq \gamma(F_l) \geq l$. On the other hand if Dominator plays his first move on the centre vertex then only $l+1-2$ vertices remain undominated in F_l. So $\gamma_g(F_l) \leq 1+l+1-2 = l$ and thus $\gamma_g(F_l) = l$.

The graph $F_l.e$ is obtained from F_l by contracting an edge e incident with the centre and a leaf. By a similar argument we can prove that $\gamma_g(F_l.e) = l$. ■

Proposition 9. *There is no graph G with an edge e such that $\gamma'_g(G) = 3$ and $\gamma'_g(G.e) = 1$.*

Proof. We know that $\gamma'_g(G) = 1$ if and only if G is complete. Assume that G is a graph with $\gamma'_g(G) = 3$ and hence G has at least two non adjacent vertices say x and y. Clearly x and y are non adjacent in $G.e$ for any edge e of G. Therefore $\gamma'_g(G.e) \geq 2$ and we conclude that there is no graph G with an edge e such that $\gamma'_g(G) = 3$ and $\gamma'_g(G.e) = 1$. ■

Proposition 10. *For any $l \geq 4$ there exists a connected no-minus graph G with an edge e such that $\gamma_g'(G) = l$ and $\gamma_g'(G.e) = l - 2$.*

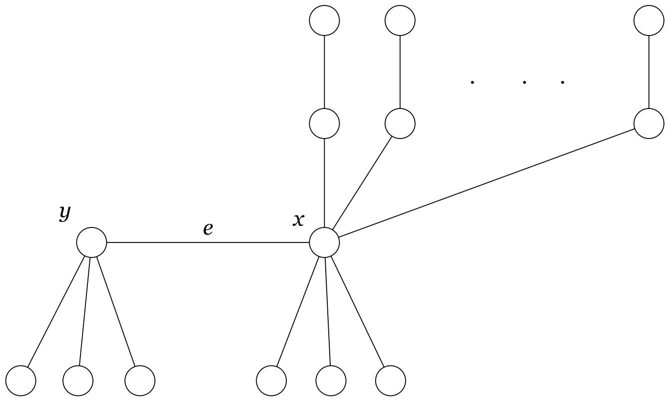

Figure 2. The graph S_k.

Proof. For $l \geq 4$, we construct the following family of connected no-minus graphs denoted by S_k, $k = l - 4 \geq 0$. Let S_0 be the graph constructed in the following way. Take two copies of $K_{1,3}$ and label their centre vertices as x and y. Join x and y by the edge e. For $k \geq 1$ the graph S_k is obtained from S_0 by identifying the end vertices of k copies of P_3 with x, see Figure 2. We claim that $\gamma_g'(S_k) = k+4$ and $\gamma_g'(S_k.e) = k + 2$. By Theorem 4 it suffices to show that $\gamma_g'(S_k) \geq k + 4$ and $\gamma_g'(S_k.e) \leq k + 2$. First we show that $\gamma_g'(S_k) \geq k + 4$ by presenting a strategy for Staller which ensures that the game ends with at least $k + 4$ moves. Staller first plays a leaf adjacent to x and we know that all vertices of S_k are either support vertices or leaves. Dominator prefers to select a support vertex to a leaf. If Dominator plays a support vertex other than x, then Staller chooses another leaf adjacent to x otherwise Staller chooses a leaf adjacent to y. Let S_k' be the residual graph after these three moves. We know that the number of support vertices of S_k is $k+2$ and the number of support vertices of S_k' is $k+1$. Therefore $\gamma_g(S_k') \geq \gamma(S_k') \geq k + 1$. Thus $\gamma_g'(S_k) = 3 + \gamma_g(S_k') \geq 3 + k + 1 = k + 4$.

