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ABOUT AN EXTREMAL PROBLEM OF BIGRAPHIC PAIRS WITH A REALIZATION CONTAINING $K_{s,t}^1$

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Abstract

Let $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$, where f_1, \ldots, f_m and g_1, \ldots, g_n are two non-increasing sequences of nonnegative integers. The pair $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ is said to be a *bigraphic pair* if there is a simple bipartite graph $G = (X \cup Y, E)$ such that f_1, \ldots, f_m and g_1, \ldots, g_n are the degrees of the vertices in X and Y, respectively. In this case, G is referred to as a *realization* of π . We say that π is a *potentially* $K_{s,t}$ -*bigraphic pair* if some realization of π contains $K_{s,t}$ (with s vertices in the part of size m and t in the part of size n). Ferrara *et al.* [*Potentially* H-*bigraphic sequences*, Discuss. Math. Graph Theory 29 (2009) 583–596] defined $\sigma(K_{s,t}, m, n)$ to be the minimum integer k such that every bigraphic pair $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ with $\sigma(\pi) =$ $f_1 + \cdots + f_m \ge k$ is potentially $K_{s,t}$ -bigraphic. They determined $\sigma(K_{s,t}, m, n)$ for $n \ge m \ge 9s^4t^4$. In this paper, we first give a procedure and two sufficient conditions to determine if π is a potentially $K_{s,t}$ -bigraphic pair. Then, we determine $\sigma(K_{s,t}, m, n)$ for $n \ge m \ge s$ and $n \ge (s+1)t^2 - (2s-1)t + s - 1$. This provides a solution to a problem due to Ferrara *et al.*

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1. INTRODUCTION

Let $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$, where f_1, \ldots, f_m and g_1, \ldots, g_n are two sequences of nonnegative integers with $f_1 \geq \cdots \geq f_m$ and $g_1 \geq \cdots \geq g_n$. We say that π is a *bigraphic pair* if there is a simple bipartite graph G with partite sets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ such that the degree of x_i is f_i and the degree of y_j is g_j . In this case, we say that G is a *realization* of π .

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Theorem 1 [2, 3]. π is a bigraphic pair if and only if $\sum_{i=1}^{m} f_i = \sum_{i=1}^{n} g_i$ and $\sum_{i=1}^{k} f_i \leq \sum_{i=1}^{n} \min\{k, g_i\}$ for k = 1, ..., m (or $\sum_{i=1}^{k} g_i \leq \sum_{i=1}^{m} \min\{k, f_i\}$ for k = 1, ..., n).

We define $\pi(f_p)$ (respectively, $\pi(g_q)$) to be the pair of two non-increasing sequences of nonnegative integers so that $\pi(f_p)$ (respectively, $\pi(g_q)$) is obtained from π by deleting f_p (respectively, g_q) and decreasing f_p (respectively, g_q) largest terms from g_1, \ldots, g_n each by 1 (respectively, from f_1, \ldots, f_m each by 1). We say that $\pi(f_p)$ (respectively, $\pi(g_q)$) is the residual pair obtained from π by laying off f_p (respectively, g_q).

Theorem 2 [4]. π is a bigraphic pair if and only if $\pi(f_p)$ (respectively, $\pi(g_q)$) is a bigraphic pair.

Let $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ be a bigraphic pair, and let $K_{s,t}$ be the complete bipartite graph with partite sets of size s and t. We say that π is a *potentially* $K_{s,t}$ -bigraphic pair if some realization of π contains $K_{s,t}$ (with s vertices in the part of size m and t in the part of size n). If some realization of π contains $K_{s,t}$ on those vertices having degree $f_1, \ldots, f_s, g_1, \ldots, g_t$, we say that π is a *potentially* $A_{s,t}$ -bigraphic pair. Ferrara et al. [1] proved that π is a potentially $A_{s,t}$ -bigraphic pair if and only if it is a potentially $K_{s,t}$ -bigraphic pair. We now give a procedure to determine if π is a potentially $K_{s,t}$ -bigraphic pair.

