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FLIPPABLE EDGES IN TRIANGULATIONS ON SURFACES

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Abstract

Concerning diagonal flips on triangulations, Gao *et al.* showed that any triangulation G on the sphere with $n \geq 5$ vertices has at least $n-2$ flippable edges. Furthermore, if G has minimum degree at least 4 and $n \geq 9$, then G has at least $2n+3$ flippable edges. In this paper, we give a simpler proof of their results, and extend them to the case of the projective plane, the torus and the Klein bottle. Finally, we give an estimation for the number of flippable edges of a triangulation on general surfaces, using the notion of irreducible triangulations.

Keywords: triangulation, diagonal flip, surface.

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1. Introduction

A *triangulation* G on a surface $\mathbb{F}$ is a simple graph embedded on $\mathbb{F}$ so that each face is triangular, except for K_3 on the sphere. We denote the vertex set, edge set and face set of G by $V(G)$, $E(G)$ and $F(G)$, respectively. A *k-cycle* means a cycle of length k. A cycle C in G is said to be *contractible* if C bounds a 2-cell

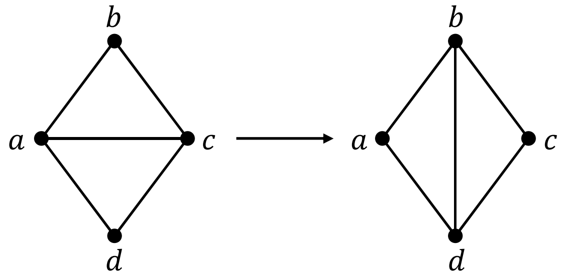

Figure 1. The diagonal flip of the edge ac.

on the surface. We say that C is *separating* if $G - V(C)$ is disconnected. Let G_i be a triangulation on a surface $\mathbb{F}_i$ and let f_i be a face of G_i, for $i = 1, 2$. A *face sum* of G_1 and G_2 with f_1 and f_2 identified is to identify the boundary 3-cycle of f_1 and that of f_2 and obtain a new triangulation on the connected sum of $\mathbb{F}_1$ and $\mathbb{F}_2$.

Let abc and adc be two faces of a triangulation G sharing the edge ac. A *diagonal flip* of ac consists in replacing ac with another diagonal bd in the quadrilateral $abcd$ as in Figure 1. We say that the edge ac is *flippable* if b and d are not adjacent in G. We do not perform a diagonal flip of any non-flippable edge.

The origin of diagonal flips in triangulations is the following.

Theorem 1 (Wagner [20])**.** *Any two triangulations on the sphere with the same number of vertices can be transformed into each other by a sequence of diagonal flips.*

Theorem 1 has been extended to the torus [5], the projective plane and the Klein bottle [16]. Arguments in the above results depend on individual surfaces, but Negami extended these results to general surfaces.

Theorem 2 (Negami [14])**.** *For any closed surface $\mathbb{F}$, there exists a natural number $N(\mathbb{F})$ such that any two triangulations G_1 and G_2 on $\mathbb{F}$ can be transformed into each other by a sequence of diagonal flips if $|V(G_1)| = |V(G_2)| \geq N(\mathbb{F})$.*

The assumption of $|V(G_1)| = |V(G_2)| \geq N(\mathbb{F})$ is needed since $N(\mathbb{F})$ does not coincide with the order of minimal triangulations on $\mathbb{F}$ in general. We can find many related researches, for example, see [4, 9, 12].

Recently, Gao *et al.* showed a lower bound for the number of flippable edges of triangulations on the sphere, as in Theorem 3.

Theorem 3 (Gao *et al.* [7])**.** *Let G be a triangulation on the sphere with n vertices, then the following hold, where all bounds are tight.*

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(a) *If* $n \geq 5$*, then* G *contains at least* $n-2$ *flippable edges.*

(b) *If* G *has minimum degree at least* 4*, then* G *contains at least* $\min\{2n+3, 3n-6\}$ *flippable edges.*

(c) *If* G *is* 4*-connected, then all edges of* G *are flippable.*

In this paper, motivated by this result, we study flippable edges in more depth.

Our first contribution is to give a simpler proof of Theorem 3. For (a), we give an inductive proof on the number of vertices which is independent of a topology of surfaces. For (b) and (c), we focus on the number of separating 3-cycles in a triangulation, and bound the number of flippable edges in triangulations with minimum degree at least 4 and 4-connected ones in the same logic, as in the following.

Theorem 4. *Let* G *be an* n*-vertex triangulation on the sphere with minimum degree at least* 4*, and let* k *be the number of separating* 3*-cycles of* G*. Then*

(i) G *has at least* $\min\{3n-k-5, 3n-6\}$ *flippable edges.*

(ii) $k \leq \max\{n-8, 0\}$.

In Theorem 4(i), if $k=0$, that is, G is 4-connected, then we have Theorem 3(c), since $|E(G)| = 3n-6$. On the other hand, if $k = \max\{n-8, 0\}$ in Theorem 4(ii), then we have Theorem 3(b) by substituting it to the result in Theorem 4(i). Moreover, we characterize n-vertex triangulations with minimum degree at least 4 which have exactly $2n+3$ flippable edges. Section 2 is devoted to Theorems 3(b) and (c).

Secondly, extending Theorem 3 to the projective plane, the torus, and the Klein bottle, we prove Theorems 5, 6, 7 stated below. The arguments in the new proof of Theorem 3 do not depend on a topology of individual surfaces, and so we can extend the results to other surfaces by the same inductive method, in which the first step of induction will be verified for the so called "irreducible triangulations", defined as follows.

Contraction of an edge $e = v_1v_2$ in a triangulation G consists in shrinking e until v_1 and v_2 coincide and in replacing each pair of the multiple edges bounding two digonal faces with a single edge as shown in Figure 2. (The inverse operation of a contraction of an edge is called *a vertex-splitting.*) An edge e of G is said to be *contractible* if the graph obtained from G by contracting e is simple. A triangulation is said to be *irreducible* if it has no contractible edge.