Let $S_k.e$ is the graph obtained from S_k by contracting the edge $e = xy$ and let w be the new vertex due to the contraction. We show that $\gamma_g'(S_k.e) \leq k+2$ by presenting a strategy for Dominator which ensures that at most $k + 2$ moves are needed to finish the game. Staller's first move is a leaf on $S_k.e$. Now Dominator plays w as his next move and the number of vertices remain undominated in $S_k.e$ is at most k. Hence $\gamma_g'(S_k.e) \leq 2 + k$ and we conclude that $\gamma_g'(S_k) = k + 4$ and $\gamma_g'(S_k.e) = k + 2$. ■

Proposition 11. *For any $l \geq 2$ there exists a connected no-minus graph G with an edge e such that $\gamma'_g(G) = l$ and $\gamma'_g(G.e) = l - 1$.*

Proof. For the general case $l \geq 2$, consider the graph P_{2l-1}. It is known that $\gamma'_g(P_{2l-1}) = l$ and $\gamma'_g(P_{2l-1}.e) = \gamma'_g(P_{2l-2}) = l - 1$ for any edge e. ■

Proposition 12. *For any $l \geq 1$ there exists a connected no-minus graph G with an edge e such that $\gamma'_g(G) = l$ and $\gamma'_g(G.e) = l$.*

Proof. For the general case $l \geq 1$, consider the graph P_{2l}. It is known that $\gamma'_g(P_{2l}) = l$ and $\gamma'_g(P_{2l}.e) = \gamma'_g(P_{2l-1}) = l$ for any edge e. ■

3.2. Edge subdivision

By Corollary 5 we have $\gamma_g(G \odot e) - \gamma_g(G) \leq 2$. We here prove that the result can be strengthened in the case of no-minus graphs, as follows.

Theorem 13. *Let G be a no-minus graph and $e \in E(G)$. The graph $G \odot e$ obtained from G by subdividing the edge e satisfies*

$$\gamma_g(G) \leq \gamma_g(G \odot e) \leq \gamma_g(G) + 1,$$
$$\gamma'_g(G) \leq \gamma'_g(G \odot e) \leq \gamma'_g(G) + 1.$$

Proof. By Corollary 5, we know that for any graph G, $\gamma_g(G) \leq \gamma_g(G \odot e)$ and $\gamma'_g(G) \leq \gamma'_g(G \odot e)$. So this is true in the case of no-minus graphs.

Now we prove that $\gamma_g(G \odot e) \leq \gamma_g(G) + 1$. Let uv be the subdivided edge and w be the vertex added in the subdivision. Consider a real Dominator start game played on $G \odot e$. At the same time Dominator imagines another Dominator start Staller-pass game played on G. Dominator copies every move of Staller in the real game except w to the imagined game and copies back his optimal response. Every move of Dominator in the imagined game is legal in the real game. If every move of Staller in the real game is also legal in the imagined game then only one vertex may remain undominated in $G \odot e$ at the end of the game, and it is either u or v or w. Thus the real game is finished within at most one move more than in the imagined game. Suppose at the k^{th} stage Staller chooses a vertex in the real game that is not a legal move in the imagined game. This is possible only if that move additionally dominates either w itself in the real game or u or v itself and u and v are already dominated in the imagined game. Let S be the set of vertices dominated in the real game after the k^{th} move and S' be the set of vertices dominated in the imagined game after the $(k-1)^{th}$ move. Clearly $S' = S - w$ and the residual graph after the k^{th} move in the real game is isomorphic with the residual graph after the $(k-1)^{th}$ move in the imagined game. Staller, but possibly not Dominator, is playing optimally in the real game and this implies

$$\gamma_g(G \odot e) \le k + \gamma_g(G \odot e|S) = k + \gamma_g(G|S').$$

Dominator, but possibly not Staller, is playing optimally in the imagined game and the k^{th} move is of Staller. Staller skips that move in the imagined game and for a no-minus graph G we have in [11] that $\gamma_g^{sp}(G) = \gamma_g(G)$. Thus

$$k - 1 + \gamma_g(G|S') \le \gamma_g^{sp}(G) = \gamma_g(G).$$

Therefore

$$\gamma_g(G \odot e) \le k + \gamma_g(G|S') = k - 1 + \gamma_g(G|S') + 1 \le \gamma_g(G) + 1.$$

This concludes the proof.

