Discussiones Mathematicae
Graph Theory 43 (2023) 437–444
https://doi.org/10.7151/dmgt.2375

ABOUT AN EXTREMAL PROBLEM OF BIGRAPHIC PAIRS WITH A REALIZATION CONTAINING $K_{s,t}$[1]

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Abstract

Let $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$, where $f_1, \ldots, f_m$ and $g_1, \ldots, g_n$ are two non-increasing sequences of nonnegative integers. The pair $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ is said to be a *bigraphic pair* if there is a simple bipartite graph $G = (X \cup Y, E)$ such that $f_1, \ldots, f_m$ and $g_1, \ldots, g_n$ are the degrees of the vertices in X and Y, respectively. In this case, G is referred to as a *realization* of π. We say that π is a *potentially $K_{s,t}$-bigraphic pair* if some realization of π contains $K_{s,t}$ (with s vertices in the part of size m and t in the part of size n). Ferrara *et al.* [*Potentially H-bigraphic sequences*, Discuss. Math. Graph Theory 29 (2009) 583–596] defined $\sigma(K_{s,t}, m, n)$ to be the minimum integer k such that every bigraphic pair $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ with $\sigma(\pi) = f_1 + \cdots + f_m \geq k$ is potentially $K_{s,t}$-bigraphic. They determined $\sigma(K_{s,t}, m, n)$ for $n \geq m \geq 9s^4t^4$. In this paper, we first give a procedure and two sufficient conditions to determine if π is a potentially $K_{s,t}$-bigraphic pair. Then, we determine $\sigma(K_{s,t}, m, n)$ for $n \geq m \geq s$ and $n \geq (s+1)t^2 - (2s-1)t + s - 1$. This provides a solution to a problem due to Ferrara *et al.*

Keywords: bigraphic pair, realization, potentially $K_{s,t}$-bigraphic pair.

2010 Mathematics Subject Classification: 05C35, 05C07.

1. Introduction

Let $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$, where $f_1, \ldots, f_m$ and $g_1, \ldots, g_n$ are two sequences of nonnegative integers with $f_1 \geq \cdots \geq f_m$ and $g_1 \geq \cdots \geq g_n$. We say that π is a *bigraphic pair* if there is a simple bipartite graph G with partite sets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ such that the degree of x_i is f_i and the degree of y_j is g_j. In this case, we say that G is a *realization* of π.

[1]Supported by National Natural Science Foundation of China (No. 11961019).

Theorem 1 [2, 3]**.** *π is a bigraphic pair if and only if $\sum_{i=1}^{m} f_i = \sum_{i=1}^{n} g_i$ and $\sum_{i=1}^{k} f_i \le \sum_{i=1}^{n} \min\{k, g_i\}$ for $k = 1, \ldots, m$ (or $\sum_{i=1}^{k} g_i \le \sum_{i=1}^{m} \min\{k, f_i\}$ for $k = 1, \ldots, n$).*

We define $\pi(f_p)$ (respectively, $\pi(g_q)$) to be the pair of two non-increasing sequences of nonnegative integers so that $\pi(f_p)$ (respectively, $\pi(g_q)$) is obtained from π by deleting f_p (respectively, g_q) and decreasing f_p (respectively, g_q) largest terms from $g_1, \ldots, g_n$ each by 1 (respectively, from $f_1, \ldots, f_m$ each by 1). We say that $\pi(f_p)$ (respectively, $\pi(g_q)$) is the *residual pair obtained from π by laying off f_p (respectively, g_q)*.

Theorem 2 [4]**.** *π is a bigraphic pair if and only if $\pi(f_p)$ (respectively, $\pi(g_q)$) is a bigraphic pair.*

Let $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ be a bigraphic pair, and let $K_{s,t}$ be the complete bipartite graph with partite sets of size s and t. We say that π is a *potentially $K_{s,t}$-bigraphic pair* if some realization of π contains $K_{s,t}$ (with s vertices in the part of size m and t in the part of size n). If some realization of π contains $K_{s,t}$ on those vertices having degree $f_1, \ldots, f_s, g_1, \ldots, g_t$, we say that π is a *potentially $A_{s,t}$-bigraphic pair.* Ferrara *et al.* [1] proved that π is a potentially $A_{s,t}$-bigraphic pair if and only if it is a potentially $K_{s,t}$-bigraphic pair. We now give a procedure to determine if π is a potentially $K_{s,t}$-bigraphic pair.

