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A NOTE ON FORCING 3-REPETITIONS IN DEGREE SEQUENCES

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Abstract

In Caro, Shapira and Yuster [*Forcing k-repetitions in degree sequences*, Electron. J. Combin. 21 (2014) #P1.24] it is proven that for any graph G with at least 5 vertices, one can delete at most 6 vertices such that the subgraph obtained has at least three vertices with the same degree. Furthermore they show that for certain graphs one needs to remove at least 3 vertices in order that the resulting graph has at least 3 vertices of the same degree.

In this note we prove that for any graph G with at least 5 vertices, one can delete at most 5 vertices such that the subgraph obtained has at least three vertices with the same degree. We also show that for any triangle-free graph G with at least 6 vertices, one can delete at most one vertex such that the subgraph obtained has at least three vertices with the same degree and this result is tight for triangle-free graphs.

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1. Introduction

All the graphs in this paper are simple, that is they have no loops and no multiple edges. A well-known elementary exercise states that every simple graph has two vertices with the same degree. Since there are graphs without 3 vertices of the same degree, it is natural to ask how many vertices one needs to remove from a graph in order that the resulting graph has at least 3 vertices of the same degree. The following theorem was proven in [1].

Theorem 1.1. *For any graph G with at least 5 vertices, one can delete at most 6 vertices such that the subgraph obtained has at least three vertices with the same degree.*

Furthermore it is shown in [1] that for certain graphs one needs to remove at least 3 vertices in order that the resulting graph has at least 3 vertices of the same degree.

More generally the following was shown in [1]. For a given positive integer k, let $C = C(k)$ denote the least integer such that any graph with n vertices has an induced subgraph with at least $n - C$ vertices, such that at least $\min\{k, n - C\}$ vertices in this subgraph are of the same degree, then $\Omega(k \log k) \leq C(k) \leq (8k)^k$. For $k = 3$ the bounds in [1] are $3 \leq C(3) \leq 6$ and the exact value was left as an open question in that paper. In this note we slightly improve Theorem 1.1, by showing that $C(3) \leq 5$. Formally we prove the following.

Theorem 1.2. *For any graph G with at least 5 vertices, one can delete at most 5 vertices such that the subgraph obtained has at least three vertices with the same degree.*

We also prove the following result for triangle-free graphs.

Theorem 1.3. *For any triangle-free graph G with at least 6 vertices, one can delete at most one vertex such that the subgraph obtained has at least three vertices with the same degree.*

Theorem 1.3 is tight as a path on 4 vertices together with an isolated vertex is a graph on 5 vertices with no three vertices of the same degree, for which no removal of a vertex creates three vertices of the same degree. Furthermore for any $n \geq 1$ the bipartite half-graph (defined below) contains $2n$ vertices and no three of them of the same degree.

Definition 1.1. The half-graph on $2n$ vertices is a bipartite graph with vertex set $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ such that x_i is adjacent to y_j if and only if $i + j > n$. Note that x_i and y_i have degree i, for $1 \leq i \leq n$.

Note that for an odd number of vertices we can take a half-graph and one isolated vertex. The resulting graph has no three vertices of the same degree.

2. Proof of Theorem 1.2

Let $G(V, E)$ be a graph on $n \geq 5$ vertices. Denote by $d(v)$ the degree of vertex v in graph G. Let $H \subset V$ be a set of 3 vertices x, y, z where $d(x) \leq d(y) \leq d(z)$. Set H is called *balanceable* if one of the following conditions hold.

1. $G[H]$ is an independent set.
2. $G[H]$ is a clique.
3. $G[H]$ contains only the edge (x, y).
4. $G[H]$ contains only the edges (x, y) and (x, z).

Let $p = d(z) - d(y)$ and $q = d(y) - d(x)$. We will need the following lemma whose proof is almost identical to the proof of Theorem 1.1 and is given only for completeness.

Lemma 2.1. *If graph G contains a balanceable set H, then one can delete at most $p + q + \max(p, q)$ vertices from G such that the subgraph obtained has at least three vertices with the same degree.*

Proof. Let H be a balanceable set in graph G with vertices x, y, z such that $d(x) \le d(y) \le d(z)$ and recall that $p = d(z) - d(y)$ and $q = d(y) - d(x)$. Throughout the proof we denote by $N(.)$ the set of neighbors of a vertex in the current G (that is, in the graph G after some vertices have possibly been deleted). Similarly, we denote by $d(.)$ the degree of a vertex in the current G. We can assume without loss of generality that set H satisfies conditions 1 or 3 of a balanceable set (otherwise we just take the complement of graph G as our graph). We will consider two cases.