For the sphere, K_4 is a unique irreducible triangulation [17]. The projective plane admits precisely two irreducible triangulations I_P^1 and I_P^2 shown in Figure 3, where I_P^1 is isomorphic to K_6 [1]. For the torus, there exist precisely 21 irreducible triangulations [10], in which I_T^1 is the smallest one whose graph is isomorphic to K_7. For the Klein bottle, there exist precisely 29 irreducible triangulations

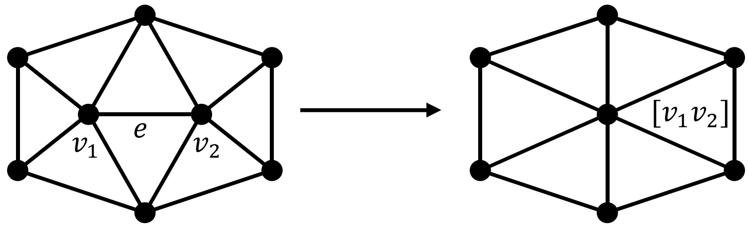

Figure 2. The contraction of the edge e.

including I_K^3 and I_K^{26} [11, 18]. For all the triangulations in Figure 3, the dotted segments are flippable edges.

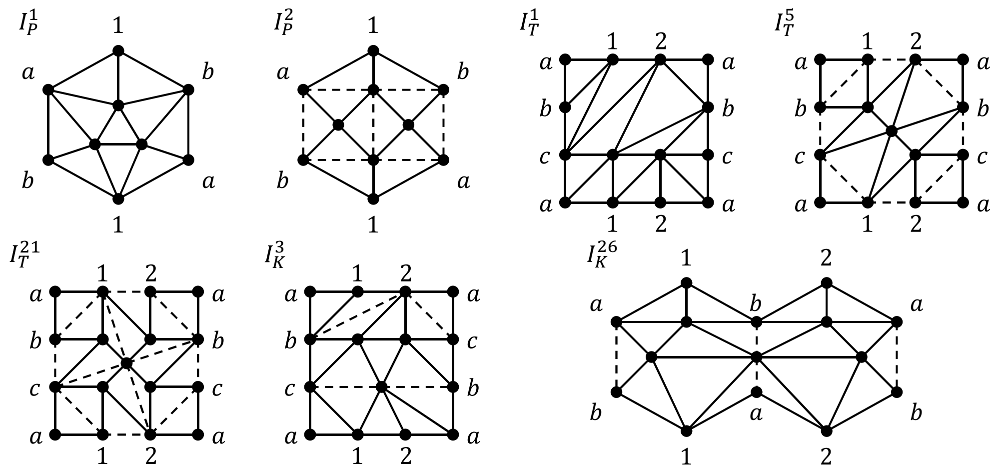

Figure 3. Triangulations I_P^1, I_P^2, I_T^1, I_T^5, I_T^{21}, I_K^3, and I_K^{26}, where we identify vertices with the same label in each triangulation.

Theorem 5. *Let G be a triangulation on the projective plane with n vertices. Then,*

(a) *G contains at least $n-4$ flippable edges if G is not isomorphic to I_P^1.*

(b) *G contains at least $2n-7$ flippable edges if G has minimum degree at least 4 and is not isomorphic to I_P^1 nor I_P^2.*

(c) *G contains at least $2n-7$ flippable edges if G is 4-connected and not isomorphic to I_P^1 nor I_P^2.*

Theorem 6. *Let G be a triangulation on the torus with n vertices. Then,*

(a) *G contains at least $n-5$ flippable edges if G is not isomorphic to I_T^1.*

(b) *G contains at least $2n-9$ flippable edges if G has minimum degree at least 4 and is not isomorphic to I_T^1, I_T^5 nor I_T^{21}.*

(c) *G contains at least $2n-9$ flippable edges if G is 4-connected and not isomorphic to I_T^1, I_T^5 nor I_T^{21}.*

Theorem 7. *Let G be a triangulation on the Klein bottle with n vertices. Then,*

(a) *G contains at least $n-5$ flippable edges if G is not isomorphic to I_K^{26}.*

(b) *G contains at least $2n-15$ flippable edges if G has minimum degree at least 4.*

(c) *G contains at least $2n-11$ flippable edges if G is 4-connected and not isomorphic to I_K^3.*

In each of Theorems 5, 6, 7, the three bounds are tight, in the sense that there exists an infinite sequence of triangulations attaining them.

In Section 3, we construct triangulations on those surfaces attaining the bounds. It should be mentioned that for the projective plane and the torus, the lower bounds for triangulations with minimum degree at least 4 and 4-connected ones coincide in Theorems 5 and 6.

Thirdly, we give an estimation for the number of flippable edges in triangulations on a given surface $\mathbb{F}$. The *Euler genus* g of a surface $\mathbb{F}$ with Euler characteristic $\chi(\mathbb{F})$ is defined as $g = 2 - \chi(\mathbb{F})$.

Theorem 8. *Let G be an n-vertex triangulation on a surface $\mathbb{F}$ with Euler genus $g \geq 1$. Then, the following hold.*

(a) *G contains at least $n-(13g-4)$ flippable edges.*

(b) *If G has minimum degree at least 4, G contains at least $2n-2(13g-4)$ flippable edges.*

2. A New Proof of Theorem 3

In this section, we prove Theorem 3. In the rest of this paper, let $E_{flip}(G)$ denote the set of flippable edges in a triangulation G. At first, we give a lemma describing a structure around a non-flippable edge in a triangulation on a surface.

Lemma 9. *Let G be a triangulation on a surface, and let ac be an edge in G shared by two faces abc and adc. Then ac is non-flippable if and only if a, b, c and d induce K_4 in G.*

Proof. The lemma directly follows from the definition. ■

We first give a simple proof of Theorem 3(a). In order to do it, we introduce the notion of "weak faces" in a triangulation, as follows.

A face abc of a triangulation G is said to be *weak* if at least one of ab, bc and ca is flippable. We denote the set of weak faces in G by $F_{weak}(G)$. Since a flippable edge is shared by two weak faces, we have $2|E_{flip}(G)| \geq |F_{weak}(G)|$.

The next lemma will be used for counting the number of weak faces by induction.

Lemma 10. *If a triangulation G on a surface $\mathbb{F}$ is obtained from another triangulation H by a single vertex-splitting, then $|F_{weak}(G)| \geq |F_{weak}(H)| + 2$.*

Proof. Let $e = v_1v_2$ be an edge of G, and let $v = [v_1v_2]$ be the image of e in H by the contraction. Let uv_1v_2 and wv_1v_2 be two faces of G sharing e. It is easy to see that all flippable edges in $E(H) - \{uv, vw\}$ are also flippable in G. Observe that at least one of v_1 and v_2, say v_1, has degree at least 4, since two vertices of degree 3 cannot be adjacent in G unless $G = K_4$. Then, uv_1 and v_1w are flippable edges of G. For otherwise, i.e., if uv_1 is non-flippable in G, then x and v_2 are adjacent, by Lemma 9, where uv_1 is shared by two facial 3-cycles uv_1x and uv_1v_2 in G. However, in this case, H has multiple edges between x and $[v_1v_2]$, a contradiction. Hence, two faces uv_1v_2 and v_1v_2w of G are weak. Since all weak faces in H are also weak in G, we have $|F_{weak}(G)| \geq |F_{weak}(H)| + 2$. ∎

Using Lemma 10, we first give a shorter proof of Theorem 3(a).