Let $f_s \ge t$ and $g_t \ge s$. We define pairs π_0, \ldots, π_s as follows. Let $\pi_0 = \pi$. Let

$$\pi_1 = \left(f_2, \dots, f_m; g_1 - 1, \dots, g_t - 1, g_{t+1}^{(1)}, \dots, g_n^{(1)}\right)$$

where $g_{t+1}^{(1)} \ge \cdots \ge g_n^{(1)}$ is a rearrangement in nonincreasing order of $g_{t+1} - 1, \ldots,$ $g_{f_1} - 1, g_{f_1+1}, \dots, g_n$. For $2 \le i \le s$, given $\pi_{i-1} = (f_i, \dots, f_m; g_1 - i + 1, \dots, g_t - i)$ $i+1, g_{t+1}^{(i-1)}, \dots, g_n^{(i-1)}$, let

$$\pi_i = \left(f_{i+1}, \dots, f_m; g_1 - i, \dots, g_t - i, g_{t+1}^{(i)}, \dots, g_n^{(i)} \right),\,$$

where $g_{t+1}^{(i)} \geq \cdots \geq g_n^{(i)}$ is a rearrangement in nonincreasing order of $g_{t+1}^{(i-1)} - 1, \ldots, g_{f_i}^{(i-1)} - 1, g_{f_i+1}^{(i-1)}, \ldots, g_n^{(i-1)}$. We now define pairs π'_0, \ldots, π'_t as follows. Let $\pi'_0 = \pi$. Let

$$\pi'_1 = \left(f_1 - 1, \dots, f_s - 1, f_{s+1}^{(1)}, \dots, f_m^{(1)}; g_2, \dots, g_n\right),$$

where $f_{s+1}^{(1)} \ge \cdots \ge f_m^{(1)}$ is a rearrangement in nonincreasing order of $f_{s+1} - 1, \ldots, f_{g_1} - 1, f_{g_1+1}, \ldots, f_m$. For $2 \le i \le t$, given $\pi'_{i-1} = (f_1 - i + 1, \ldots, f_s - i + 1, f_{s+1}^{(i-1)}, \ldots, f_m^{(i-1)}; g_i, \ldots, g_n)$, let

$$\pi'_{i} = \left(f_{1} - i, \dots, f_{s} - i, f_{s+1}^{(i)}, \dots, f_{m}^{(i)}; g_{i+1}, \dots, g_{n}\right),\$$

where $f_{s+1}^{(i)} \ge \cdots \ge f_m^{(i)}$ is a rearrangement in nonincreasing order of $f_{s+1}^{(i-1)} - 1, \dots, f_{g_i}^{(i-1)} - 1, f_{g_i+1}^{(i-1)}, \dots, f_m^{(i-1)}$.

Theorem 3. Let $f_s \ge t$ and $g_t \ge s$. Then π is a potentially $K_{s,t}$ -bigraphic pair if and only if π_s (respectively, π'_t) is a bigraphic pair.

We also give two sufficient conditions of π that is potentially $K_{s,t}$ -bigraphic.

Theorem 4. Let $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ be a bigraphic pair with $f_s \ge t$ and $g_t \ge s$. If $n \ge f_{s+1} + t$ and $g_{f_{s+1}+t} \ge s - 1$, then π is a potentially $K_{s,t}$ -bigraphic pair.

Theorem 5. Let $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ be a bigraphic pair with $f_s \ge t$ and $g_t \ge s$. If $m \ge g_{t+1} + s$ and $f_{g_{t+1}+s} \ge t-1$, then π is a potentially $K_{s,t}$ -bigraphic pair.

Ferrara *et al.* [1] investigated an extremal problem of potentially $K_{s,t}$ -bigraphic pairs. They defined $\sigma(K_{s,t}, m, n)$ to be the minimum integer k such that every bigraphic pair $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ with $\sigma(\pi) = f_1 + \cdots + f_m \ge k$ is potentially $K_{s,t}$ -bigraphic. They determined $\sigma(K_{s,t}, m, n)$ when m and n are sufficiently large in terms of s and t. This problem can be viewed as a "potential" degree sequence relaxation of the (forcible) Turán problem.

Theorem 6 [1]. If $1 \le s \le t$ and $n \ge m \ge 9s^4t^4$, then $\sigma(K_{s,t}, m, n) = n(s-1) + m(t-1) - (t-1)(s-1) + 1$.

In [1], Ferrara *et al.* also proposed a problem as follows.

Problem 1. This would be useful if one were interested in finding smaller bounds on the n and m necessary to assure Theorem 6.

As an application of Theorems 4 and 5, we determine $\sigma(K_{s,t}, m, n)$ for $n \ge m \ge s$ and $n \ge (s+1)t^2 - (2s-1)t + s - 1$ (Theorem 7), which is a solution to Problem 1.