Let $f_s \ge t$ and $g_t \ge s$. We define pairs $\pi_0, \ldots, \pi_s$ as follows. Let $\pi_0 = \pi$. Let

$$\pi_1 = \left(f_2, \ldots, f_m; g_1 - 1, \ldots, g_t - 1, g_{t+1}^{(1)}, \ldots, g_n^{(1)}\right),$$

where $g_{t+1}^{(1)} \ge \cdots \ge g_n^{(1)}$ is a rearrangement in nonincreasing order of $g_{t+1} - 1, \ldots, g_{f_1} - 1, g_{f_1+1}, \ldots, g_n$. For $2 \le i \le s$, given $\pi_{i-1} = \left(f_i, \ldots, f_m; g_1 - i + 1, \ldots, g_t - i + 1, g_{t+1}^{(i-1)}, \ldots, g_n^{(i-1)}\right)$, let

$$\pi_i = \left(f_{i+1}, \ldots, f_m; g_1 - i, \ldots, g_t - i, g_{t+1}^{(i)}, \ldots, g_n^{(i)}\right),$$

where $g_{t+1}^{(i)} \ge \cdots \ge g_n^{(i)}$ is a rearrangement in nonincreasing order of $g_{t+1}^{(i-1)} - 1, \ldots, g_{f_i}^{(i-1)} - 1, g_{f_i+1}^{(i-1)}, \ldots, g_n^{(i-1)}$.

We now define pairs $\pi'_0, \ldots, \pi'_t$ as follows. Let $\pi'_0 = \pi$. Let

$$\pi'_1 = \left(f_1 - 1, \ldots, f_s - 1, f_{s+1}^{(1)}, \ldots, f_m^{(1)}; g_2, \ldots, g_n\right),$$

where $f_{s+1}^{(1)} \ge \cdots \ge f_m^{(1)}$ is a rearrangement in nonincreasing order of $f_{s+1} - 1, \ldots, f_{g_1} - 1, f_{g_1+1}, \ldots, f_m$. For $2 \le i \le t$, given $\pi'_{i-1} = \left(f_1 - i + 1, \ldots, f_s - i + 1, f_{s+1}^{(i-1)}, \ldots, f_m^{(i-1)}; g_i, \ldots, g_n\right)$, let

$$\pi'_i = \left(f_1 - i, \ldots, f_s - i, f_{s+1}^{(i)}, \ldots, f_m^{(i)}; g_{i+1}, \ldots, g_n\right),$$

where $f_{s+1}^{(i)} \geq \cdots \geq f_m^{(i)}$ is a rearrangement in nonincreasing order of $f_{s+1}^{(i-1)}-1, \ldots, f_{g_i}^{(i-1)} - 1, f_{g_i+1}^{(i-1)}, \ldots, f_m^{(i-1)}$.

Theorem 3. *Let $f_s \geq t$ and $g_t \geq s$. Then π is a potentially $K_{s,t}$-bigraphic pair if and only if π_s (respectively, π'_t) is a bigraphic pair.*

We also give two sufficient conditions of π that is potentially $K_{s,t}$-bigraphic.

Theorem 4. *Let $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ be a bigraphic pair with $f_s \geq t$ and $g_t \geq s$. If $n \geq f_{s+1} + t$ and $g_{f_{s+1}+t} \geq s - 1$, then π is a potentially $K_{s,t}$-bigraphic pair.*

Theorem 5. *Let $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ be a bigraphic pair with $f_s \geq t$ and $g_t \geq s$. If $m \geq g_{t+1} + s$ and $f_{g_{t+1}+s} \geq t - 1$, then π is a potentially $K_{s,t}$-bigraphic pair.*

Ferrara *et al.* [1] investigated an extremal problem of potentially $K_{s,t}$-bigraphic pairs. They defined $\sigma(K_{s,t}, m, n)$ to be the minimum integer k such that every bigraphic pair $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ with $\sigma(\pi) = f_1 + \cdots + f_m \geq k$ is potentially $K_{s,t}$-bigraphic. They determined $\sigma(K_{s,t}, m, n)$ when m and n are sufficiently large in terms of s and t. This problem can be viewed as a "potential" degree sequence relaxation of the (forcible) Turán problem.