The same argument also holds for Staller start game domination number. ■

Proposition 14. *For any $l \ge 1$, and $l \ne 2$, there is a connected no-minus graph G with an edge e such that $\gamma_g(G) = l$ and $\gamma_g(G \odot e) = l$.*

Proof. For the case $l = 1$, consider the graph $G = K_2$. It is clear that $\gamma_g(G \odot e) = 1$ and $\gamma_g(G) = 1$.

For the general case when $l \ge 3$, consider the graph G_{l-3} in Proposition 6 and it is known that $\gamma_g(G_{l-3}) = l$, $l \ge 3$. The edge e as the edge joining the vertices x and y in the graph G_{l-3}. Consider the graph $G_{l-3} \odot e$ and claim that $\gamma_g(G_{l-3} \odot e) = l$. We present a strategy for Dominator which yields $\gamma_g(G_{l-3} \odot e) \le l$. Dominator selects his first move as x and by the Continuation Principle y is not an optimal first move of Staller because leaves are adjacent to y. So Dominator selects y after the first move of Staller and there are at most $l-3$ vertices remain undominated. Therefore $\gamma_g(G_{l-3} \odot e) \le 3 + l - 3 = l$. By Theorem 13 we have $l = \gamma_g(G_{l-3}) \le \gamma_g(G_{l-3} \odot e)$. Thus $\gamma_g(G_{l-3} \odot e) = l$. ■

Proposition 15. *There is no connected graph G with $\gamma_g(G) = 2$ and $\gamma_g(G \odot e) = 2$.*

Proof. Let G be a connected graph with $\gamma_g(G) = 2$. So there exists a vertex $v \in V(G)$ such that $V(G) - N[v]$ is a clique and G has no universal vertex. Let $G \odot e$ be the graph obtained from G by subdividing any edge $e = uv$ to $u - w - v$. Suppose that if Dominator first chooses the vertex w, then either u or v or both vertices in $G \odot e$ are legal moves for Staller and Staller selects one of this move. So after the move of Staller there are undominated vertices in $G \odot e$ because G has no universal vertex. Therefore assume that an optimal move of Dominator is a vertex x other than w in $G \odot e$. If $x \ne u, v$ then x is not adjacent to w and at least one more vertex say y other than w of $G \odot e$. This is because of G has no universal vertex. Now either y and w are adjacent in $G \odot e$ or not. If y and w are not adjacent then Staller chooses y and after this move there are undominated vertices in $G \odot e$. Also if y and w are adjacent then Staller chooses a vertex which is adjacent to y and not to w. So after this move of Staller in $G \odot e$ there are undominated vertices in $G \odot e$. On the other hand if $x = u$ or $x = v$ then Staller chooses w as next move. Here also $G \odot e$ has undominated vertices after the first two moves because G has no universal vertex. So it concludes that $G \odot e$ has at least 3 moves for a D game and hence $\gamma_g(G \odot e) \geq 3$ and hence there is no connected graph G with $\gamma_g(G) = 2$ and $\gamma_g(G \odot e) = 2$. ∎

Note 16. For the case $l = 2$, there is a no-minus graph G with $\gamma_g(G) = 2$ and $\gamma_g(G \odot e) = 2$. Consider the graph $G = K_2 \cup K_2$. It is clear that $\gamma_g(G) = 2$ and $\gamma_g(G \odot e) = 2$ for any edge e of G.

Proposition 17. *For any $l \geq 1$, there is a connected no-minus graph G with an edge e such that $\gamma_g(G) = l$ and $\gamma_g(G \odot e) = l + 1$.*

Proof. For $l \geq 1$, construct the graph G from a star $K_{1,l}$ by subdividing each edge except one. Clearly $\gamma_g(G) = l$ and let $G \odot e$ be the graph obtained from G by subdividing the remaining edge and we get $\gamma_g(G \odot e) = l + 1$. ∎

Note 18. It is obvious that a graph G is complete if and only if $\gamma'_g(G) = 1$. So there is no graph G with $\gamma'_g(G) = \gamma'_g(G \odot e) = 1$.