Case 1. $d(x) < d(y) = d(z)$. In this case $p = 0$. If $(N(z) \setminus N(x)) \cap (N(y) \setminus N(x)) \ne \emptyset$, we can delete a vertex of this intersection and decrease the degrees of y and z by 1 without affecting the degree of x (notice that $x \notin (N(z) \setminus N(x)) \cap (N(y) \setminus N(x))$ as H satisfies conditions 1 or 3 of a balanceable set). Otherwise, if $(N(z) \setminus N(x)) \cap (N(y) \setminus N(x)) = \emptyset$ we can delete a vertex of $N(z) \setminus N(x)$ and a vertex of $(N(y) \setminus N(x)) \setminus \{x\}$ and decrease the degrees of y and z by 1 without affecting the degree of x. Observe that in any case we delete at most two vertices. Repeating this process at most q times we eventually obtain $d(x) = d(y) = d(z)$. Overall we have deleted at most $2q \le p + q + \max(p, q)$ vertices.

Case 2. $d(x) \le d(y) < d(z)$. Let us first equate $d(z)$ and $d(y)$ by deleting some $u \in N(z) \setminus N(y)$. Observe that $u \ne x$. We always prefer to delete a vertex u that is non-adjacent to x, as long as there is such a vertex u. Overall, we have deleted p vertices. The problem is that in the current graph we may have that $d(x)$ also decreased by some amount $r \le p$. Suppose first that $r = 0$. As in the previous case, we may need to delete $2q$ additional vertices to equate the degrees of y and z to that of x so in total we removed $p + 2q \le p + q + \max(p, q)$ vertices. If $r > 0$, then this means that at some point, when we deleted a vertex u, that vertex also had to be adjacent to x. Hence, in the current graph $(N(z) \setminus N(x)) \subseteq (N(y) \setminus N(x))$. So, we may simply delete $r + q$ additional vertices (all of them from $N(z) \setminus N(x)$) to equate the degrees of y and z to that of x. Overall, we deleted $p + r + q \leq 2p + q \leq p + q + \max(p, q)$ vertices and the proof follows. ■

We call a balanceable set H good if $p + q \leq 3$ and $\max(p, q) \leq 2$. Hence by Lemma 2.1 it is sufficient to find a good set in G in order to finish our proof, as in this case we have $p + q + \max(p, q) \leq 5$.

We will need the following lemma proven in [2].

Lemma 2.2. *Let G be any graph of order n and let m be any positive integer less than n. Then G contains a set X of $m+1$ vertices such that $|d(u)-d(v)| \leq m-1$ for all $u, v \in X$*

Hence by Lemma 2.2 graph G has a set X of 5 vertices such that $|d(u) - d(v)| \leq 3$ for all $u, v \in X$. Assume that the vertices of X are v_1, v_2, v_3, v_4, v_5. Furthermore assume that $d(v_1) \leq d(v_2) \leq \cdots \leq d(v_5)$. We also may assume that $G[X]$ contains at least two of the edges in the set $\{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5)\}$, for otherwise this would be true in the complement of the graph G. Now let $d(v_1) = d$. By the definition of X we know that $d(v_5) \leq d + 3$. We also may assume the following.

Lemma 2.3. *$d + 1 \leq d(v_3) \leq d + 2$.*

Proof. If $d(v_3) = d$, then $d(v_1) = d(v_2) = d(v_3) = d$ and we are done. Furthermore if $d(v_3) = d + 3$, then $d(v_3) = d(v_4) = d(v_5) = d + 3$ and we are done. ■

Corollary 1. *If $H \subset X$ is a balanceable set such that $v_3 \in H$, then H is a good set.*

Proof. In this case $\max(p, q) \leq 2$. Furthermore $p+q \leq 3$ by the definition of X. ■

Finally we will need the following simple lemma.

Lemma 2.4. *For any $2 \leq i \leq 4$, if $(v_i, v_{i+1}) \notin E$ and $(v_{i-1}, v_i) \in E$, then the set $\{v_{i-1}, v_i, v_{i+1}\}$ is a good set.*

Proof. Each such set for $2 \leq i \leq 4$ is a balanceable set by definition, furthermore each such set contains the vertex v_3 and hence is good by Corollary 1. ■

Let $E' = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5)\}$. Recall that we may assume that $G[X]$ contains at least two of the edges of E'. We will consider two cases.