Theorem 6 [1]**.** *If $1 \leq s \leq t$ and $n \geq m \geq 9s^4t^4$, then $\sigma(K_{s,t}, m, n) = n(s-1) + m(t-1) - (t-1)(s-1) + 1$.*

In [1], Ferrara *et al.* also proposed a problem as follows.

Problem 1. This would be useful if one were interested in finding smaller bounds on the n and m necessary to assure Theorem 6.

As an application of Theorems 4 and 5, we determine $\sigma(K_{s,t}, m, n)$ for $n \geq m \geq s$ and $n \geq (s+1)t^2 - (2s-1)t + s - 1$ (Theorem 7), which is a solution to Problem 1.

Theorem 7. *If $1 \leq s \leq t$, $n \geq m \geq s$ and $n \geq (s+1)t^2 - (2s-1)t + s - 1$, then $\sigma(K_{s,t}, m, n) = n(s-1) + m(t-1) - (t-1)(s-1) + 1$.*

2. Proofs of Theorems 3–5 and 7

***Proof of Theorem* 3.** We only need to prove that π is a potentially $A_{s,t}$-bigraphic pair if and only if π_s is a bigraphic pair. The proof for π'_t is similar. If π is a potentially $A_{s,t}$-bigraphic pair, then π has a realization G with partite sets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ such that $d_G(x_i) = f_i$ for $1 \leq i \leq m$, $d_G(y_j) = g_j$

for $1 \le j \le n$ and the subgraph of G induced by $\{x_1, \dots, x_s\} \cup \{y_1, \dots, y_t\}$ is a $K_{s,t}$. We now show that π has such a realization G such that x_1 is adjacent to vertices $y_{t+1}, \dots, y_{f_1}$. Otherwise, we may choose such a realization H of π so that the number of vertices adjacent to x_1 in $\{y_{t+1}, \dots, y_{f_1}\}$ is maximum. Let $y_i \in \{y_{t+1}, \dots, y_{f_1}\}$ and $x_1 y_i \notin E(H)$, and let $y_j \in \{y_{f_1+1}, \dots, y_n\}$ and $x_1 y_j \in E(H)$. We may assume $g_i > g_j$. Hence there is a vertex x_t such that $y_i x_t \in E(H)$ and $y_j x_t \notin E(H)$. Clearly, $H' = (H \setminus \{x_1 y_j, y_i x_t\}) \cup \{x_1 y_i, y_j x_t\}$ is a realization of π such that $d_{H'}(x_i) = f_i$ for $1 \le i \le m$, $d_{H'}(y_j) = g_j$ for $1 \le j \le n$ and the subgraph of H' induced by $\{x_1, \dots, x_s\} \cup \{y_1, \dots, y_t\}$ is a $K_{s,t}$, and H' has the number of vertices adjacent to x_1 in $\{y_{t+1}, \dots, y_{f_1}\}$ larger than that of H. This contradicts to the choice of H. Clearly, π_1 is the degree sequence pair of $G - x_1$ and is a potentially $A_{s-1,t}$-bigraphic pair. Repeating this method, we can see that π_i is a potentially $A_{s-i,t}$-bigraphic pair for $i = 2, \dots, s$ in turn. In particular, π_s is a bigraphic pair.