Theorem 7. If $1 \le s \le t$, $n \ge m \ge s$ and $n \ge (s+1)t^2 - (2s-1)t + s - 1$, then $\sigma(K_{s,t}, m, n) = n(s-1) + m(t-1) - (t-1)(s-1) + 1$.

2. Proofs of Theorems 3–5 and 7

Proof of Theorem 3. We only need to prove that π is a potentially $A_{s,t}$ bigraphic pair if and only if π_s is a bigraphic pair. The proof for π'_t is similar. If π is a potentially $A_{s,t}$ -bigraphic pair, then π has a realization G with partite sets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ such that $d_G(x_i) = f_i$ for $1 \le i \le m$, $d_G(y_j) = g_j$ for $1 \leq j \leq n$ and the subgraph of G induced by $\{x_1, \ldots, x_s\} \cup \{y_1, \ldots, y_t\}$ is a $K_{s,t}$. We now show that π has such a realization G such that x_1 is adjacent to vertices y_{t+1}, \ldots, y_{f_1} . Otherwise, we may choose such a realization H of π so that the number of vertices adjacent to x_1 in $\{y_{t+1}, \ldots, y_{f_1}\}$ is maximum. Let $y_i \in \{y_{t+1}, \ldots, y_{f_1}\}$ and $x_1y_i \notin E(H)$, and let $y_j \in \{y_{f_1+1}, \ldots, y_n\}$ and $x_1y_j \in E(H)$. We may assume $g_i > g_j$. Hence there is a vertex x_t such that $y_ix_t \in E(H)$ and $y_jx_t \notin E(H)$. Clearly, $H' = (H \setminus \{x_1y_j, y_ix_t\}) \cup \{x_1y_i, y_jx_t\}$ is a realization of π such that $d_{H'}(x_i) = f_i$ for $1 \leq i \leq m$, $d_{H'}(y_j) = g_j$ for $1 \leq j \leq n$ and the subgraph of H' induced by $\{x_1, \ldots, x_s\} \cup \{y_1, \ldots, y_t\}$ is a $K_{s,t}$, and H'has the number of vertices adjacent to x_1 in $\{y_{t+1}, \ldots, y_{f_1}\}$ larger than that of H. This contradicts to the choice of H. Clearly, π_1 is the degree sequence pair of $G - x_1$ and is a potentially $A_{s-1,t}$ -bigraphic pair. Repeating this method, we can see that π_i is a potentially $A_{s-i,t}$ -bigraphic pair for $i = 2, \ldots, s$ in turn. In particular, π_s is a bigraphic pair.

Suppose that π_s is a bigraphic pair and is realized by bipartite graph G_s with partite sets $\{x_{s+1}, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ such that $d_{G_s}(x_i) = f_i$ for $s+1 \leq i \leq m$, $d_{G_s}(y_i) = g_i - s$ for $1 \leq i \leq t$ and $d_{G_s}(y_i) = g_i^{(s)}$ for $t+1 \leq i \leq n$. For $i = s, \ldots, 1$ in turn, form G_{i-1} from G_i by adding a new vertex x_i that is adjacent to y_1, \ldots, y_t and also adjacent to vertices of G_i with degrees $g_{t+1}^{(i-1)} - 1, \ldots, g_{f_i}^{(i-1)} - 1$. Then, for each i, G_i has the degree sequence pair given by π_i , and G_i contains $K_{s-i,t}$ on vertices $x_{i+1}, \ldots, x_s, y_1, \ldots, y_t$ whose degrees are $f_{i+1}, \ldots, f_s, g_1 - i, \ldots, g_t - i$ so that $\{x_{i+1}, \ldots, x_s\}$ and $\{y_1, \ldots, y_t\}$ is the partite sets of $K_{s-i,t}$. In particular, G_0 has the degree sequence pair given by π and contains $K_{s,t}$ on vertices $x_1, \ldots, x_s, y_1, \ldots, y_t$ whose degrees are $f_1, \ldots, f_s, g_1, \ldots, g_t$ so that $\{x_1, \ldots, x_s\}$ and $\{y_1, \ldots, y_t\}$ is the partite sets of $K_{s,t}$.

In order to prove Theorem 4, we need the following lemmas.

