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DAISY HAMMING GRAPHS

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Abstract

Daisy graphs of a rooted graph G with the root r were recently introduced as a generalization of daisy cubes, a class of isometric subgraphs of hypercubes. In this paper we first address a problem posed in [A. Taranenko, *Daisy cubes: A characterization and a generalization*, European J. Combin. 85 (2020) #103058] and characterize rooted graphs G with the root r for which all daisy graphs of G with respect to r are isometric in G, assuming the graph G satisfies the rooted triangle condition. We continue the investigation of daisy graphs G (generated by X) of a Hamming graph $\mathcal{H}$ and characterize those daisy graphs generated by X of cardinality 2 that are isometric in $\mathcal{H}$. Finally, we give a characterization of isometric daisy graphs of a Hamming graph $K_{k_1}\Box\cdots\Box K_{k_n}$ with respect to 0^n in terms of an expansion procedure.

Keywords: daisy graphs, expansion, isometric subgraphs.

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1. Introduction and Preliminary Results

All graphs $G=(V,E)$ in this paper are undirected and without loops or multiple edges. The *distance* $d_G(u,v)$ between two vertices u and v is the length of a shortest u,v-path, and the *interval* $I_G(u,v)$ between u and v consists of all the vertices on all shortest u,v-paths, that is, $I_G(u,v)=\{x\in V(G)\,|\,d_G(u,x)+d_G(x,v)=d_G(u,v)\}$. For a set U of vertices of a graph G we denote by $\langle U\rangle_G$

the subgraph of G induced by the set U. The index G may be omitted when the graph will be clear from the context. A subgraph H of G is called *isometric* if $d_H(u,v) = d_G(u,v)$, for all $u,v \in V(H)$.

The *Cartesian product* $G = G_1 \Box \cdots \Box G_n$ of n graphs $G_1, \ldots, G_n$ has the n-tuples $(x_1, \ldots, x_n)$ as its vertices (with vertex x_i from G_i) and an edge between two vertices $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ if and only if, for some i, the vertices x_i and y_i are adjacent in G_i, and $x_j = y_j$, for the remaining $j \neq i$ [6]. The Cartesian product of n copies of K_2 is a *hypercube* or *n-cube* Q_n. If all the factors in a Cartesian product are complete graphs then G is called a *Hamming graph.* The Hamming graph $\mathcal{H} = K_{k_1} \Box \cdots \Box K_{k_n}$ will be denoted by $\mathcal{H}_{k_1,\ldots,k_n}$. Isometric subgraphs of hypercubes are called *partial cubes* and isometric subgraphs of Hamming graphs are called *partial Hamming graphs.* Note, a tuple $(x_1, \ldots, x_n)$ may be written in a shorter form as $x_1 \cdots x_n$.

For any positive integer n the set $\{1, \ldots, n\}$ is denoted by $[n]$ and the set $\{0, 1, \ldots, n-1\}$ by $[n]_0$. Let $k_1, \ldots, k_n$ be positive integers and let $V = \prod_{i=1}^{n} [k_i]_0$. The *Hamming distance*, $H(u,v)$, of two vectors $u, v \in V$ is the number of coordinates in which they differ. Note, a Hamming graph $\mathcal{H}_{k_1,\ldots,k_n}$ is the graph with the vertex set $\prod_{i=1}^{n} [k_i]_0$, such that the Hamming distance and the distance function of the graph coincide. Let $v = v_1 \cdots v_n \in V(\mathcal{H}_{k_1,\ldots,k_n})$. If $x_1 \cdots x_n \in I_{\mathcal{H}_{k_1,\ldots,k_n}}(v, 0^n)$, then $x_i \in \{0, v_i\}$, for any $i \in [n]$.