Suppose that π_s is a bigraphic pair and is realized by bipartite graph G_s with partite sets $\{x_{s+1}, \dots, x_m\}$ and $\{y_1, \dots, y_n\}$ such that $d_{G_s}(x_i) = f_i$ for $s+1 \le i \le m$, $d_{G_s}(y_i) = g_i - s$ for $1 \le i \le t$ and $d_{G_s}(y_i) = g_i^{(s)}$ for $t+1 \le i \le n$. For $i = s, \dots, 1$ in turn, form G_{i-1} from G_i by adding a new vertex x_i that is adjacent to $y_1, \dots, y_t$ and also adjacent to vertices of G_i with degrees $g_{t+1}^{(i-1)} - 1, \dots, g_{f_i}^{(i-1)} - 1$. Then, for each i, G_i has the degree sequence pair given by π_i, and G_i contains $K_{s-i,t}$ on vertices $x_{i+1}, \dots, x_s, y_1, \dots, y_t$ whose degrees are $f_{i+1}, \dots, f_s, g_1 - i, \dots, g_t - i$ so that $\{x_{i+1}, \dots, x_s\}$ and $\{y_1, \dots, y_t\}$ is the partite sets of $K_{s-i,t}$. In particular, G_0 has the degree sequence pair given by π and contains $K_{s,t}$ on vertices $x_1, \dots, x_s, y_1, \dots, y_t$ whose degrees are $f_1, \dots, f_s, g_1, \dots, g_t$ so that $\{x_1, \dots, x_s\}$ and $\{y_1, \dots, y_t\}$ is the partite sets of $K_{s,t}$. ■

In order to prove Theorem 4, we need the following lemmas.

Lemma 8 [5]. *Theorem* 1 *remains valid if* $\sum_{i=1}^k f_i \le \sum_{i=1}^n \min\{k, g_i\}$ *is assumed only for those* k *for which* $f_k > f_{k+1}$ *or* $k = m$ *(or* $\sum_{i=1}^k g_i \le \sum_{i=1}^m \min\{k, f_i\}$ *is assumed only for those* k *for which* $g_k > g_{k+1}$ *or* $k = n$*).*

Lemma 9. *Let* $\pi = (f_1, \dots, f_m; g_1, \dots, g_n)$ *be a bigraphic pair with* $f_s \ge t$, $g_t \ge s$, $m - 1 \ge g_1 \ge \cdots \ge g_t = \cdots = g_{f_1+1} \ge g_{f_1+2} \ge \cdots \ge g_n$ *and* $n - 1 \ge f_1 \ge \cdots \ge f_s = \cdots = f_{g_1+1} \ge f_{g_1+2} \ge \cdots \ge f_m$. *For* $\pi_i = \big(f_{i+1}, \dots, f_m; g_1 - i, \dots, g_t - i, g_{t+1}^{(i)}, \dots, g_n^{(i)}\big)$ *with* $0 \le i \le s$, *let* $t_i = \max\big\{j | g_{t+1}^{(i)} - g_{t+j}^{(i)} \le 1\big\}$. *Then*

(1) $t_s \ge t_{s-1} \ge \cdots \ge t_0 \ge f_1 + 1 - t$.

(2) *For* i, $1 \le i \le s$, *we have* $g_{t+k}^{(i)} = g_{t+k}^{(i-1)}$ *for* $k > t_i$. *Thus,* $g_{t+k}^{(s)} = g_{t+k}$ *for* $k > t_s$.

Proof. (1) Clearly, $t+t_0 \geq f_1+1$, i.e., $t_0 \geq f_1+1-t$. Moreover, $g^{(i-1)}_{t+1} - g^{(i-1)}_{t+t_{i-1}} \leq 1$, which implies $g^{(i)}_{t+1} - g^{(i)}_{t+t_{i-1}} \leq 1$ for $1 \leq i \leq s$. Hence $t_i \geq t_{i-1}$ for $1 \leq i \leq s$.

(2) By $\min\left\{g^{(i-1)}_{t+1} - 1, \ldots, g^{(i-1)}_{f_i} - 1, g^{(i-1)}_{f_i+1}, \ldots, g^{(i-1)}_{t+t_{i-1}}\right\} \geq g^{(i-1)}_{t+1} - 2 \geq g^{(i-1)}_{t+t_{i-1}+1} \geq \cdots \geq g^{(i-1)}_n$, we have $g^{(i)}_{t+t_{i-1}+k'} = g^{(i-1)}_{t+t_{i-1}+k'}$ for $k' \geq 1$. Thus $g^{(i)}_{t+k} = g^{(i-1)}_{t+k}$ for $k > t_i$. ■