Proposition 19. *For any $l \geq 2$, there is a connected no-minus graph G with an edge e such that $\gamma'_g(G) = \gamma'_g(G \odot e) = l$.*

Proof. For the general case $l \geq 2$, consider the graph $G = P_{2l-1}$. It is known that $\gamma'_g(P_{2l-1}) = l$. We know that $P_{2l-1} \odot e = P_{2l}$ for any edge e of P_{2l-1} and hence $\gamma'_g(P_{2l}) = \gamma'_g(P_{2l-1} \odot e) = l$. ∎

Proposition 20. *For $l \geq 1$, there is a connected no-minus graph G with an edge e such that $\gamma'_g(G) = l$ and $\gamma'_g(G \odot e) = l + 1$.*

Proof. For the general case $l \geq 2$, consider the graph $G = P_{2l}$. It is known that $\gamma'_g(P_{2l}) = l$. We know that $P_{2l} \odot e = P_{2l+1}$ for any edge e of P_{2l} and hence $\gamma'_g(P_{2l+1}) = \gamma'_g(P_{2l} \odot e) = l + 1$. ∎

4. Edge Subdivision in the General Case

For all no-minus graphs we have $0 \leq \gamma_g(G \odot e) - \gamma_g(G) \leq 1$ and $0 \leq \gamma'_g(G \odot e) - \gamma'_g(G) \leq 1$ by Theorem 13 but in general $0 \leq \gamma_g(G \odot e) - \gamma_g(G) \leq 2$ and $0 \leq \gamma'_g(G \odot e) - \gamma'_g(G) \leq 2$. Here we discuss all possibilities of graphs realizing $\gamma_g(G \odot e) = \gamma_g(G) + 2$ and $\gamma'_g(G \odot e) = \gamma'_g(G) + 2$. Note that all graphs with $\gamma_g(G) \leq 2$ are no-minus graphs. Hence in the following we consider only $l \geq 3$.

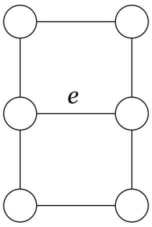

Figure 3. The domino graph D.

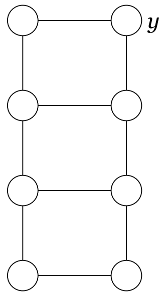

Figure 4. The graph F.

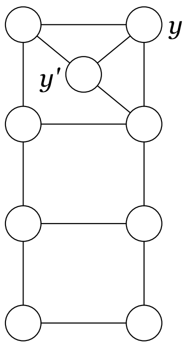

Figure 5. The graph F'.

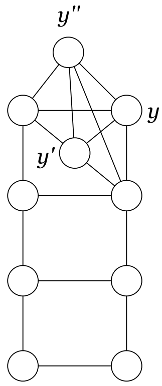

Figure 6. The graph F''.

Proposition 21. *For any $l \geq 3$ there is a connected graph G with an edge e such that $\gamma_g(G) = l$ and $\gamma_g(G \odot e) = l + 2$.*

Proof. We present two families of graphs U_k and V_k that realize odd and even values of l, respectively.

Construct U_0 in the following way. Take the disjoint union of C_6 and $K_{1,2}$ having x as its centre. We get U_0 by connecting x with one of the vertices of C_6 say y. The graph U_k, $k \geq 1$ is obtained from U_0 by identifying one end vertex of $2k$ copies of P_3 with x. We set e to be the edge between x and y, see Figure 7.