Lemma 8 [5]. Theorem 1 remains valid if $\sum_{i=1}^{k} f_i \leq \sum_{i=1}^{n} \min\{k, g_i\}$ is assumed only for those k for which $f_k > f_{k+1}$ or k = m (or $\sum_{i=1}^{k} g_i \leq \sum_{i=1}^{m} \min\{k, f_i\}$ is assumed only for those k for which $g_k > g_{k+1}$ or k = n).

Lemma 9. Let $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ be a bigraphic pair with $f_s \ge t, g_t \ge s$, $m-1 \ge g_1 \ge \cdots \ge g_t = \cdots = g_{f_1+1} \ge g_{f_1+2} \ge \cdots \ge g_n$ and $n-1 \ge f_1 \ge \cdots \ge f_s = \cdots = f_{g_1+1} \ge f_{g_1+2} \ge \cdots \ge f_m$. For $\pi_i = (f_{i+1}, \ldots, f_m; g_1 - i, \ldots, g_t - i, g_{t+1}^{(i)}, \ldots, g_n^{(i)})$ with $0 \le i \le s$, let $t_i = \max \{j | g_{t+1}^{(i)} - g_{t+j}^{(i)} \le 1\}$. Then

- (1) $t_s \ge t_{s-1} \ge \dots \ge t_0 \ge f_1 + 1 t.$
- (2) For $i, 1 \leq i \leq s$, we have $g_{t+k}^{(i)} = g_{t+k}^{(i-1)}$ for $k > t_i$. Thus, $g_{t+k}^{(s)} = g_{t+k}$ for $k > t_s$.

 $\begin{array}{l} \textit{Proof.} \ (1) \ \text{Clearly}, t+t_0 \geq f_1+1, \text{ i.e., } t_0 \geq f_1+1-t. \ \text{Moreover}, g_{t+1}^{(i-1)}-g_{t+t_{i-1}}^{(i-1)} \\ \leq 1, \text{ which implies } g_{t+1}^{(i)}-g_{t+t_{i-1}}^{(i)} \leq 1 \ \text{for } 1 \leq i \leq s. \ \text{Hence} \ t_i \geq t_{i-1} \ \text{for } 1 \leq i \leq s. \\ (2) \ \text{By} \ \min \left\{ g_{t+1}^{(i-1)}-1, \ldots, g_{f_i}^{(i-1)}-1, g_{f_i+1}^{(i-1)}, \ldots, g_{t+t_{i-1}}^{(i-1)} \right\} \geq g_{t+1}^{(i-1)}-2 \geq g_{t+t_{i-1}+1}^{(i-1)} \geq \cdots \geq g_n^{(i-1)}, \text{ we have } g_{t+t_{i-1}+k'}^{(i)} = g_{t+t_{i-1}+k'}^{(i-1)} \ \text{for } k' \geq 1. \ \text{Thus } g_{t+k}^{(i)} = g_{t+k}^{(i-1)} \ \text{for } k > t_i. \end{array}$

Lemma 10. Let $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ be a bigraphic pair with $f_s \geq t$, $g_t \geq s, m-1 \geq g_1 \geq \cdots \geq g_t = \cdots = g_{f_1+1} \geq g_{f_1+2} \geq \cdots \geq g_n$ and $n-1 \geq f_1 \geq \cdots \geq f_s = \cdots = f_{g_1+1} \geq f_{g_1+2} \geq \cdots \geq f_m$. If $n \geq f_{s+1} + t$ and $g_{f_{s+1}+t} \geq s-1$, then π is a potentially $A_{s,t}$ -bigraphic pair.