Lemma 10. *Let* $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ *be a bigraphic pair with* $f_s \geq t$, $g_t \geq s$, $m-1 \geq g_1 \geq \cdots \geq g_t = \cdots = g_{f_1+1} \geq g_{f_1+2} \geq \cdots \geq g_n$ *and* $n-1 \geq f_1 \geq \cdots \geq f_s = \cdots = f_{g_1+1} \geq f_{g_1+2} \geq \cdots \geq f_m$. *If* $n \geq f_{s+1}+t$ *and* $g_{f_{s+1}+t} \geq s-1$, *then* π *is a potentially* $A_{s,t}$*-bigraphic pair.*

Proof. It is trivial for $s=1$. Assume $s \geq 2$. By Theorem 3, we only need to check that $\pi_s = (f_{s+1}, \ldots, f_m; g_1-s, \ldots, g_t-s, g^{(s)}_{t+1}, \ldots, g^{(s)}_n)$ is a bigraphic pair. Clearly, $f_{s+1}+\cdots+f_m = (g_1-s)+\cdots+(g_t-s)+g^{(s)}_{t+1}+\cdots+g^{(s)}_n$. Denote $p = \max\{i \mid f_{s+i} = f_s\}$. Then $s+p \geq g_1+1$, i.e., $p \geq g_1+1-s$. By Lemma 8, it is enough to check that $\sum_{i=1}^k f_{s+i} \leq \sum_{i=1}^t \min\{k, g_i-s\} + \sum_{i=t+1}^n \min\left\{k, g^{(s)}_i\right\}$ for $p \leq k \leq m-s$. Denote $x = g^{(s)}_{t+1}$ and $y = g_t - s$. If $k \geq x$, by $k \geq p \geq g_1+1-s > g_i - s$ for $1 \leq i \leq t$, then $\sum_{i=1}^t \min\{k, g_i-s\} + \sum_{i=t+1}^n \min\left\{k, g^{(s)}_i\right\} = \sum_{i=1}^t (g_i - s) + \sum_{i=t+1}^n g^{(s)}_i = f_{s+1}+\cdots+f_m \geq \sum_{i=1}^k f_{s+i}$. Assume $p \leq k \leq x-1$. Clearly, $y+s = g_t \geq g_{t+1} \geq x$, i.e., $(x-1)-y \leq s-1$. This implies $k-y \leq s-1$. Moreover, by Lemma 9, $g^{(s)}_{f_{s+1}+t} \geq x-1$ if $t+t_s \geq f_{s+1}+t$, and $g^{(s)}_{f_{s+1}+t} = g_{f_{s+1}+t} \geq s-1$ if $t+t_s < f_{s+1}+t$. Hence $g^{(s)}_{f_{s+1}+t} \geq \min\{x-1, s-1\} \geq k-y$. Thus by $t+t_s \geq f_1+1 > f_{s+1}$, we have $\sum_{i=1}^t \min\{k, g_i-s\} + \sum_{i=t+1}^n \min\left\{k, g^{(s)}_i\right\} = \sum_{i=1}^t (g_i - s) + \sum_{i=t+1}^{f_{s+1}} \min\{k, g^{(s)}_i\} + \sum_{i=f_{s+1}+1}^{f_{s+1}+t} \min\left\{k, g^{(s)}_i\right\} + \sum_{i=f_{s+1}+t+1}^n \min\left\{k, g^{(s)}_i\right\} \geq yt + (f_{s+1}-t)k + (k-y)t = f_{s+1}k \geq \sum_{i=1}^k f_{s+i}$. ■