***A shorter proof of Theorem* 3(a).** Let G be an n-vertex triangulation on the sphere with $n \geq 5$, and we prove that $|F_{weak}(G)| \geq 2n-4$. By Steinitz's result[17], every triangulation on the sphere can be transformed into the only irreducible triangulation K_4 by contractions of edges. Hence, if $n = 5$, then G is obtained from K_4 by a single vertex-splitting, and we see that G has exactly three flippable edges, and all six faces are weak. Thus, $|F_{weak}(G)| \geq 2n-4$ when $n = 5$. If $n \geq 6$, then G is obtained from H with $|V(H)| = n - 1$ by a single vertex-splitting. By induction hypothesis and Lemma 10, we have $|F_{weak}(G)| \geq |F_{weak}(H)| + 2 \geq 2(n-1) - 4 + 2 = 2n - 4$. Hence, we have $|E_{flip}(G)| \geq \frac{1}{2}|F_{weak}(G)| \geq n - 2$. ∎

Next, we give a proof to Theorem 3(b) and (c) simultaneously, by focusing on the number k of separating 3-cycles.

Let $k \geq 1$, and let $T_1, \ldots, T_{k-1}$ be $k - 1$ copies of a triangulation on the sphere isomorphic to K_4. For each i, let s_i and f_i be two distinct faces of T_i, and let e_i be the edge of T_i shared by the two faces other than s_i and t_i. Let O_0 and O_k be two copies of a triangulation on the sphere isomorphic to an octahedron, and let f_0 (respectively, f_k) be a face of O_0 (respectively, O_k). Let L_1 be the triangulation obtained from O_0 and O_k by a face sum of f_0 and f_k. For $k \geq 2$, let L_k be a triangulation obtained from $O_0, T_1, \ldots, T_{k-1}, O_k$ by a face sum of f_0 and s_1, a face sum of t_i and s_{i+1} for $i = 1, \ldots, k-2$, and a face sum of t_{k-1} and f_k. (See Figure 4.) Then L_k is a triangulation on the sphere with minimum degree at least 4 which has $n = k + 8 \geq 9$ vertices and k separating 3-cycles. Moreover, L_k contains $k - 1 = n - 9$ non-flippable edges, which are $e_1, \ldots, e_{k-1}$. Hence, we have $|E_{flip}(L_k)| \geq 3n - 6 - (n - 9) = 2n + 3$. For $k \geq 1$, let $\mathcal{L}_k$ denote the set of triangulations L_k constructed by the above procedures.

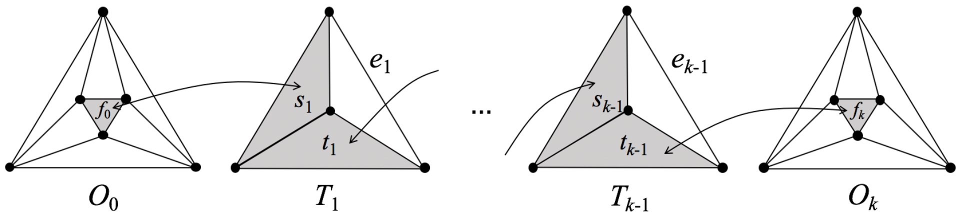

Figure 4. Triangulation L_k consisting of $O_0, T_1, \ldots, T_{k-1}, O_k$.

Lemma 11. *Let G be an n-vertex triangulation on the sphere with minimum degree at least 4. Let k be the number of separating 3-cycles of G. Then*

(1) *G has at most $\max\{k-1, 0\}$ non-flippable edges,*

(2) *$k \leq \max\{n-8, 0\}$, where the equality holds if and only if $n \leq 8$ or G is isomorphic to a member of $\mathcal{L}_k$ with $k = n-8 \geq 1$.*

Proof. (1) We use induction on k. If $k = 0$, then G is 4-connected. In this case, since G does not contain K_4 as a subgraph, G has no non-flippable edge, by Lemma 9.

Suppose $k \geq 1$, and let $C = abc$ be a separating 3-cycle of G. Let G_1 and G_2 be two subgraphs of G such that $V(G_1) \cup V(G_2) = V(G)$ and $V(G_1) \cap V(G_2) = V(C)$, where we may suppose that G_1 is *innermost*, that is, G_1 has no separating 3-cycle. We note that G_2 is also a triangulation on the sphere. By Jordan Curve Theorem, G_2 is a graph on a punctured surface with boundary cycle C. Then we paste a 2-cell on C, and obtain a triangulation G_2 on the sphere. Let k_i denote the number of separating 3-cycles in G_i, for $i = 1, 2$. Then we have $k_2 = k - 1$, since G_1 has no separating 3-cycle, and since C is no longer a separating 3-cycle in G_2. Since G_1 is 4-connected, every edge in G_1 is flippable, as described in the first paragraph. Moreover, since every edge $e \in E(C)$ is flippable in G, all edges of G contained in G_1 are flippable in G.

Now we count the number of non-flippable edges in G_2 in the following two cases on whether G_2 has minimum degree at least 4 or not.

Case 1. G_2 has minimum degree at least 4. Observe that for any edge $e \notin E(C)$ in G_2, e is flippable in G_2 if and only if e is flippable in G, since G_2 is an induced subgraph of G. Moreover, $e \in E(C)$ is flippable in G, even if e is non-flippable in G_2. Hence the number of non-flippable edges of G is less than or equal to that of G_2. By induction hypothesis, since G_2 has precisely k_2 separating 3-cycles, G_2 has at most $\max\{k_2 - 1, 0\}$ non-flippable edges. Hence the number of non-flippable edges of G is at most

$$\max\{k_2 - 1, 0\} = \max\{k - 2, 0\} \leq \max\{k - 1, 0\},$$

where the equality holds if and only if $k = 1$.

