

## OPTIMAL ERROR-DETECTING OPEN-LOCATING-DOMINATING SET ON THE INFINITE TRIANGULAR GRID

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### Abstract

Let  $G$  be a graph and  $S \subseteq V(G)$  represent a subset of vertices having installed “detectors,” each of which is capable of sensing an “intruder” in its open-neighborhood. The open-locating-code of  $v \in V(G)$  is the set of neighboring detectors,  $N(v) \cap S$ . The set  $S$  is said to be an open-locating-dominating set if every open-locating-code is unique and non-empty. In this paper we focus on error-detecting open-locating-dominating sets on the infinite triangular grid, present a solution with density  $\frac{1}{2}$ , and prove it is optimal.

**Keywords:** domination, open-locating-dominating set, error-detection, triangular grid, density.

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### 1. INTRODUCTION

Let  $G$  be an (undirected) graph with vertex set  $V(G)$  and edge set  $E(G)$ . The *open neighborhood* of a vertex  $v \in V(G)$ , denoted by  $N(v)$ , is the set of vertices adjacent to  $v$ :  $N(v) \equiv \{w \in V(G) : vw \in E(G)\}$ .

**Definition 1** [6]. Vertex  $v \in V(G)$  *openly dominates* its neighbors, that is, every vertex in  $N(v)$  and vertex set  $D \subseteq V(G)$  is an *open-dominating* (also called *total dominating*) set if every vertex is dominated by at least one  $v \in D$ , that is,  $V(G) = \bigcup_{v \in D} N(v)$ .

**Definition 2.** Given an open-dominating set  $S \subseteq V(G)$  and a vertex  $v \in V(G)$ , the *open-locating code* of  $v$ , denoted  $\mathcal{L}_S(v)$ , is defined as the set of all open-neighbors of  $v$  which are in  $S$ :  $\mathcal{L}_S(v) \equiv N(v) \cap S$ .

**Definition 3** [4]. A vertex is said to be *k-dominated* by an open-dominating set  $S \subseteq V(G)$  if it is adjacent to exactly  $k$  vertices in  $S$ , that is,  $v \in V(G)$  is *k-dominated* if and only if  $|\mathcal{L}_S(v)| = k$ .

**Definition 4.** An open-dominating set  $S \subseteq V(G)$  is *k-distinguishing* if for any distinct vertices  $u, v \in V(G)$  we have that  $|\mathcal{L}_S(u) \Delta \mathcal{L}_S(v)| \geq k$ , where  $\Delta$  denotes the symmetric difference.

**Definition 5.** An open-dominating set  $S \subseteq V(G)$  is *k<sup>#</sup>-distinguishing* if for any distinct vertices  $u, v \in V(G)$  we have that  $|\mathcal{L}_S(u) - \mathcal{L}_S(v)| \geq k$  or  $|\mathcal{L}_S(v) - \mathcal{L}_S(u)| \geq k$ .

Note that Definitions 4 and 5 are pre-existing concepts from other papers which were phrased in different notation [4, 7].

*Distinguishing sets* provide a way to uniquely identify some target vertex in the graph, traditionally known as the “intruder” [4, 5]. In this context, the distinguishing set represents the subset of vertices which have some form of detector installed on them. This could represent any type of sensor detecting any type of event. The notion of a distinguishing set is a general term which includes many different types of sets with various properties; at least 424 papers have been published on such related concepts [10].

The goal in constructing a distinguishing set is to use as few detectors as possible while still covering the entire graph. The problem of finding *open-locating-dominating* (OLD) sets is a conservative instance of this general problem, which only requires the sensors be able to detect an intruder up to one vertex away and does not require them to be able to detect an intruder at their position. Making such a conservative assumption about the capabilities of the sensors is important in contexts where an intruder could pose significant harm.

**Definition 6** [9]. An open-dominating set  $S \subseteq V(G)$  is called an *open-locating-dominating* (OLD) set if the open-locating codes for all vertices are unique.