***Proof of Theorem* 4.** We use induction on $s+t$. It is trivial for $s=1$ or $t=1$. Assume $s \geq 2$ and $t \geq 2$. If $f_1 = n$ or there exists an integer k with $t \leq k \leq f_1$ such that $g_k > g_{k+1}$, then the residual pair $\pi(f_1) = (f_2, \ldots, f_m; g'_1, \ldots, g'_n)$ obtained from π by laying off f_1 satisfies $g'_t \geq g_t - 1 \geq s-1$, $g'_{f_{s+1}+t} \geq g_{f_{s+1}+t} - 1 \geq s-2$ and $g'_1 = g_1 - 1, \ldots, g'_t = g_t - 1$. By Theorem 2 and the induction hypothesis, $\pi(f_1)$ is a potentially $A_{s-1,t}$-bigraphic pair, and hence π is a potentially $A_{s,t}$-bigraphic pair. So we may assume $f_1 \leq n-1$ and $g_1 \geq \cdots \geq g_t = \cdots = g_{f_1+1} \geq g_{f_1+2} \geq \cdots \geq g_n$. If $g_1 = m$ or there exists an integer k with $s \leq k \leq g_1$ such that $f_k > f_{k+1}$, then the residual pair $\pi(g_1) = (f'_1, \ldots, f'_m; g_2, \ldots, g_n)$ obtained from π by laying off g_1 satisfies $f'_s \geq f_s - 1 \geq t-1$, $n-1 \geq f_{s+1} + (t-1) \geq f'_{s+1} + (t-1)$, $g_{1+(f'_{s+1}+(t-1))} \geq g_{f_{s+1}+t} \geq s-1$ and $f'_1 = f_1 - 1, \ldots, f'_s = f_s - 1$. By Theorem 2 and the induction hypothesis, $\pi(g_1)$ is a potentially $A_{s,t-1}$-bigraphic pair, and hence π is a potentially $A_{s,t}$-bigraphic pair. So we may further assume $g_1 \leq m-1$ and $f_1 \geq \cdots \geq f_s = \cdots = f_{g_1+1} \geq f_{g_1+2} \geq \cdots \geq f_m$. Thus by Lemma 10, π is a potentially $A_{s,t}$-bigraphic pair. ■

***Proof of Theorem* 5.** By symmetry, the proof of Theorem 5 is similar to that of Theorem 4. ■

As an application of Theorems 4 and 5, we now prove Theorem 7. We first need some lemmas.

Lemma 11 [5]. *If $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ is a bigraphic pair with $f_s \geq 2t-1$ and $g_t \geq 2s-1$, then π is a potentially $K_{s,t}$-bigraphic pair.*

Lemma 12. *Let $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ be a bigraphic pair with $f_s \geq t$ and $g_t \geq s$. If $n \geq st - s + t$ and $g_n \geq s-1$, then π is a potentially $K_{s,t}$-bigraphic pair.*

Proof. If $f_{s+1} \leq n - t$, i.e., $n \geq f_{s+1} + t$, by Theorem 4, π is a potentially $A_{s,t}$-bigraphic pair. Assume $f_{s+1} \geq n - (t-1)$. Let G be a realization of π with partite sets $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ such that $d_G(x_i) = f_i$ for $1 \leq i \leq m$ and $d_G(y_j) = g_j$ for $1 \leq j \leq n$. For each i, denote $N_G(x_i)$ to be the neighbor set of x_i in G. Clearly, $|N_G(x_i)| = d_G(x_i) = f_i \geq n-(t-1)$ for $1 \leq i \leq s$. Moreover, we have the following Claim 1.

Claim 1. $|N_G(x_1) \cap \cdots \cap N_G(x_s)| \geq n - s(t-1)$.

Proof. Clearly, $|N_G(x_1)| \geq n - (t-1)$. Assume $2 \leq k \leq s$ and $|N_G(x_1) \cap \cdots \cap N_G(x_{k-1})| \geq n - (k-1)(t-1)$. Denote $A = N_G(x_1) \cap \cdots \cap N_G(x_{k-1})$. Then $|N_G(x_1) \cap \cdots \cap N_G(x_k)| = |A \cap N_G(x_k)| = |A| + |N_G(x_k)| - |A \cup N_G(x_k)| \geq (n-(k-1)(t-1)) + (n-(t-1)) - n = n - k(t-1)$. This proves Claim 1. □

By Claim 1, $|N_G(x_1) \cap \cdots \cap N_G(x_s)| \geq n - s(t-1) \geq t$. This implies that G contains $K_{s,t}$. Hence, π is a potentially $K_{s,t}$-bigraphic pair. ■

Lemma 13. *Let $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ be a bigraphic pair with $f_s \geq t$ and $g_t \geq s$. If $m \geq st - t + s$ and $f_m \geq t-1$, then π is a potentially $K_{s,t}$-bigraphic pair.*