We claim that $\gamma_g(U_k) = 2k + 3$ and $\gamma_g(U_k \odot e) = 2k + 5$, for all $k \geq 0$. By Corollary 5, it suffices to show that $\gamma_g(U_k) \leq 2k + 3$ and $\gamma_g(U_k \odot e) \geq 2k + 5$. First we prove $\gamma_g(U_k) \leq 2k + 3$ by presenting a strategy for Dominator which ensures that the game ends with at most $2k + 3$ moves. Dominator starts the

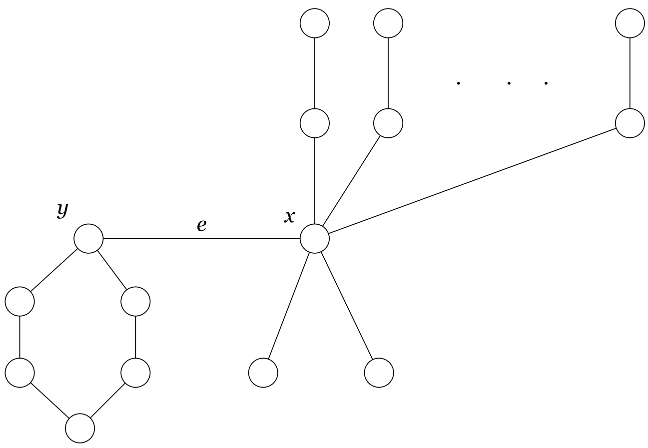

Figure 7. The graph U_k.

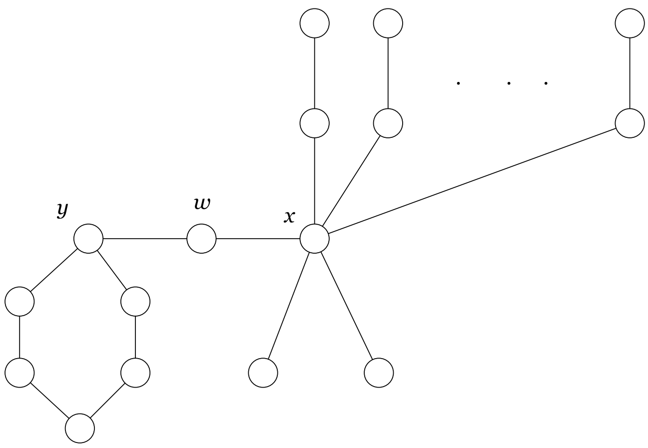

Figure 8. The graph $U_k \odot e$.

game by playing the vertex x. Any move of Staller on one of the $2k$ attached paths is followed by a move of Dominator on some other path in the same 2k attached paths, so that all vertices of this $2k$ paths are dominated. Therefore Staller is forced to be the first to play in the subgraph C_6 and it is known that $\gamma'_g(C_6|y) = 2$. Hence Dominator can ensure that at most 1 + 2k + 2 = 2k + 3 moves are needed to finish the game. Thus we get $\gamma_g(U_k) \leq 2k + 3$.

Now we show that $\gamma_g(U_k \odot e) \geq 2k + 5$ by presenting a strategy for Staller which ensures that at least $2k+5$ moves are needed to finish the game. We set w as the new vertex obtained due to the subdivision of the edge e which is adjacent to x and y. Whenever Dominator plays on one of the $2k$ attached paths then Staller

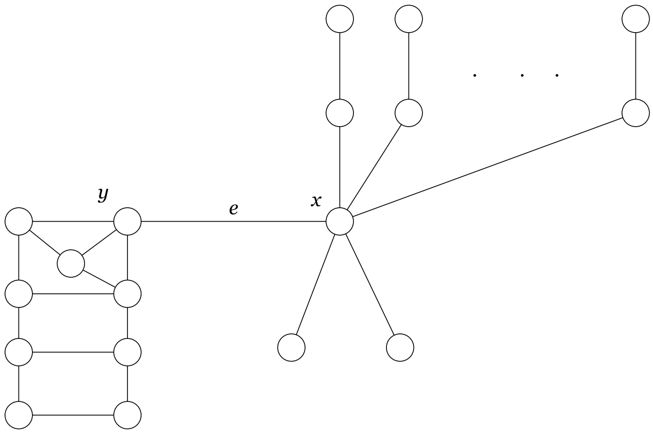

Figure 9. The graph V_k.