Case 1. $G[X]$ contains at least three of the edges of E'. By Lemma 2.4 we may assume that $G[X]$ contains the edges $(v_2, v_3), (v_3, v_4), (v_4, v_5)$. Now if $(v_3, v_5) \in E$, then $\{v_3, v_4, v_5\}$ is a good set, while if $(v_3, v_5) \notin E$, then $\{v_2, v_3, v_5\}$ is a good set.

Case 2. $G[X]$ contains exactly two edges of E'. By Lemma 2.4 we may assume that $G[X]$ contains the edges $(v_3, v_4), (v_4, v_5)$ but not the edges $(v_1, v_2), (v_2, v_3)$. Now if $(v_3, v_5) \in E$, then $\{v_3, v_4, v_5\}$ is a good set. Hence we may assume that $(v_3, v_5) \notin E$. Furthermore if $(v_1, v_3) \notin E$, then $\{v_1, v_2, v_3\}$ is a good set. Hence we may assume that $(v_1, v_3) \in E$. Now since $(v_1, v_3) \in E$ and $(v_3, v_5) \notin E$ we conclude that $\{v_1, v_3, v_5\}$ is a good set.

This finishes the proof.

3. Proof of Theorem 1.3

We will need the following theorem proven in [3].

Theorem 3.1. *Let G be a triangle-free graphs without three vertices of the same degree and without isolated vertices. Let the vertices of G be $v_1, \dots, v_n$ such that $d(v_1) \le d(v_2) \le \cdots \le d(v_n)$. Then $d(v_1) = 1$ and for all $1 \le i \le n-2$ we have $d(v_{i+2}) - d(v_i) = 1$.*

We start by proving the following.

Theorem 3.2. *For any triangle-free graph G with at least 5 vertices and no isolated vertices, one can delete at most one vertex such that the subgraph obtained has at least three vertices with the same degree.*

Proof. Suppose that G contains $n \ge 5$ vertices and the vertices of G are $v_1, \dots, v_n$ such that $d(v_1) \le d(v_2) \le \cdots \le d(v_n)$. By Theorem 3.1 we may consider two cases.

Case 1. $d(v_1) = 1$ and $d(v_2) = 1$ and for all $1 \le i \le n-2$ we have $d(v_{i+2}) - d(v_i) = 1$. In this case the degree sequence of G starts with $1, 1, 2, 2, 3$.

If v_1 is not adjacent to v_i for some $2 \le i \le 4$, we can remove v_1 and obtain a graph with three vertices of the same degree. If v_2 is not adjacent to v_i for some $3 \le i \le 4$ or $i = 1$, we can remove v_2 and obtain a graph with three vertices of the same degree. Hence we need to only consider the case where v_n is not adjacent to both v_1 and v_2. Thus the neighbor v_j of v_n with the smallest index j satisfies $j \ge 3$, so we can remove v_n and obtain a graph with three vertices of the same degree.

Case 2. $d(v_1) = 1$ and $d(v_2) = 2$ and for all $1 \le i \le n-2$ we have $d(v_{i+2}) - d(v_i) = 1$. As the sum of degrees in a graph is even we may assume that $n \ge 8$ in this case. Hence the degree sequence of G starts with $1, 2, 2, 3, 3, 4, 4, 5$.

If v_1 is not adjacent to v_i for some $2 \le i \le 3$, we can remove v_1 and obtain a graph with three vertices of the same degree. Assume without loss of generality that v_1 is adjacent to v_2 (as $d(v_3) = d(v_2) = 2$). Removing v_1 we obtain a graph

G' in which the degree sequence starts with $1, 2, 3$ and this graph must contain three vertices of the same degree by Theorem 3.1. ■

Theorem 1.3 follows as if graph G has two isolated vertices we can remove a neighbor of a vertex of degree 1 to obtain three vertices of degree 0.

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References

[1] Y. Caro, A. Shapira and R. Yuster, *Forcing k-repetitions in degree sequences*, Electron. J. Combin. **21** (2014) #P1.24.
https://doi.org/10.37236/3503

[2] P. Erdős, G. Chen, C.C. Rousseau and R.H. Schelp, *Ramsey problems involving degrees in edge-colored complete graphs of vertices belonging to monochromatic subgraphs*, European J. Combin. **14** (1993) 183–189.
https://doi.org/10.1006/eujc.1993.1023

[3] P. Erdős, S. Fajtlowicz and W. Staton, *Degree sequences in triangle-free graphs*, Discrete Math. **92** (1991) 85–88.
https://doi.org/10.1016/0012-365X(91)90269-8

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