Proof. By symmetry, the proof of Lemma 13 is similar to that of Lemma 12. ■

Lemma 14. *Let $1 \leq s \leq t$, $m \geq s$, $n \geq t$ and $n \geq m$. If $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ is a bigraphic pair with $\sigma(\pi) \geq n(s-1)+m(t-1)+n(t-1)-2(t-1)(s-1)+1$, then π is a potentially $K_{s,t}$-bigraphic pair.*

Proof. If $f_s \leq 2t-2$, then $\sigma(\pi) \leq n(s-1) + (m-s+1)(2t-2) = n(s-1) + m(t-1)+m(t-1)-2(t-1)(s-1) \leq n(s-1)+m(t-1)+n(t-1)-2(t-1)(s-1)$, a contradiction. Hence $f_s \geq 2t-1$. If $g_t \leq 2s-2$, then $\sigma(\pi) \leq m(t-1)+(n-t+1)(2s-2) = n(s-1)+m(t-1)+n(s-1)-2(t-1)(s-1) \leq n(s-1)+m(t-1)+n(t-1)-2(t-1)(s-1)$, a contradiction. Hence $g_t \geq 2s-1$. Thus by Lemma 11, π is a potentially $K_{s,t}$-bigraphic pair. ■

Lemma 15. *Let* $1 \leq s \leq t$, $n \geq m \geq s$ *and* $n = (st-s+t)+k$ *with* $0 \leq k \leq (t-1)(st-2s+t+1)$. *If* $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ *is a bigraphic pair with* $\sigma(\pi) \geq n(s-1)+m(t-1)-(t-1)(s-1)+1+(t-1)(st-2s+t+1)-k$, *then* π *is a potentially* $K_{s,t}$*-bigraphic pair.*

Proof. We use induction on k. If $k=0$, then $n = st-s+t$ and $\sigma(\pi) \geq n(s-1)+m(t-1)-(t-1)(s-1)+1+(t-1)(st-2s+t+1) = n(s-1)+m(t-1)+n(t-1)-2(t-1)(s-1)+1$. By Lemma 14, π is a potentially $K_{s,t}$-bigraphic pair. Suppose that $1 \leq k \leq (t-1)(st-2s+t+1)$. Then $\sigma(\pi) \geq n(s-1)+m(t-1)-(t-1)(s-1)+1$. It is straightfoward to show that $f_s \geq t$ and $g_t \geq s$. If $g_n \geq s-1$, then by Lemma 12, π is a potentially $K_{s,t}$-bigraphic pair. Assume $g_n \leq s-2$. If $n \geq m+1$, then the residual pair $\pi(g_n) = (f'_1, \ldots, f'_m; g_1, \ldots, g_{n-1})$ obtained from π by laying off g_n satisfies $\sigma(\pi(g_n)) = \sigma(\pi)-g_n \geq n(s-1)+m(t-1)-(t-1)(s-1)+1+(t-1)(st-2s+t+1)-k-(s-2) = (n-1)(s-1)+m(t-1)-(t-1)(s-1)+1+(t-1)(st-2s+t+1)-(k-1)$. By Theorem 2 and the induction hypothesis, $\pi(g_n)$ is a potentially $K_{s,t}$-bigraphic pair, and hence so is π. Further assume $n=m$. Then $m \geq st-s+t \geq st-t+s$. If $f_m \geq t-1$, then by Lemma 13, π is a potentially $K_{s,t}$-bigraphic pair. If $f_m \leq t-2$, let $\pi(g_n, f'_m) = (f'_1, \ldots, f'_{m-1}; g'_1, \ldots, g'_{n-1})$ be the residual pair obtained from $\pi(g_n)$ by laying off f'_m, by $f'_m \leq f_m$, then $\sigma(\pi(g_n, f'_m)) = \sigma(\pi)-g_n-f'_m \geq n(s-1)+m(t-1)-(t-1)(s-1)+1+(t-1)(st-2s+t+1)-k-(s-2)-(t-2) \geq (n-1)(s-1)+(m-1)(t-1)-(t-1)(s-1)+1+(t-1)(st-2s+t+1)-(k-1)$. By Theorem 2 and the induction hypothesis, it follows that $\pi(g_n, f'_m)$ is a potentially $K_{s,t}$-bigraphic pair. Thus, both $\pi(g_n)$ and π are potentially $K_{s,t}$-bigraphic. ■