There are several more-restrictive specializations of OLD sets, such as *redundant open-locating-dominating* (RED:OLD) sets, which are resilient to a detector being destroyed or going offline [7]. In this paper, we explore an even more restrictive, but more powerful specialization of OLD sets known as *error-detecting open-locating-dominating* (DET:OLD) sets, which are described by Definition 8. In addition to having all the properties of RED:OLD sets, DET:OLD sets are capable of correctly identifying an intruder even when at most one sensor incorrectly reports that there is no intruder in its detection range [7]. Note that this

is different from a RED:OLD set, in which the malfunctioning sensor does not report anything at all. In this way, DET:OLD sets allow for uniquely locating an intruder in a way which is resilient to up to one false negative.

**Definition 7** [9]. An open-dominating set  $S \subseteq V(G)$  is called a *redundant open-locating-dominating* (RED:OLD) set if for all  $v \in S$ ,  $S - \{v\}$  is an OLD set.

**Definition 8** [9]. An open-dominating set  $S \subseteq V(G)$  is called an *error-detecting open-locating-dominating* (DET:OLD) if an “intruder” is correctly located even when at most one of the detectors reports a false negative.

Although Definitions 6–8 are useful to understand the concepts and purposes for these sets, they are perhaps less conducive to actually constructing or verifying that a given set meets its requirements. Instead, we use Theorems 9–11, which have been proven to be equivalent constraints.

**Theorem 9.** *An open-dominating set is an OLD set if and only if every pair of vertices is 1-distinguished.*

**Theorem 10** [7,9]. *An open-dominating set is a RED:OLD set if and only if all vertices are at least 2-dominated and all pairs are 2-distinguished.*

**Theorem 11** [7,9]. *An open-dominating set is a DET:OLD set if and only if all vertices are at least 2-dominated and all pairs are  $2^\#$ -distinguished.*

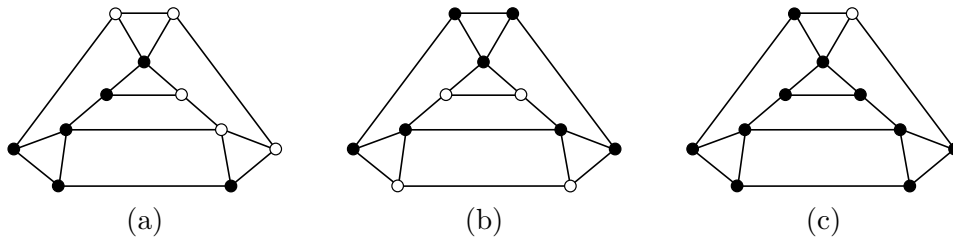


Figure 1. Optimal OLD (a), RED:OLD (b), and DET:OLD (c) sets on the same (finite) graph. Shaded vertices represent detectors.

For finite graphs, the notations  $OLD(G)$ ,  $RED:OLD(G)$ , and  $DET:OLD(G)$  represent the cardinality of the smallest possible OLD, RED:OLD, and DET:OLD sets on graph  $G$ , respectively [7]. From Figure 1, which shows optimal solutions on the given graph which we will call  $G$ , we see that  $OLD(G)=6$ ,  $RED:OLD(G)=7$ , and  $DET:OLD(G)=10$ .

For infinite graphs, instead of the cardinality, we measure via the *density* of the subset, which is defined as the ratio of the number of detectors to the total number of vertices. The notations  $OLD\%(G)$ ,  $RED:OLD\%(G)$ , and  $DET$ :