Case 2. G_2 has a vertex v of degree 3. Since G has minimum degree at least 4, v lies on C in G_2. So we let $v = a$ and let a', b, c be the three neighbors of a in G_2. Note that Case 2 happens only when $k \geq 2$ since G has two separating 3-cycles abc and $a'bc$. In this case, we contract the edge aa' in G, and let G' be the resulting triangulation, where we let $[aa']$ denote the image of aa' in G' by the contraction. This operation merges the two separating 3-cycles abc and $a'bc$ in G into a single separating 3-cycle $[aa']bc$ in G', but this introduces no new separating 3-cycle. Hence, if we let k' be the number of separating 3-cycles in G', then $k' = k - 1 (\geq 1)$. Since a, b and c have degree at least 4 in G_1 by the 4-connectedness, G' has minimum degree at least 4. Hence, by induction hypothesis, G' has at most $k' - 1$ non-flippable edges. By the contraction of aa' in G, the single non-flippable edge aa' in G disappears in G', and no new non-flippable edge is not produced in G', and hence G has at most $(k'-1)+1 = k-1$ non-flippable edges.

By Cases 1 and 2, the number of flippable edges in G is at most $\max\{k-1, 0\}$, and we are done.

(2) Using the same induction as in (1), we prove that $n \geq k+8$ for any $k \geq 1$ by the same case analysis in Cases 1 and 2.

In Case 1, let $n_i = |V(G_i)|$ for $i = 1, 2$, where $n_1 + n_2 - 3 = n$. Observe that the smallest 4-connected triangulation is an octahedron, which has six vertices. Since G_1 is 4-connected, we have $n_1 \geq 6$. On the other hand, we have $k_2 = k - 1$. If $k_2 = 0$, then we also have $n_2 \geq 6$ by the same argument as for G_1. Hence, when $k = 1$,

$$n = n_1 + n_2 - 3 \geq 6 + 6 - 3 = 9,$$

where the equality holds when both G_1 and G_2 are isomorphic to the octahedron, and hence $G = L_1 \in \mathcal{L}_1$. If $k_2 \geq 1$, then $n_2 \geq k_2 + 8 = (k-1) + 8 = k + 7$ by induction hypothesis. Therefore,

$$n = n_1 + n_2 - 3 \geq 6 + (k+7) - 3 = k + 10 > k + 8.$$

In Case 2, we note that $k \geq 2$. Then, if we let $n' = |V(G')|$, then we have $n = n' + 1$ and $k' = k - 1 \geq 1$. Hence, by induction hypothesis, G' satisfies $n' \geq k' + 8$, in which the equality holds if and only if $G' \in \mathcal{L}_{k'}$. Therefore, $n = n' + 1 \geq (k' + 8) + 1 = k + 8$ for $k \geq 2$, where the equality holds in G if and only if $n' = k' + 8$ in G'. In this case, we have $G \in \mathcal{L}_k$, since G is obtained from $G' \in \mathcal{L}_{k'}$ by a splitting of a vertex in a separating 3-cycle of G' which increases the number of separating 3-cycles by one. ■

Next, we prove Theorem 3(b) and (c) using Lemma 11, and characterize triangulations attaining the equality in (b).

• Theorem 3(b). Let G be a triangulation on the sphere with minimum degree at least 4. By Lemma 11(1), since the number of non-flippable edges of G is at most $\max\{k-1, 0\}$, we have

$$|E_{flip}(G)| \geq 3n - 6 - \max\{k-1, 0\} \geq \min\{3n - k - 5, 3n - 6\}.$$

Then, since $k \leq \max\{n-8, 0\}$ by Lemma 11(2), we have

$$|E_{flip}(G)| \geq \min\{3n - k - 5, 3n - 6\} \geq \min\{2n + 3, 3n - 6\}.$$

• Theorem 3(c). In Lemma 11, if G is 4-connected, then $k = 0$, and hence, by Lemma 11(1), every edge in G is flippable, which proves Theorem 3(c).

• Characterization of triangulations G with minimum degree at least 4 with $|E_{flip}(G)| = 2n + 3$. Observe that a required n-vertex triangulation G satisfies both equalities of Lemma 11(1) and (2) for some $k \geq 1$. Hence we have $G \in \mathcal{L}_k$ by Lemma 11(2), and clearly G satisfies the equality in Lemma 11(1) too.

3. PROOF OF THE THEOREMS FOR NON-SPHERICAL SURFACES

In this section, let $\mathbb{S}$, $\mathbb{P}$, $\mathbb{T}$ and $\mathbb{K}$ denote the sphere, the projective plane, the torus and the Klein bottle, respectively.

We first show Theorems 5, 6 and 7 by induction on the number of vertices, by a similar method to that in Theorem 3(a). We note that Lemma 10 holds for triangulations on all surfaces.

***Proof of Theorem* 5(a).** Let G be a triangulation on $\mathbb{P}$ which is not isomorphic to I_P^1. We show that $|F_{weak}(G)| \geq 2n - 8$ by induction on the number of vertices.

Contracting edges, we can transform G into either I_P^2 or a triangulation, say T, obtained from I_P^1 by a single vertex-splitting, since I_P^1 and I_P^2 are the two irreducible triangulations on $\mathbb{P}$. Figure 5 shows I_P^1 and I_P^2 and the two candidates of T, and we verify that all the three have seven vertices and at least six weak faces. Hence, if $|V(G)| = 7$, then $|F_{weak}(G)| \geq 2n - 8$. By the same induction by Lemma 10 as in the proof of Theorem 3(a), we have $|E_{flip}(G)| \geq \frac{1}{2}|F_{weak}(G)| \geq n - 4$. ■

Theorems 6(a) and 7(a) can be proved in the same way. For $\mathbb{T}$, we should verify that the result holds for all irreducible triangulations except I_T^1 and all triangulations obtained from I_T^1 by a single vertex-splitting. For $\mathbb{K}$, we should do it for all irreducible triangulations except I_K^{26} and the ones obtained from I_K^{26} by a single vertex-splitting. We leave these tasks to readers. At the end of the paper, we attach an Appendix with the table for the number of flippable edges in all

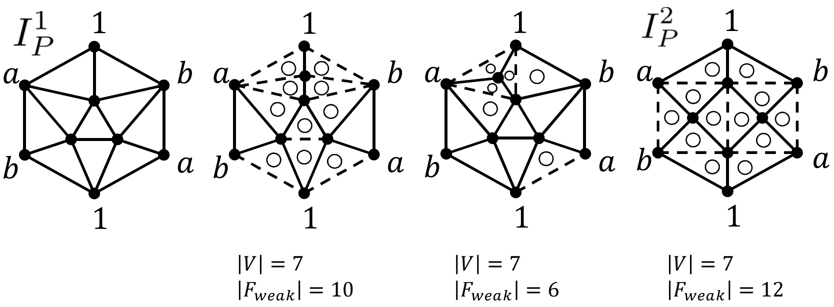

Figure 5. Irreducible triangulations I_P^1 and I_P^2 on $\mathbb{P}$ and ones obtained from I_P^1 by a single vertex-splitting. In the figures, the faces with a circle are weak.

irreducible triangulations on $\mathbb{P}$, $\mathbb{T}$ and $\mathbb{K}$, and we list the figures of all irreducible triangulations on $\mathbb{T}$ and $\mathbb{K}$ with the indication of flippable edges.