Proof. It is trivial for s = 1. Assume $s \ge 2$. By Theorem 3, we only need to check that $\pi_s = (f_{s+1}, \ldots, f_m; g_1 - s, \ldots, g_t - s, g_{t+1}^{(s)}, \ldots, g_n^{(s)})$ is a bigraphic pair. Clearly, $f_{s+1} + \cdots + f_m = (g_1 - s) + \cdots + (g_t - s) + g_{t+1}^{(s)} + \cdots + g_n^{(s)}$. Denote $p = \max\{i \mid f_{s+i} = f_s\}$. Then $s + p \ge g_1 + 1$, i.e., $p \ge g_1 + 1 - s$. By Lemma 8, it is enough to check that $\sum_{i=1}^k f_{s+i} \le \sum_{i=1}^t \min\{k, g_i - s\} + \sum_{i=t+1}^n \min\{k, g_i^{(s)}\}$ for $p \le k \le m - s$. Denote $x = g_{t+1}^{(s)}$ and $y = g_t - s$. If $k \ge x$, by $k \ge p \ge g_1 + 1 - s > g_i - s$ for $1 \le i \le t$, then $\sum_{i=1}^t \min\{k, g_i - s\} + \sum_{i=t+1}^n \min\{k, g_i^{(s)}\} = \sum_{i=1}^t (g_i - s) + \sum_{i=t+1}^n g_i^{(s)} = f_{s+1} + \cdots + f_m \ge \sum_{i=1}^k f_{s+i}$. Assume $p \le k \le x - 1$. Clearly, $y + s = g_t \ge g_{t+1} \ge x$, i.e., $(x-1) - y \le s - 1$. This implies $k - y \le s - 1$. Moreover, by Lemma 9, $g_{f_{s+1}+t}^{(s)} \ge x - 1$ if $t + t_s \ge f_{s+1} + t$. Hence $g_{f_{s+1}+t}^{(s)} \ge \min\{x - 1, s - 1\} \ge k - y$. Thus by $t + t_s \ge f_1 + 1 > f_{s+1}$, we have $\sum_{i=1}^t \min\{k, g_i - s\} + \sum_{i=t+1}^n \min\{k, g_i^{(s)}\} = \sum_{i=1}^t (g_i - s) + \sum_{i=t+1}^{f_{s+1}} \min\{k, g_i^{(s)}\} + \sum_{i=t+1}^{n} \min\{k, g_i^{(s)}\} = \sum_{i=1}^t (g_i - s) + \sum_{i=t+1}^{f_{s+1}} \min\{k, g_i^{(s)}\} + \sum_{i=t+1}^n \min\{k, g_i^{(s)}\} = \sum_{i=1}^t (g_i - s) + \sum_{i=t+1}^{f_{s+1}} \min\{k, g_i^{(s)}\} + \sum_{i=t+1}^n \min\{k, g_i^{(s)}\} = \sum_{i=1}^t (g_i - s) + \sum_{i=t+1}^{f_{s+1}} \min\{k, g_i^{(s)}\} + \sum_{i=t+1}^n \min\{k, g_i^{(s)}\} = \sum_{i=1}^t (g_i - s) + \sum_{i=t+1}^{f_{s+1}} \min\{k, g_i^{(s)}\} + \sum_{i=t+1}^{f_{s+1}} \min\{k, g_i^{(s)}\} = \sum_{i=1}^t (g_i - s) + \sum_{i=t+1}^{f_{s+1}} \min\{k, g_i^{(s)}\} + \sum_{i=t+1}^{f_{s+1}} \min\{k, g_i^{(s)}\} = \sum_{i=1}^t (g_i - s) + \sum_{i=t+1}^{f_{s+1}} \min\{k, g_i^{(s)}\} + \sum_{i=t+1}^{f_{s+1}} \min\{k, g_i^{(s)}\} = \sum_{i=1}^t (g_i - s) + \sum_{i=t+1}^{f_{s+1}} \min\{k, g_i^{(s)}\} + \sum_{i=t+1}^{f_{s+1}} \min\{k, g_i^{(s)}\} = \sum_{i=1}^t (g_i - s) + \sum_{i=t+1}^{f_{s+1}} \min\{k, g_i^{(s)}\} + \sum_{i=t+1}^{f_{s+1}} \min\{k, g_i^{(s)}\} = \sum_{i=1}^t (g_i - s) + \sum_{i=t+1}^{f_{s+1}} \min\{k, g_i^{(s)}\} + \sum_{i=t+1}^{f_{s+$