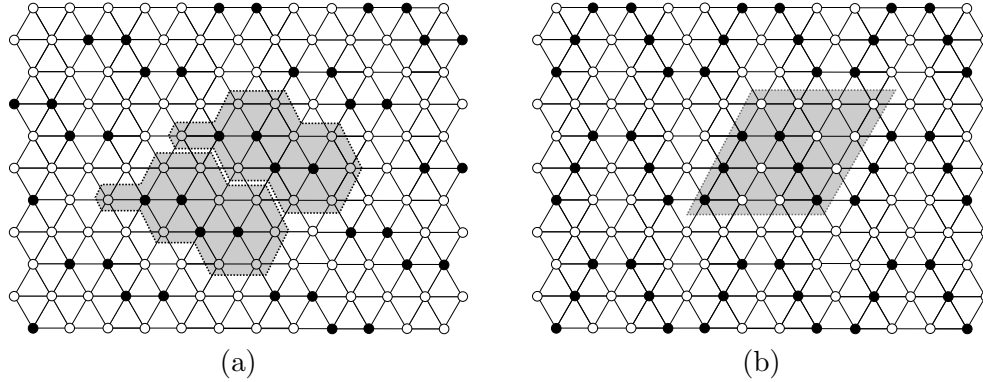


Figure 2. Optimal OLD (a) and RED:OLD (b) sets on the TRI graph.

$\text{OLD}\%(G)$  represent the minimum density of such a set on  $G$  [4]. Note that the notion of density is also defined for finite graphs.

One common method for constructing distinguishing sets on infinite graphs is through the use of *tessellation*. To do this, one begins by constructing some tile shape spanning a finite number of vertices. One or more detectors is then selected in the tile. Lastly, the tile is duplicated, along with its detectors, in order to cover every vertex in the infinite graph with no overlapping. Thus, there are an infinite number of detectors and an infinite number of vertices, but the density remains equal to the density in the selected shape, which is trivial to determine.

In this paper, we will explore the optimal construction of a DET:OLD set on the *infinite triangular grid*, which for convenience is shortened to TRI. The TRI graph is a regular graph of degree six (or 6-regular). While it can be depicted as a square grid with additional edges along a diagonal, it is often easier to think of it in terms of triangles or hexagons.

Much study has been done concerning various graphical parameters and distinguishing sets on the TRI graph [1–3, 7]. For instance, optimal constructions of OLD sets [3] and RED:OLD sets [7] on the TRI graph have already been explored. For examples of these optimal constructions, see Figure 2. These are constructed via tessellation, with the tiles used to create them shaded and outlined. The OLD set (a) has a tile size of 13 and uses 4 detectors, and thus has a density of  $\frac{4}{13}$ . The RED:OLD set (b) has density  $\frac{6}{16} = \frac{3}{8}$ . As these constructions have been proven to be optimal, this means that  $\text{OLD}\%(\text{TRI}) = \frac{4}{13}$  and  $\text{RED:OLD}\%(\text{TRI}) = \frac{3}{8}$ .

In order to find the value for  $\text{DET:OLD}\%(\text{TRI})$ , we will begin by demonstrating a particular tessellated solution—effectively establishing an upper bound for  $\text{DET:OLD}\%(\text{TRI})$ —and then employ theory to establish a lower bound.