Next we deal with triangulations with minimum degree at least 4. An *octahedron-addition* is an operation defined as below. Let $a_1a_2a_3$ be a face of a triangulation. Put a 3-cycle $v_iv_jv_k$ inside the face $a_1a_2a_3$ and add edges a_iv_j for all $i \neq j$ with $i, j \in \{1, 2, 3\}$. A 4-*splitting* is a vertex-splitting such that the resulting triangulation has minimum degree at least 4.

The following result guarantees that these two operations generate all triangulations on any surface with minimum degree at least 4.

Theorem 12 (Nakamoto *et al.* [13])**.** *Every triangulation on a non-spherical surface* (*respectively, the sphere*) *with minimum degree at least* 4 *can be obtained from an irreducible triangulation* (*respectively, an octahedron*) *by a sequence of* 4-*splittings and octahedron-additions, preserving the minimum degree at least* 4.

Using Theorem 12, we estimate the increase of the number of flippable edges by a 4-splitting and an octahedron-addition in the next lemma.

Lemma 13. *Let G and H be triangulations on a closed surface with minimum degree at least* 4.

- *If G can be obtained from H by a* 4-*splitting, then* $|E_{flip}(G)| \geq |E_{flip}(H)| + 2$.
- *If G can be obtained from H by an octahedron-addition, then* $|E_{flip}(G)| \geq |E_{flip}(H)| + 9$.

Proof. The former part can be proved similarly to Lemma 10, but we have to note that both v_1 and v_2 have degree at least 4 in G. Therefore, four edges v_1u, v_1w, v_2u, v_2w are flippable in G, but uv and vw might be flippable in H. Hence, we have $|E_{flip}(G)| \geq |E_{flip}(H)| + 2$.

In the latter part, if G is obtained from H by a single octahedron-addition, then nine edges in $E(G) - E(H)$ are flippable in G and hence, we have $|E_{flip}(G)| \geq |E_{flip}(H)| + 9$. ■

Now we show Theorem 5(b) using Lemma 13.

***Proof of Theorem* 5(b).** Let G be an n-vertex triangulation on $\mathbb{P}$ which is not isomorphic to I_P^1 nor I_P^2. We prove that $|E_{flip}(G)| \geq 2n - 7$.

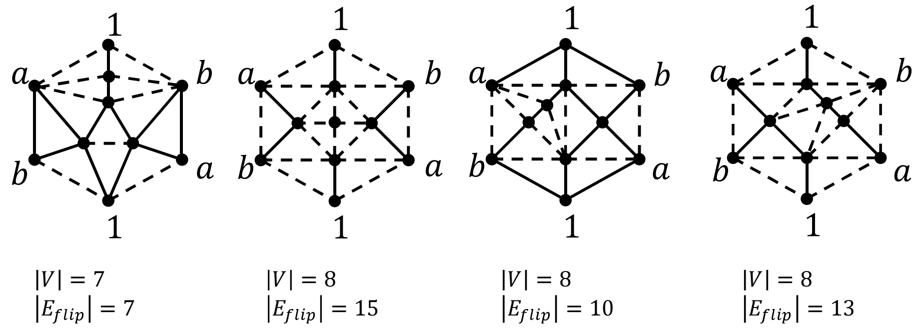

Figure 6. All candidates of H and their flippable edges.

For the base cases, we deal with triangulations, denoted by H, obtained from I_P^1 and I_P^2 by a single 4-splitting or a single octahedron-addition. Figure 6 shows all candidates of H by a single 4-splitting, up to isomorphism, and we can verify the claim holds for all possible H. Suppose that G can be obtained from H by a 4-splitting. By induction hypothesis and Lemma 13, we have $|E_{flip}(G)| \geq |E_{flip}(H)| + 2 \geq 2(n-1) - 7 + 2 = 2n - 7$, and our claim holds. (When applying Lemma 13, we note that the 4-splitting produces less flippable edges than the octahedron-addition with respect to the ratio $\frac{|E_{flip}(G)|}{|E(G)|}$, and so we do not deal with the latter.) ■

The proofs for Theorems 6(b) and 7(b) are similar to that for Theorem 5(b), and hence we omit them. Readers should do the tasks using the materials in Appendix.

We show the tightness of Theorems 5, 6 and 7. The initial set up is common with the tightness part of Theorem 3 in [7]. Let T' be any triangulation on the plane with $m \geq 3$ vertices, and let C be the 3-cycle of T' bounding the infinite face, where we note that $|E(T')| = 3m-6$. We put a single vertex in the interior of each finite face of T', and join it to three vertices of the corresponding boundary 3-cycle. The resulting plane triangulation, denoted by T, has $m+(2m-5) = 3m-5$ vertices. We note that every edge of T incident to an added vertex to T' is not flippable, and this observation will be important in the following argument.

In the following constructions, we use the irreducible triangulations I_P^1 for $\mathbb{P}$, I_T^1 for $\mathbb{T}$, I_K^3 and I_K^{26} for $\mathbb{K}$.

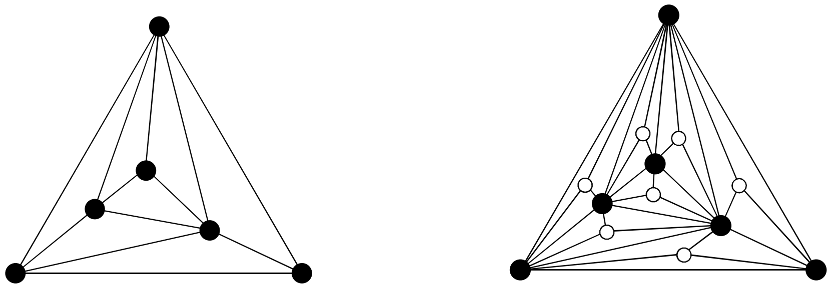

Figure 7. Examples of plane triangulations T' and T.