Proof of Theorem 4. We use induction on s + t. It is trivial for s = 1 or t = 1. Assume $s \ge 2$ and $t \ge 2$. If $f_1 = n$ or there exists an integer k with $t \le k \le f_1$ such that $g_k > g_{k+1}$, then the residual pair $\pi(f_1) = (f_2, \ldots, f_m; g'_1, \ldots, g'_n)$ obtained from π by laying off f_1 satisfies $g'_t \ge g_t - 1 \ge s - 1$, $g'_{f_{s+1}+t} \ge g_{f_{s+1}+t} - 1 \ge s - 2$ and $g'_1 = g_1 - 1, \ldots, g'_t = g_t - 1$. By Theorem 2 and the induction hypothesis, $\pi(f_1)$ is a potentially $A_{s-1,t}$ -bigraphic pair, and hence π is a potentially $A_{s,t}$ -bigraphic pair. So we may assume $f_1 \le n - 1$ and $g_1 \ge \cdots \ge g_t = \cdots = g_{f_1+1} \ge g_{f_1+2} \ge \cdots \ge g_n$. If $g_1 = m$ or there exists an integer k with $s \le k \le g_1$ such that $f_k > f_{k+1}$, then the residual pair $\pi(g_1) = (f'_1, \ldots, f'_m; g_2, \ldots, g_n)$ obtained from π by laying off g_1 satisfies $f'_s \ge f_s - 1 \ge t - 1$, $n - 1 \ge f_{s+1} + (t - 1) \ge f'_{s+1} + (t - 1)$, $g_{1+(f'_{s+1}+(t-1))} \ge g_{f_{s+1}+t} \ge s - 1$ and $f'_1 = f_1 - 1, \ldots, f'_s = f_s - 1$. By Theorem 2 and the induction hypothesis, $\pi(g_1)$ is a potentially $A_{s,t-1}$ -bigraphic pair, and $f'_1 = f_1 - 1, \ldots, f'_s = f_s - 1$. hence π is a potentially $A_{s,t}$ -bigraphic pair. So we may further assume $g_1 \leq m-1$ and $f_1 \geq \cdots \geq f_s = \cdots = f_{g_1+1} \geq f_{g_1+2} \geq \cdots \geq f_m$. Thus by Lemma 10, π is a potentially $A_{s,t}$ -bigraphic pair.

Proof of Theorem **5.** By symmetry, the proof of Theorem 5 is similar to that of Theorem 4. ■

As an application of Theorems 4 and 5, we now prove Theorem 7. We first need some lemmas.

Lemma 11 [5]. If $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ is a bigraphic pair with $f_s \ge 2t - 1$ and $g_t \ge 2s - 1$, then π is a potentially $K_{s,t}$ -bigraphic pair.

Lemma 12. Let $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ be a bigraphic pair with $f_s \ge t$ and $g_t \ge s$. If $n \ge st - s + t$ and $g_n \ge s - 1$, then π is a potentially $K_{s,t}$ -bigraphic pair.

Proof. If $f_{s+1} \leq n-t$, i.e., $n \geq f_{s+1}+t$, by Theorem 4, π is a potentially $A_{s,t}$ -bigraphic pair. Assume $f_{s+1} \geq n - (t-1)$. Let G be a realization of π with partite sets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ such that $d_G(x_i) = f_i$ for $1 \leq i \leq m$ and $d_G(y_j) = g_j$ for $1 \leq j \leq n$. For each i, denote $N_G(x_i)$ to be the neighbor set of x_i in G. Clearly, $|N_G(x_i)| = d_G(x_i) = f_i \geq n - (t-1)$ for $1 \leq i \leq s$. Moreover, we have the following Claim 1.

Claim 1. $|N_G(x_1) \cap \cdots \cap N_G(x_s)| \ge n - s(t-1).$

Proof. Clearly, $|N_G(x_1)| \ge n - (t-1)$. Assume $2 \le k \le s$ and $|N_G(x_1) \cap \cdots \cap N_G(x_{k-1})| \ge n - (k-1)(t-1)$. Denote $A = N_G(x_1) \cap \cdots \cap N_G(x_{k-1})$. Then $|N_G(x_1) \cap \cdots \cap N_G(x_k)| = |A \cap N_G(x_k)| = |A| + |N_G(x_k)| - |A \cup N_G(x_k)| \ge (n - (k-1)(t-1)) + (n - (t-1)) - n = n - k(t-1)$. This proves Claim 1. \Box

By Claim 1, $|N_G(x_1) \cap \cdots \cap N_G(x_s)| \ge n - s(t-1) \ge t$. This implies that G contains $K_{s,t}$. Hence, π is a potentially $K_{s,t}$ -bigraphic pair.

Lemma 13. Let $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ be a bigraphic pair with $f_s \ge t$ and $g_t \ge s$. If $m \ge st - t + s$ and $f_m \ge t - 1$, then π is a potentially $K_{s,t}$ -bigraphic pair.

Proof. By symmetry, the proof of Lemma 13 is similar to that of Lemma 12.