## 2. UPPER BOUND

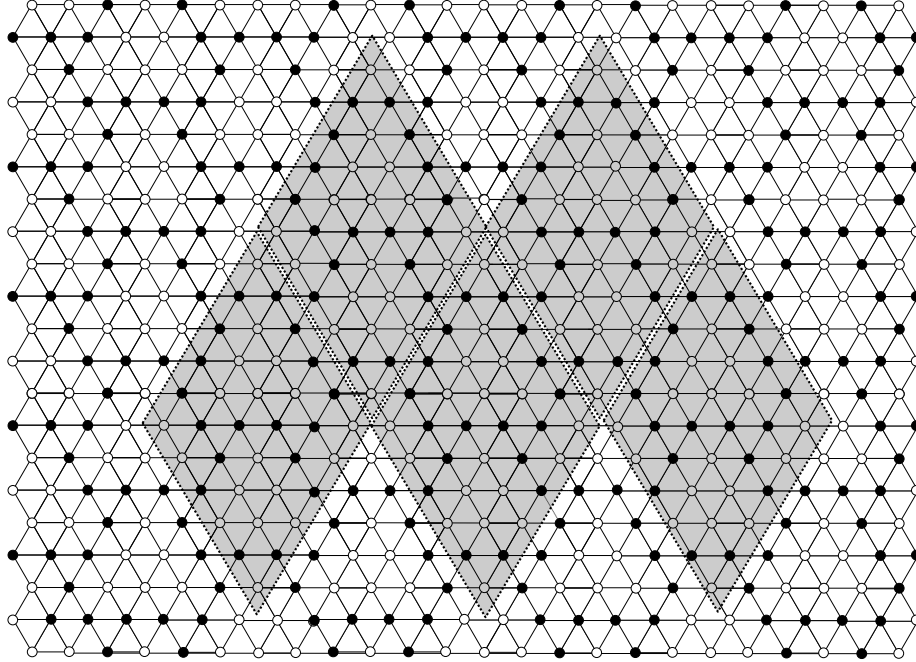


Figure 3. A 36-vertex tile DET:OLD tessellation demonstrating a density of  $\frac{1}{2}$ . Shaded vertices are in the DET:OLD set.

Seo and Slater [7] previously found an upper bound of  $\frac{5}{9}$  for the DET:OLD set problem on the TRI graph. We have improved the bound by constructing a 36-vertex tile with 18 detectors which can be tessellated to form a DET:OLD set with density  $\frac{1}{2}$ . Figure 3 shows this particular solution; it is easy to verify that every vertex is at least 2-dominated and all pairs are  $2^\#$ -distinguished, thus satisfying the requirements for a DET:OLD set given in Theorem 11. Therefore,  $\text{DET:OLD}\%(\text{TRI}) \leq \frac{1}{2}$ .

## 3. LOWER BOUND

For constructing a lower bound, we will use the notion of a *share argument* introduced by Slater [8] and used extensively by Seo and Slater [4–7]. In essence, a share argument inverts the problem, so instead of trying to find a lower bound on the density of a dominating set directly, we find an upper bound on the average *share* of a vertex in the set, i.e., its contribution in the domination of its neighbors. Once an upper bound for the average share is constructed, its reciprocal acts as

a lower bound for the density.

For an open-dominating set  $S \subseteq V(G)$  of  $G$ , a vertex  $u \in S$ , and a vertex  $v \in N(u)$ , let  $sh_v(u) \equiv \frac{1}{|\mathcal{L}_S(v)|}$  denote  $u$ 's share of  $v$ , that is  $u$ 's contribution to the domination of  $v$ . Further, we let  $sh(u) \equiv \sum_{w \in N(u)} sh_w(u)$  denote the total share of  $u$ , or  $u$ 's total contribution to the domination of its neighbors.

Figure 4 shows an example graph with its open-dominating set, where the vertices with detectors are shown as shaded vertices. As stated above, share is only defined for a vertex which is a detector, but all of its neighbors, detectors or not, factor into its share value. Consider vertex 7, which is adjacent to vertices 5, 6, 8, and 9. Vertices 5 and 9 are 3-dominated, so  $sh_5(7) = sh_9(7) = \frac{1}{3}$ . Vertices 6 and 8 are 2-dominated, so  $sh_6(7) = sh_8(7) = \frac{1}{2}$ . Thus, the (total) share of vertex 7 is determined as  $sh(7) \equiv sh_5(7) + sh_6(7) + sh_8(7) + sh_9(7) = \frac{1}{3} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} = \frac{5}{3}$ . As a second example, consider vertex 3, which has neighbor vertices 2, 4, 5, 6, and 9. It can be easily checked that the share of vertex 3,  $sh(3) = \frac{1}{2} + 1 + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \frac{8}{3}$ .

In any DET:OLD set  $S \subseteq V(TRI)$ , each vertex is by definition at least 2-dominated. Since the TRI graph is 6-regular, for any vertex  $u \in S$  we have  $sh(u) \leq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3$ . Thus, we have a trivial bound:  $\text{DET:OLD}\%(TRI) \geq \frac{1}{3}$ .