TIGHTNESS OF THEOREM 5. (a) Let T_P^1 be a triangulation on $\mathbb{P}$ with $3m-2$ vertices obtained from I_P^1 and T by identifying C and a facial cycle of I_P^1. Flippable edges in T_P^1 are those in T', and hence the number of flippable edges of T_P^1 is $3m-6$. Since $n = 3m-2$, T_P^1 contains $n-4$ flippable edges.

(b,c) Let T_P^2 be a triangulation on $\mathbb{P}$ with minimum degree at least 4 which is constructed, as follows. Let xy be an edge of I_P^1, and let xyz and xyc be two faces of I_P^1 sharing xy. Replace xy with a path $P = xv_1v_2\cdots v_my$, and join v_i to z and to c for $i = 1,\dots,m$, as shown in Figure 8. Then T_P^2 has $m+6 = n$ vertices, and the flippable edges of T_P^2 are v_iz, v_ic for each i, and xz, zy, xc, cy, ab, and hence $2m+5 = 2n-7$ flippable edges. Since T_P^2 is 4-connected, T_P^2 is also an example for the case (c).

TIGHTNESS OF THEOREM 6. A triangulation on $\mathbb{T}$ and that with minimum degree at least 4 attaining the equality in (a), (b,c) can be obtained from I_T^1 by the same construction as the two triangulations in TIGHTNESS OF THEOREM 5, respectively.

TIGHTNESS OF THEOREM 7. Let abc be the 3-cycle of I_K^{26} which separates I_K^{26} into two Möbius bands, and let abd be a face.

(a) Identify the outer face C of T with abd of I_K^{26}, and let T_K^1 be the resulting triangulation on $\mathbb{K}$ with $3m+1$ vertices. As in Figure 8, flippable edges of T_K^1 are exactly those of T' and bc, ca, where we note that ab is contained in T. Hence the number of flippable edges is $3m-4$ edges. Since $n = 3m+1$, T_K^1 has $3m-4 = n-5$ flippable edges.

(b) Apply a 4-splitting in I_K^{26} as in the figure, and subdivide the new edge by m vertices. The resulting triangulation, denoted by T_K^2, has minimum degree at least 4, but is not 4-connected. Then T_K^2 has $m+10$ vertices, and $2m+5$ flippable edges. Since $n = m+10$, T_K^2 has $2n-15$ flippable edges.

(c) Apply a 4-splitting in I_K^3 as in the figure, and subdivide the new edge by m vertices. The resulting triangulation on $\mathbb{K}$, denoted by T_K^3, is 4-connected. Then T_K^3 has $m+9$ vertices, and $2m+7$ flippable edges. Since $n = m+9$, T_K^3

has $2n - 11$ flippable edges.

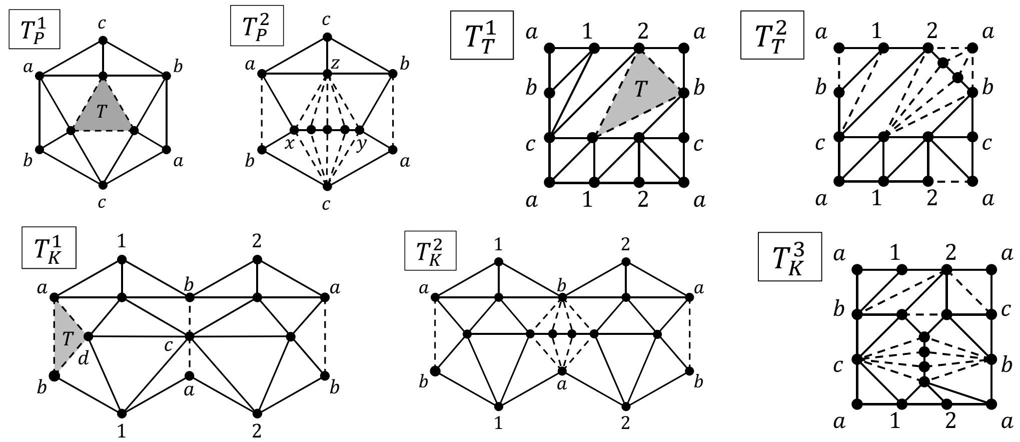

Figure 8. Triangulations attaining bounds in Theorems 5, 6 and 7 in which the dotted segments represent flippable edges.

Finally, we show Theorem 8. For general surfaces, it is known that each surface admits only finitely many irreducible triangulations [2], which is shown by bounding the number of vertices, for example, see [3, 6, 15]. The best upper bound known so far is the following, which will be used in our proof of Theorem 8.

Theorem 14 (Joret *et al.* [8])**.** *Every irreducible triangulation of a surface with Euler genus* $g \geq 1$ *has at most* $13g - 4$ *vertices.*

We prove Theorem 8.

***Proof of Theorem* 8.** We show Theorem 8(a) by induction on the number of vertices. Let G be a triangulation on a non-spherical surface $\mathbb{F}$ with Euler genus $g \geq 1$. If G is irreducible, then we clearly have $|E_{flip}(G)| \geq 0$. Hence G satisfies $|E_{flip}(G)| \geq |V(G)| - (13g - 4)$, since $|V(G)| \leq 13g - 4$ by Theorem 14. The induction step of is completely the same as in the proof of Theorem 5(a).

Theorem 8(b) can be proved similarly to that of Theorem 5(b), and so we omit a proof. ■

4. Remarks

In this paper, we estimate the number of flippable edges of triangulations on surfaces in various settings.

Gao *et al.* [7] have established a result for triangulations on the sphere, and given three distinct lower bounds for the number of flippable edges in ordinary triangulations, the ones with minimum degree at least 4 and 4-connected ones in Theorem 3(a), (b) and (c) respectively, where all these bounds are best possible. Then, in this paper, we extend these results to other surfaces by induction on the number of vertices based on an edge contraction. This argument is independent of topology of surfaces, and some methods in the spherical case can be applied to a non-spherical case, but some are not.

So, in this section, we would like to make a remark about the following three points.

• Although both Theorem 5(b) and Theorem 3(b) deal with triangulations with minimum degree at least 4, the proof of Theorem 5(b) is much simpler than that of Theorem 3(b). Why cannot we give a similar proof to Theorem 3(b)?