Lemma 14. Let $1 \leq s \leq t$, $m \geq s$, $n \geq t$ and $n \geq m$. If $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ is a bigraphic pair with $\sigma(\pi) \geq n(s-1) + m(t-1) + n(t-1) - 2(t-1)(s-1) + 1$, then π is a potentially $K_{s,t}$ -bigraphic pair.

Proof. If $f_s \le 2t - 2$, then $\sigma(\pi) \le n(s-1) + (m-s+1)(2t-2) = n(s-1) + m(t-1) + m(t-1) - 2(t-1)(s-1) \le n(s-1) + m(t-1) + n(t-1) - 2(t-1)(s-1)$,

a contradiction. Hence $f_s \ge 2t - 1$. If $g_t \le 2s - 2$, then $\sigma(\pi) \le m(t-1) + (n-t+1)(2s-2) = n(s-1) + m(t-1) + n(s-1) - 2(t-1)(s-1) \le n(s-1) + m(t-1) + n(t-1) - 2(t-1)(s-1)$, a contradiction. Hence $g_t \ge 2s - 1$. Thus by Lemma 11, π is a potentially $K_{s,t}$ -bigraphic pair.

Lemma 15. Let $1 \le s \le t$, $n \ge m \ge s$ and n = (st - s + t) + k with $0 \le k \le (t - 1)(st - 2s + t + 1)$. If $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ is a bigraphic pair with $\sigma(\pi) \ge n(s-1) + m(t-1) - (t-1)(s-1) + 1 + (t-1)(st - 2s + t + 1) - k$, then π is a potentially $K_{s,t}$ -bigraphic pair.

Proof. We use induction on k. If k = 0, then n = st - s + t and $\sigma(\pi) \geq st - s + t$ n(s-1) + m(t-1) - (t-1)(s-1) + 1 + (t-1)(st-2s+t+1) = n(s-1) + nm(t-1) + n(t-1) - 2(t-1)(s-1) + 1. By Lemma 14, π is a potentially $K_{s,t}$ -bigraphic pair. Suppose that $1 \leq k \leq (t-1)(st-2s+t+1)$. Then $\sigma(\pi) \ge n(s-1) + m(t-1) - (t-1)(s-1) + 1$. It is straightforward to show that $f_s \geq t$ and $g_t \geq s$. If $g_n \geq s-1$, then by Lemma 12, π is a potentially $K_{s,t}$ bigraphic pair. Assume $g_n \leq s-2$. If $n \geq m+1$, then the residual pair $\pi(g_n) =$ $(f'_1,\ldots,f'_m;g_1,\ldots,g_{n-1})$ obtained from π by laying off g_n satisfies $\sigma(\pi(g_n)) =$ $\sigma(\pi) - g_n \ge n(s-1) + m(t-1) - (t-1)(s-1) + 1 + (t-1)(st-2s+t+1) - k - (s-2) = n(s-1) + n(s$ (n-1)(s-1) + m(t-1) - (t-1)(s-1) + 1 + (t-1)(st-2s+t+1) - (k-1).By Theorem 2 and the induction hypothesis, $\pi(q_n)$ is a potentially $K_{s,t}$ -bigraphic pair, and hence so is π . Further assume n = m. Then $m \ge st - s + t \ge st - t + s$. If $f_m \ge t-1$, then by Lemma 13, π is a potentially $K_{s,t}$ -bigraphic pair. If $f_m \le t-2$, let $\pi(g_n, f'_m) = (f'_1, \dots, f'_{m-1}; g'_1, \dots, g'_{n-1})$ be the residual pair obtained from $\pi(g_n)$ by laying off f'_m , by $f'_m \leq f_m$, then $\sigma(\pi(g_n, f'_m)) = \sigma(\pi) - g_n - f'_m \geq 0$ $n(s-1)+m(t-1)-(t-1)(s-1)+1+(t-1)(st-2s+t+1)-k-(s-2)-(t-2)\geq 0$ (n-1)(s-1)+(m-1)(t-1)-(t-1)(s-1)+1+(t-1)(st-2s+t+1)-(k-1). By Theorem 2 and the induction hypothesis, it follows that $\pi(g_n, f'_m)$ is a potentially $K_{s,t}$ -bigraphic pair. Thus, both $\pi(g_n)$ and π are potentially $K_{s,t}$ -bigraphic.

Lemma 16. Let $1 \le s \le t$, $n \ge m \ge s$ and $n \ge (s+1)t^2 - (2s-1)t + s - 1$. If $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ is a bigraphic pair with $\sigma(\pi) \ge n(s-1) + m(t-1) - (t-1)(s-1) + 1$, then π is a potentially $K_{s,t}$ -bigraphic pair.