**Lemma 12.** *For any graph  $G$  and for any DET:OLD set  $S \subseteq V(G)$ , for any two distinct vertices  $u, v \in V(G)$  where  $|\mathcal{L}_S(u) \cap \mathcal{L}_S(v)| = k$ , at least one of them must be at least  $(k + 2)$ -dominated.*

**Proof.** Suppose  $G$  is a graph and  $S \subseteq V(G)$  is a DET:OLD set on  $G$ . Let  $u, v \in V(G)$  be two distinct vertices with  $|\mathcal{L}_S(u) \cap \mathcal{L}_S(v)| = c$ . Suppose neither  $u$  nor  $v$  is at least  $(c + 2)$ -dominated. This means that  $|\mathcal{L}_S(u)| \leq c + 1$  and  $|\mathcal{L}_S(v)| \leq c + 1$ . Then  $|\mathcal{L}_S(u) - \mathcal{L}_S(v)| \leq 1$  and  $|\mathcal{L}_S(v) - \mathcal{L}_S(u)| \leq 1$ , which means  $u$  and  $v$  are not  $2^\#$ -distinguished. This contradicts that  $S$  is a DET:OLD set on  $G$ . ■

**Proposition 13.** *Let  $S$  be a DET:OLD set for a graph  $G$ . Then, a vertex in  $S$  may have at most one 2-dominated neighbor.*

**Proof.** Let  $x \in S$  and  $u, v \in N(x)$  with  $u \neq v$ . Because  $G$  is undirected, we observe  $x \in N(u) \cap N(v)$ , which implies  $|N(u) \cap N(v)| \geq 1$ . Therefore, by Lemma 12, at least one of  $u$  or  $v$  must be at least 3-dominated. ■

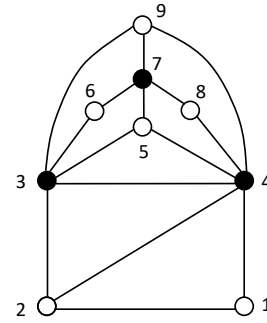


Figure 4.  $\{3, 4, 7\}$  is an open-dominating set [4].

By Proposition 13, we can impose a stricter bound on the maximum share: for all  $u \in S$ ,  $sh(u) \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{13}{6}$ . Thus, we establish a better lower bound:  $DET:OLD\%(TRI) \geq \frac{6}{13}$ . While this bound is certainly better than our trivial bound of  $\frac{1}{3}$ , it is not sufficient to close the gap between the lower and upper bounds.

The particular solution to the DET:OLD set problem on the TRI graph depicted in Figure 3 shows that the share of every detector vertex is exactly 2. Next, we characterize two conditions which make the share of a vertex in a DET:OLD(TRI) set at most 2. This result will be heavily utilized in proving Theorem 15.

**Observation 14.** *If a detector  $u \in S$  has at least two neighbors which are at least 4-dominated then  $sh(u) \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} = 2$ . Also, if at least one neighbor is 6-dominated then  $sh(u) \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{6} = 2$ .*

Now we prove that the value of 2 is indeed an upper bound for the average share of all the vertices in any DET:OLD set on the TRI graph.

**Theorem 15.** *Let  $S \subseteq V(TRI)$  be a DET:OLD set on the TRI graph. Then, the average share of all vertices in  $S$  is no more than 2.*

**Proof.** To prove Theorem 15 we will consider every possible non-isomorphic case for a “center” detector vertex  $x \in S$  being dominated by its neighbors. These cases are shown in Figure 5. We will demonstrate that all of these cases result in the average share being no more than 2.

*Case 1.* Case 1 covers 8 sub-cases from Figure 5: 1–1, 1–2, 1–3, 1–4, 1–5, 1–6, 1–7, and 1–8. In all of these we have that  $\{1, 2\} \subseteq S$ . We see that vertices 2 and 6 are dominated by two common detectors, so by Lemma 12 we have that at least one of them is at least 4-dominated. Similarly, we can use vertices 1 and 3 to show that at least one of them is also at least 4-dominated. By Observation 14, we now have that  $sh(x) \leq 2$  for any of these sub-cases.