In the proof of Theorem 5(b), applying Lemma 13, we consider only a 4-splitting to generate triangulations with minimum degree at least 4, since an octahedron-addition produces more flippable edges. Moreover, this method gives a best possible lower bound for the number of flippable edges in triangulations on $\mathbb{P}$ in Theorem 5(b), as shown in the above construction of T_P^2. This also gives a best possible lower bounds for $\mathbb{T}$ and $\mathbb{K}$ in Theorems 6(b) and 7(b).

For the sphere $\mathbb{S}$, we observe that the octahedron O is the smallest triangulation with minimum degree at least 4, which has six vertices and twelve edges, all flippable. If we let G be a triangulation obtained from O by m 4-splittings, then we get $|V(G)| = n = m + 6$ and $|E_{flip}(G)| \geq 2m + 12 = 2n$ by Lemma 13. However, this bounds is not tight, since the lower bound is $2n + 3$ in Theorem 3(b). Therefore, in order to get a tight bound in Theorem 3(b), we need to investigate a worst combination of 4-splittings and octahedron-additions. (Note that $\mathcal{L}_k$ cannot be obtained from O only by 4-splittings, since 4-splittings preserve the 4-connectedness of triangulations.)

• Why do the lower bounds for the cases of "minimum degree at least 4" and "4-connected" coincide for $\mathbb{P}$ and $\mathbb{T}$ in Theorems 5(b) and 6(b)?

Observe that all irreducible triangulations on $\mathbb{P}$ and $\mathbb{T}$ are 4-connected. In the proof of Theorem 5(b), we estimate the number of flippable edges in a triangulation G on $\mathbb{P}$ which is obtained from an irreducible one only by 4-splittings. Since a 4-splitting preserves the 4-connectedness, G must be 4-connected. Since this gives a best possible bound, a "4-connected" triangulation on $\mathbb{P}$ attains the lower bound in Theorem 5(b). The same holds for $\mathbb{T}$, but does not for the Klein bottle $\mathbb{K}$, since there is an irreducible triangulation with a non-contractible separating 3-cycle. Actually, the two lower bounds do not coincide in Theorem 7(b) and (c) for $\mathbb{K}$.

• Why can a 4-connected triangulation on a non-spherical surface admit non-flippable edges, as shown in Theorems 5(c), 6(c) and 7(c)?

For the sphere, if a triangulation G is 4-connected, then every edge of G is flippable, by Theorem 3(c). This can be explained by a subgraph isomorphic to K_4 as described in Lemma 9. If an edge $e = ac$ shared by two faces abc and acd is not flippable, then b and d are adjacent in the graph, and then a, b, c and d induce K_4. If a triangulation G on the sphere has the flippable edge e and at least five vertices, then by Jordan Curve Theorem, either abd or bcd forms a separating 3-cycle in G, and so G is not 4-connected. On the other hand, in a triangulation G' on a non-spherical surface $\mathbb{F}$, both of 3-cycles abd or bcd can be non-contractible on $\mathbb{F}$, and e can be non-flippable on $\mathbb{F}$ even if G' is 4-connected.

However, forbidding non-contractible 3-cycles, we see that all edges of a 4-connected triangulations on any non-spherical surface are flippable, since the graph can no longer contain K_4 as a subgraph. Therefore, the condition for all edges being flippable in triangulations on the sphere is "4-connected", but the corresponding condition for any non-spherical surface is "4-connected and no non-contractible 3-cycle".

Our methods in this paper depend on the concrete structures of irreducible triangulations on $\mathbb{P}$, $\mathbb{T}$ and $\mathbb{K}$. Can we say something on flippable edges in triangulations on non-spherical surfaces, without the lists of irreducible triangulations? We already know that Slanke [19] determined the complete lists of irreducible triangulations on some other surfaces, but their numbers for each surface are very large. We wonder if we can deal with those lists efficiently to count the number of flippable edges.

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APPENDIX

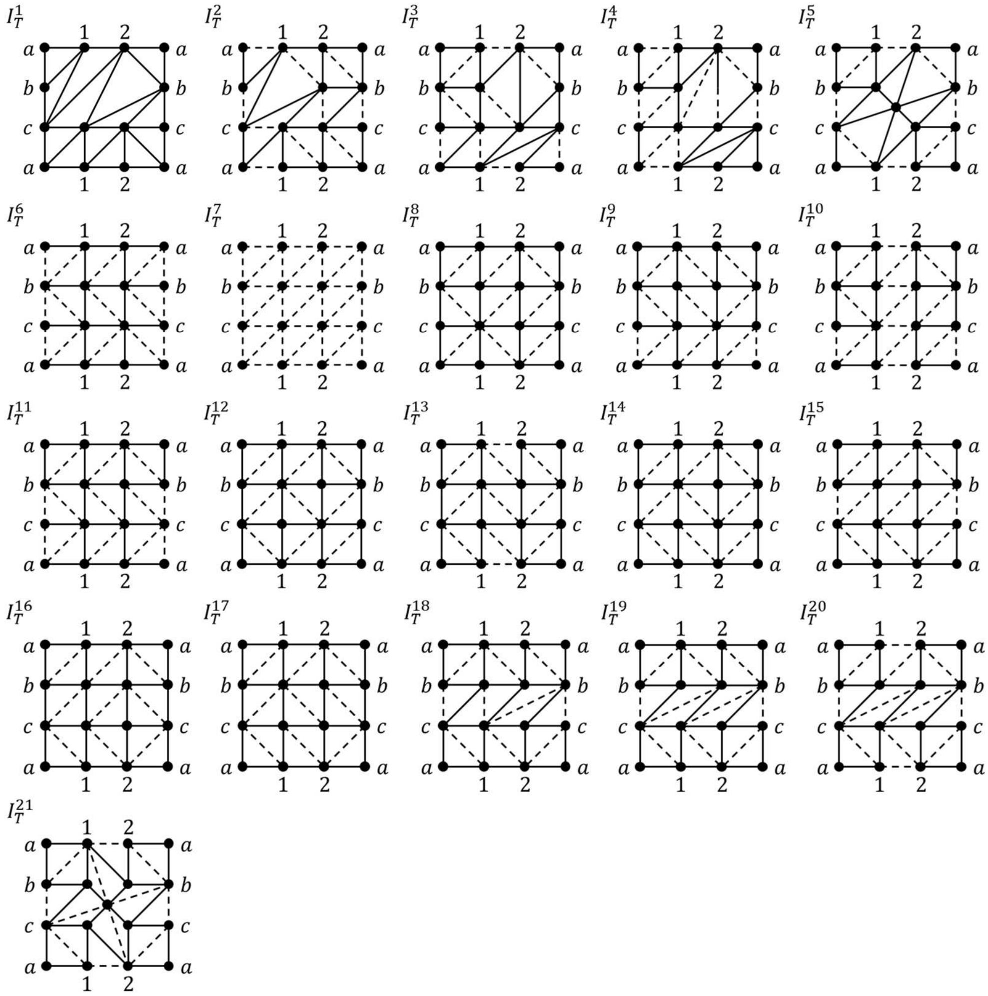

Figure 9. Irreducible triangulations on the torus. The dotted segments represent flippable edges.