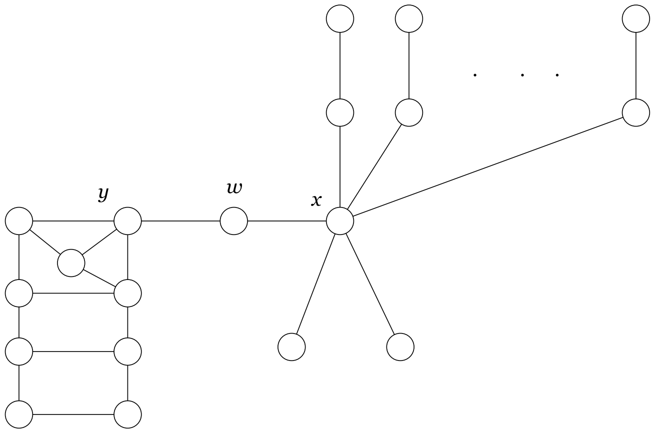

Figure 10. The graph $V_k \odot e$.

follows a move on some other path in the $2k$ attached paths. If Dominator plays on x, then Staller responds by playing on w. On the other hand if Dominator plays on w then Staller selects a leaf adjacent to x. (This is a legal move because x is not selected and the leaves which are adjacent to x are not dominated yet.) By this strategy Staller forces Dominator to be the first to play in the subgraph C_6 and it is known that $\gamma_g(C_6|y) = 3$. On the other hand if Dominator starts to play a vertex in C_6 then Staller selects a vertex adjacent to the vertex selected by Dominator in C_6 and two more vertices remain undominated in C_6. Now there are two possibilities: either Dominator selects a vertex in C_6 that dominates the remaining undominated vertices in C_6 or selects a vertex from $\{x, w\}$. In the first case Staller responds by playing on w and the game is finished with $2k+1$ moves

($2k$ moves in the attached paths and one for x). In the other case Staller selects a vertex in C_6 which dominates only one new vertex. So one more move is needed to dominate all vertices in C_6 and $2k$ moves are needed in the $2k$ attached paths. Thus in any case there is at least $2k+5$ moves needed to finish the game and concludes the proof.

The family V_k realizes the case when l is even. Construct V_0 in the following way. Take disjoint union of the graph F' in Figure 5 and $K_{1,2}$ having x as its centre. We get V_0 by connecting the vertex y of F' with x. The graph V_k, $k \geq 1$, is obtained from V_0 by identifying one end vertex of $2k$ copies of P_3 with x. We set e to be the edge between x and y. By using a similar argument as in the previous case we get $\gamma_g(V_k) \leq 2k+4$ and $\gamma_g(V_k \odot e) \geq 2k+6$. ■

Proposition 22. *For any $l \geq 2$ there is a connected graph G with an edge e such that $\gamma_g'(G) = l$ and $\gamma_g'(G \odot e) = l+2$.*

Proof. For the case when $l = 2$, consider the domino graph D and set the edge e as the chord, see Figure 3. It is known that $\gamma_g'(D) = 2$ while after subdividing the edge e we get $\gamma_g'(D \odot e) = 4$.

For the case $l = 3$, construct the graph F'' from F by attaching two vertices y' and y'' as true twins of y consecutively, see Figure 6. It is known from [11] that $\gamma_g(F) = 4$ and $\gamma_g'(F) = 3$ and the game domination number remains the same after attaching true twins. Therefore $\gamma_g'(F) = \gamma_g'(F'') = 3$ and we set e as the edge between y' and y''. After subdividing the edge e we get $\gamma_g'(F'' \odot e) = 5$.

For the general case $l \geq 5$, we present two different infinite families U_k and V_k realizing even and odd l respectively. By using analogous arguments as in the previous case we claim that $\gamma_g'(U_k) = 2k+4$ and $\gamma_g'(U_k \odot e) = 2k+6$ and $\gamma_g'(V_k) = 2k+5$ and $\gamma_g'(V_k \odot e) = 2k+7$ where e is the edge joining x and y. ■

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