*Case 2.* Consider Case 2 from Figure 5. Vertices 2 and 6 are dominated by two common detectors, so Lemma 12 gives us that at least one of them is at least 4-dominated. If both of them are at least 4-dominated, then Observation 14 immediately yields that  $sh(x) \leq 2$ . Thus, we need only consider when exactly one of vertices 2 or 6 is 4-dominated. Due to symmetry, without loss of generality we can assume vertex 2 is 3-dominated. Then, we can apply Lemma 12 to vertices 2 and 4 to yield a second vertex which is at least 4-dominated, implying that  $sh(x) \leq 2$ .

*Case 3.* Consider Case 3 from Figure 5. Vertices 2 and 6 are dominated by two common detectors, so Lemma 12 gives us that at least one of them is at least

4-dominated. Similarly, we can use vertices 3 and 5 to show that at least one of them is also at least 4-dominated. Thus, Observation 14 gives us that  $sh(x) \leq 2$ .

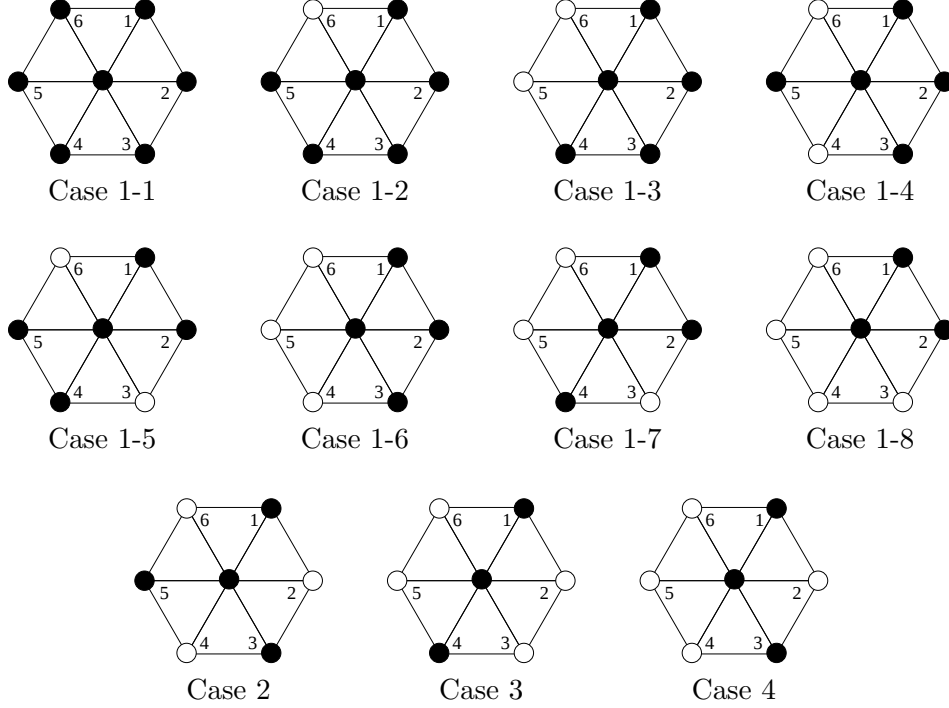


Figure 5. The 11 non-isomorphic cases for  $\mathcal{L}_S(x)$  in a DET:OLD set  $S \subseteq V(TRI)$ .

*Case 4.* Lastly, we consider Case 4 from Figure 5. In order to continue with this case, we must extend our view of the graph to radius 2, as shown in Figure 6. If vertex 2 is 3-dominated, which implies  $8 \notin S$ ,  $9 \notin S$ , and  $10 \notin S$ , then we can apply Lemma 12 to vertex pairs  $\{2,4\}$  and  $\{2,6\}$  to show that both vertices 4 and 6 must be at least 4-dominated. By Observation 14, this would immediately give us that  $sh(x) \leq 2$ . Furthermore, if vertex 2 is 6-dominated, then by Observation 14 we have  $sh(x) \leq 2$ . Thus, we need only consider when vertex 2 is 4- or 5-dominated.