Projective plane

graph	vertices	flippable edges
I_P^1	6	0
I_P^2	7	6

Torus

graph	vertices	flippable edges
I_T^1	7	0
I_T^2	8	8
I_T^3	8	7
I_T^4	8	10
I_T^5	8	6
I_T^6	9	18
I_T^7	9	27
I_T^8	9	9
I_T^9	9	12
I_T^{10}	9	15
I_T^{11}	9	15
I_T^{12}	9	9
I_T^{13}	9	12
I_T^{14}	9	9
I_T^{15}	9	12
I_T^{16}	9	9
I_T^{17}	9	9
I_T^{18}	9	9
I_T^{19}	9	9
I_T^{20}	9	11
I_T^{21}	10	10

Klein bottole, 4-connected

graph	vertices	flippable edges
I_K^1	8	6
I_K^2	8	6
I_K^3	8	4
I_K^4	8	8
I_K^5	8	6
I_K^6	8	5
I_K^7	9	8
I_K^8	9	10
I_K^9	9	10
I_K^{10}	9	9
I_K^{11}	9	10
I_K^{12}	9	11
I_K^{13}	9	9
I_K^{14}	9	20
I_K^{15}	9	11
I_K^{16}	9	9
I_K^{17}	9	9
I_K^{18}	9	10
I_K^{19}	9	10
I_K^{20}	9	10
I_K^{21}	9	12
I_K^{22}	9	15
I_K^{23}	9	15
I_K^{24}	9	14
I_K^{25}	10	13

Klein bottole, non 4-connected

graph	vertices	flippable edges
I_K^{26}	9	3
I_K^{27}	10	13
I_K^{28}	11	9
I_K^{29}	11	13

Table 1. The number of flippable edges of irreducible triangulations on $\mathbb{P}, \mathbb{T}$ and $\mathbb{K}$.

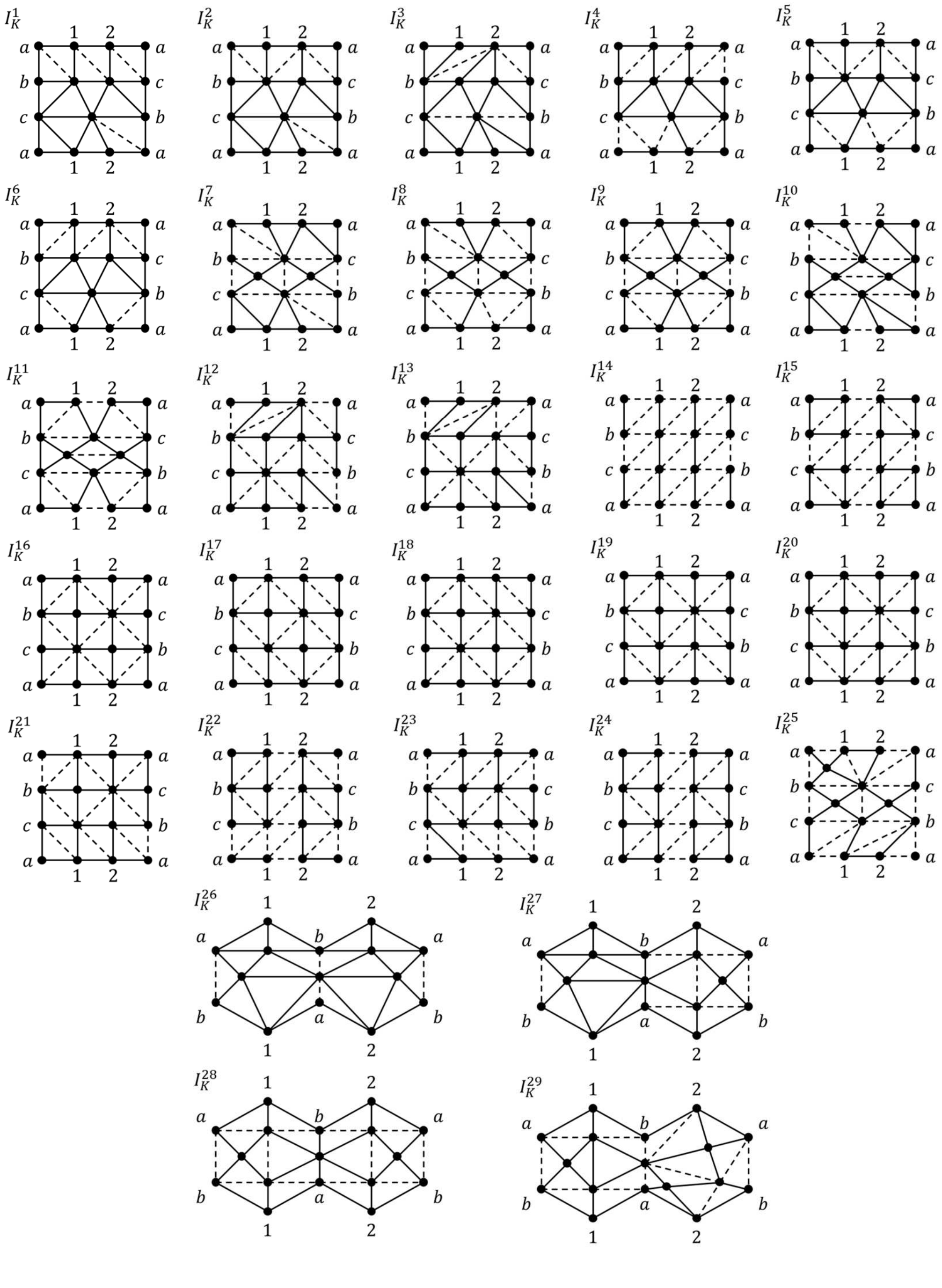

Figure 10. Irreducible triangulations on the Klein bottle. The dotted segments represent flippable edges.