Lemma 16. *Let* $1 \leq s \leq t$, $n \geq m \geq s$ *and* $n \geq (s+1)t^2-(2s-1)t+s-1$. *If* $\pi = (f_1, \ldots, f_m; g_1, \ldots, g_n)$ *is a bigraphic pair with* $\sigma(\pi) \geq n(s-1)+m(t-1)-(t-1)(s-1)+1$, *then* π *is a potentially* $K_{s,t}$*-bigraphic pair.*

Proof. We use induction on n. Clearly, the result for $n = (s+1)t^2-(2s-1)t+s-1$ follows from Lemma 15 by letting $k = (t-1)(st-2s+t+1)$. Assume $n \geq (s+1)t^2-(2s-1)t+s$. Clearly, $f_s \geq t$, $g_t \geq s$ and $n \geq st-s+t$. If $g_n \geq s-1$, then by Lemma 12, π is a potentially $K_{s,t}$-bigraphic pair. Assume $g_n \leq s-2$. If $n \geq m+1$, then the residual pair $\pi(g_n) = (f'_1, \ldots, f'_m; g_1, \ldots, g_{n-1})$ obtained from π by laying off g_n satisfies $\sigma(\pi(g_n)) = \sigma(\pi)-g_n \geq n(s-1)+m(t-1)-(t-1)(s-1)+1-(s-2) \geq (n-1)(s-1)+m(t-1)-(t-1)(s-1)+1$. By Theorem 2 and the induction hypothesis, $\pi(g_n)$ is a potentially $K_{s,t}$-bigraphic pair, and hence so is π. Further assume $n=m$. Then $m \geq st-s+t \geq st-t+s$. If $f_m \geq t-1$, then by Lemma 13, π is a potentially $K_{s,t}$-bigraphic pair. If $f_m \le t-2$, let $\pi(g_n, f'_m) = (f'_1, \ldots, f'_{m-1}; g'_1, \ldots, g'_{n-1})$ be the residual pair obtained from $\pi(g_n)$ by laying off f'_m, by $f'_m \le f_m$, then $\sigma(\pi(g_n, f'_m)) = \sigma(\pi) - g_n - f'_m \ge n(s-1) + m(t-1) - (t-1)(s-1)+1-(s-2)-(t-2) \ge (n-1)(s-1)+(m-1)(t-1)-(t-1)(s-1)+1$. By Theorem 2 and the induction hypothesis, it follows that $\pi(g_n, f'_m)$ is a potentially $K_{s,t}$-bigraphic pair. Thus, both $\pi(g_n)$ and π are potentially $K_{s,t}$-bigraphic. ■

Proof of Theorem 7. Ferrara *et al.* [1] considered the bigraphic pair $\pi = (n^{s-1}, (t-1)^{m-s+1}; m^{s-1}, (t-1)^{m-s+1}, (s-1)^{n-m})$, where the symbol x^y stands for y consecutive terms, each equal to x. Clearly, π is not a potentially $K_{s,t}$-bigraphic pair. Thus $\sigma(K_{s,t}, m, n) \ge \sigma(\pi) + 1 = n(s-1) + m(t-1) - (t-1)(s-1) + 1$. The upper bound directly follows from Lemma 16. ■

Remark. The lower bound $n \ge (s+1)t^2 - (2s-1)t + s - 1$ in Theorem 7 is not the best lower bound. However, we will investigate a lower bound on $n + m$ so that the extremal function value $\sigma(K_{s,t}, m, n) = n(s-1) + m(t-1) - (t-1)(s-1) + 1$, which implies a smaller bound on the n or m necessary to assure Theorem 7. It also would be a meaningful further research for $n \ge m \ge s$ and $s > t$. We will consider this problem in our future studies.

Acknowledgement

The authors would like to thank the referees for their helpful suggestions and comments.

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Received 14 July 2020
Revised 23 October 2020
Accepted 23 October 2020