Suppose vertex 8 or vertex 10 is not in  $S$ , that is  $\{8,10\} \not\subseteq S$ . Due to symmetry, without loss of generality let  $10 \notin S$ . Vertex 2 is currently at least 4-dominated due to already handling when it is 3-dominated. Vertex 5 is currently only 1-dominated in Figure 6, but it has to be at least 2-dominated. If it is at least 4-dominated then we have our second neighbor of  $x$  which is at least 4-dominated and are done. Therefore, we need only check when it is 2- or 3-dominated. If  $14 \in S$ , then we can apply Lemma 12 to vertices 4 and 5 to show that another

vertex exists which is at least 4-dominated. Similarly, if  $16 \in S$ , then we can use vertices 5 and 6 to arrive at a second vertex which is at least 4-dominated. In either event, Observation 14 gives us that  $sh(x) \leq 2$ . Otherwise we can assume  $15 \in S$ ,  $14 \notin S$ , and  $16 \notin S$ , which means vertex 5 is 2-dominated. Because vertex 5 is already 2-dominated, by Proposition 13 vertex 3 must be 3-dominated, which implies  $\{11, 12\} \subseteq S$ . We then see that applying Lemma 12 to vertices 3 and 4 gives us a second vertex which is at least 4-dominated, which implies  $sh(x) \leq 2$ .

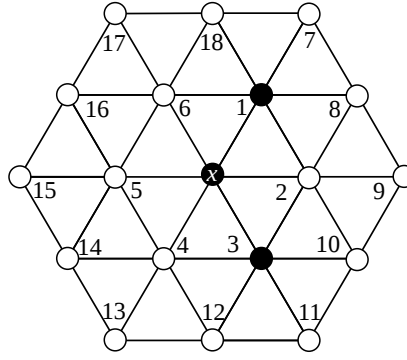


Figure 6. Configuration 4, extended to radius 2 around  $x$ .

The only remaining configuration is where both vertices 8 and 10 are detectors, that is  $\{8, 10\} \subseteq S$  and vertex 2 is at least 5-dominated. If vertex 2 is 6-dominated, then  $sh(x) \leq 2$ , so we assume vertex 2 is 5-dominated and  $9 \notin S$ . Again, vertex 5 must be 2- or 3-dominated; 4-dominated is unnecessary to test, as Observation 14 would immediately yield that  $sh(x) \leq 2$ . As discussed in the preceding paragraph, if  $16 \in S$  or  $14 \in S$  we can use Lemma 12 to arrive at a second vertex which is at least 4-dominated. The only remaining possibility is where  $15 \in S$ ,  $14 \notin S$ , and  $16 \notin S$ . Vertex 6 cannot be 2-dominated because there is already vertex 5 being 2-dominated. If vertex 6 is 4-dominated then we are done, so we need only consider when vertex 6 is 3-dominated, which implies that  $|\{17, 18\} \cap S| = 1$ .

If  $18 \in S$  we can apply Lemma 12 to vertices 1 and 6 to give us a second vertex which is at least 4-dominated, so  $sh(x) \leq 2$ . Thus, we assume  $17 \in S$  and  $18 \notin S$ . We then see that vertex 1 must be 3-dominated because vertex 5 is already 2-dominated; so  $7 \in S$ . We can apply the same arguments with symmetry to show that  $11 \in S$ ,  $12 \notin S$ , and  $13 \in S$ .

Figure 7 shows the last remaining configuration for Case 4 and the most complicated, as  $sh(x) \equiv sh_1(x) + sh_2(x) + sh_3(x) + sh_4(x) + sh_5(x) + sh_6(x) = \frac{1}{3} + \frac{1}{5} + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} + \frac{1}{3} = \frac{61}{30} > 2$ . Vertices 1–18 are known: shaded represent detectors, non-shaded are not detectors. However, vertices 19–23 are unknown. They are included here because in order to resolve this configuration we must

find an upper bound for  $sh(3)$ , which requires us to expand around vertex 3 with radius 2. The strategy is to show that even in this configuration the average share is still no larger than 2.