Proof. We use induction on n. Clearly, the result for $n = (s+1)t^2 - (2s-1)t + s - 1$ follows from Lemma 15 by letting k = (t-1)(st-2s+t+1). Assume $n \ge (s+1)t^2 - (2s-1)t + s$. Clearly, $f_s \ge t$, $g_t \ge s$ and $n \ge st-s+t$. If $g_n \ge s-1$, then by Lemma 12, π is a potentially $K_{s,t}$ -bigraphic pair. Assume $g_n \le s-2$. If $n \ge m+1$, then the residual pair $\pi(g_n) = (f'_1, \ldots, f'_m; g_1, \ldots, g_{n-1})$ obtained from π by laying off g_n satisfies $\sigma(\pi(g_n)) = \sigma(\pi) - g_n \ge n(s-1) + m(t-1) - (t-1)(s-1) + 1 - (s-2) \ge (n-1)(s-1) + m(t-1) - (t-1)(s-1) + 1$. By Theorem 2 and the induction hypothesis, $\pi(g_n)$ is a potentially $K_{s,t}$ -bigraphic pair, and hence so is π . Further assume n = m. Then $m \ge st-s+t \ge st-t+s$. If $f_m \ge t-1$, then by Lemma 13, π is a potentially $K_{s,t}$ -bigraphic pair. If $f_m \leq t-2$, let $\pi(g_n, f'_m) = (f'_1, \ldots, f'_{m-1}; g'_1, \ldots, g'_{n-1})$ be the residual pair obtained from $\pi(g_n)$ by laying off f'_m , by $f'_m \leq f_m$, then $\sigma(\pi(g_n, f'_m)) = \sigma(\pi) - g_n - f'_m \geq n(s-1) + m(t-1) - (t-1)(s-1) + 1 - (s-2) - (t-2) \geq (n-1)(s-1) + (m-1)(t-1) - (t-1)(s-1) + 1$. By Theorem 2 and the induction hypothesis, it follows that $\pi(g_n, f'_m)$ is a potentially $K_{s,t}$ -bigraphic pair. Thus, both $\pi(g_n)$ and π are potentially $K_{s,t}$ -bigraphic.

Proof of Theorem 7. Ferrara *et al.* [1] considered the bigraphic pair $\pi = (n^{s-1}, (t-1)^{m-s+1}; m^{s-1}, (t-1)^{m-s+1}, (s-1)^{n-m})$, where the symbol x^y stands for y consecutive terms, each equal to x. Clearly, π is not a potentially $K_{s,t}$ -bigraphic pair. Thus $\sigma(K_{s,t}, m, n) \geq \sigma(\pi) + 1 = n(s-1) + m(t-1) - (t-1)(s-1) + 1$. The upper bound directly follows from Lemma 16.

Remark. The lower bound $n \ge (s+1)t^2 - (2s-1)t + s - 1$ in Theorem 7 is not the best lower bound. However, we will investigate a lower bound on n + m so that the extremal function value $\sigma(K_{s,t}, m, n) = n(s-1) + m(t-1) - (t-1)(s-1) + 1$, which implies a smaller bound on the n or m necessary to assure Theorem 7. It also would be a meaningful further research for $n \ge m \ge s$ and s > t. We will consider this problem in our future studies.

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References

- M.J. Ferrara, M.S. Jacobson, J.R. Schmitt and M. Siggers, *Potentially H-bigraphic sequences*, Discuss. Math. Graph Theory **29** (2009) 583–596. https://doi.org/10.7151/dmgt.1466
- [2] D. Gale, A theorem on flows in networks, Pacific J. Math. 7 (1957) 1073–1082. https://doi.org/10.2140/pjm.1957.7.1073
- H.J. Ryser, Combinatorial properties of matrices of zeros and ones, Canad. J. Math. 9 (1957) 371–377. https://doi.org/10.4153/CJM-1957-044-3
- [4] J.H. Yin, An extremal problem on bigraphic pairs with an A-connected realization, Discrete Math. 339 (2016) 2018–2026. https://doi.org/10.1016/j.disc.2016.02.014
- [5] J.H. Yin, A note on potentially $K_{s,t}$ -bigraphic pairs, Util. Math. **100** (2016) 407–410.

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