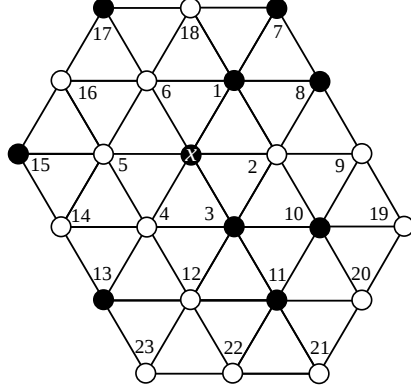


Figure 7. Final sub-case of Case 4, extended to radius 2 around vertex 3.

From Figure 7 we see that vertex 3 has one 5-dominated neighbor, namely vertex 2. Additionally, we see that applying Lemma 12 to vertices 4 and 12 gives us that vertex 12 is at least 4-dominated. Therefore, we have that  $sh(3) \leq \frac{1}{5} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{3} + \frac{1}{2} = \frac{39}{20}$ . By symmetry, we also have that  $sh(1) = sh(3) \leq \frac{39}{20}$ .

Consider the average share of two vertices:  $\frac{1}{2} [sh(x) + sh(3)] = \frac{1}{2} [\frac{61}{30} + \frac{39}{20}] = \frac{239}{120} < 2$ . As every other possible configuration around a given detector results in a share of no more than 2, if we can prove that this averaging technique is sound in any general construction, then we will have proven that the average share of all detectors in  $S$  is no larger than 2. To prove this, it would be sufficient to show that any arbitrary detector  $y \in S$  with  $sh(y) > 2$  ( $x$ -like) has a neighbor detector  $z \in N(y) \cap S$  with  $sh(z) < 2$  ( $3$ -like) and  $\{v \in N(z) \cap S : sh(v) > 2\} = \{y\}$ . As Figure 7 is the only legal construction for a vertex with share exceeding 2, we know  $N(z) \cap S = \{y, v_1, v_2\}$ , where  $y, v_1$ , and  $v_2$  correspond to vertices 3, 10, and 11, respectively. Additionally, from Case 1 we know that if a detector vertex  $u$  has two neighbor detectors which are adjacent to one another, then  $sh(u) \leq 2$ . We observe that  $\{z, v_2\} \subseteq N(v_1) \cap S$  and  $v_2 \in N(z)$ , so  $z$  is adjacent to  $v_2$ . Similarly,  $\{z, v_1\} \subseteq N(v_2) \cap S$  and  $v_1 \in N(z)$  so  $z$  is adjacent to  $v_1$ . Hence  $sh(v_1) \leq 2$  and  $sh(v_2) \leq 2$ . Therefore,  $\{v \in N(z) \cap S : sh(v) > 2\} = \{y\}$ , as desired. Thus, we can conclude that in any DET:OLD set on the TRI graph the average vertex share is no more than 2, completing the proof. ■

We obtain the following lower bound for  $DET:OLD\%(TRI)$  from Theorem 15.

**Corollary 16.**  $DET:OLD\%(TRI) \geq \frac{1}{2}$ .

## 4. CONCLUSION

We have given a construction of a DET:OLD set on the infinite triangular grid which demonstrates a density of  $\frac{1}{2}$ , giving us that  $\text{DET:OLD}\%(\text{TRI}) \leq \frac{1}{2}$ . Additionally, Corollary 16 gives us that  $\text{DET:OLD}\%(\text{TRI}) \geq \frac{1}{2}$ . Therefore, by antisymmetry,  $\text{DET:OLD}\%(\text{TRI}) = \frac{1}{2}$ , and the provided construction in Figure 3 is proven to be optimal.

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