A recent paper by Klavžar and Mollard [8] introduced a new family of graphs called daisy cubes. The daisy cube $Q_n(X)$ is the subgraph of Q_n induced by the union of the intervals $I(x, 0^n)$ over all $x \in X \subseteq V(Q_n)$. Daisy cubes are shown to be partial cubes (i.e., isometric subgraphs of hypercubes) and include some other previously well known classes of cube-like graphs, e.g. Fibonacci cubes [7] and Lucas cubes [11, 12]. Regarding daisy cubes, several results have already appeared in the literature. Vesel [14] has shown that a cube-complement of a daisy cube is also a daisy cube. Moreover, daisy cubes also appear in chemical graph theory in connection with resonance graphs. Žigert Pleteršek has shown in [16] that resonance graphs of the so-called kinky benzenoid systems are daisy cubes and Brezovnik *et al.* [3] characterized catacondensed even ring systems of which resonance graphs are daisy cubes.

Taranenko [13] characterized daisy cubes by means of special kind of peripheral expansions and thus proved that daisy cubes are tree-like partial cubes [2]. In the same paper a generalization of daisy cubes to arbitrary rooted graphs was introduced. These graphs are called *daisy graphs* of rooted graphs with respect to the root. A sufficient but not a necessary condition for a rooted graph G in which every daisy graph of G with respect to the root is isometric in G was presented. We improve this result with another sufficient condition for this and also prove that both conditions together with an additional one provide a characterization of such graphs. We present these and related results in Section 2. In Section 3 we focus on daisy graphs of Hamming graphs (with respect to a chosen root), called *daisy Hamming graphs*. Since hypercubes are a special case of Hamming graphs and daisy cubes are a special case of daisy graphs, a natural question that arises is: what properties do isometric daisy Hamming graphs have. Studying the properties of these graphs we obtain a characterization of isometric daisy Hamming graphs in terms of a specific kind of expansion.

We continue this section with some notations and preliminary results.

Definition. [9] Let G be a graph and (u, v, w) a triple of vertices of G. A triple (x, y, z) of vertices of G is a *pseudo-median* of the triple (u, v, w) if it satisfies all of the following conditions.

1. (i) There is a shortest u, v-path in G that contains both x and y;
   (ii) There is a shortest v, w-path in G that contains both y and z;
   (iii) There is a shortest u, w-path in G that contains both x and z;
2. $d(x, y) = d(y, z) = d(x, z)$;
3. $d(x, y)$ is minimal under the first two conditions.

The distance $d(x, y)$ is called the *size* of the pseudo-median (x, y, z).

Pseudo-median of a triple (u, v, w) of size 0, is called a *median* of (u, v, w). Let G be a graph and (u, v, w) a triple of vertices of G. A triple (x, y, z) of vertices of G is a *quasi-median* of the triple (u, v, w) if it is a pseudo-median of (u, v, w) and if (u, v, w) has no pseudo-median different from (x, y, z). Note that any triple (u, v, w) of vertices $u = u_1 \cdots u_n$, $v = v_1 \cdots v_n$, $w = w_1 \cdots w_n$ of a Hamming graph $\mathcal{H}_{k_1,\ldots,k_n}$ has a quasi-median (x, y, z), that can be obtained in the following way. If u_i, v_i and w_i are pairwise distinct, then $x_i = u_i$, $y_i = v_i$, $z_i = w_i$. If u_i, v_i and w_i are not all pairwise distinct with at least two of u_i, v_i, w_i equal to p_i, then $x_i = y_i = z_i = p_i$. The size of this quasi-median is the number of coordinates in which u, v and w are all distinct [9].

A binary expansion was first defined in [10] and a generalization of binary expansion using more covering sets was first introduced in [9]. We will use the definition of general expansion introduced by Chepoi [4], as follows.

Definition. [4] Let G be a connected graph and let $W_1, W_2, \ldots, W_n$ be subsets of $V(G)$ such that

1. $W_i \cap W_j \neq \emptyset$, for all $i, j \in [n]$;
2. $\bigcup_{i=1}^{n} W_i = V(G)$;
3. There are no edges between sets $W_i \setminus W_j$ and $W_j \setminus W_i$, for all $i, j \in [n]$;
4. Subgraphs $\langle W_i \rangle, \langle W_i \cup W_j \rangle$ are isometric in G, for all $i, j \in [n]$.

Then to each vertex $x \in V(G)$ we associate a set $\{i_1, i_2, \ldots, i_t\}$ of all indices i_j, where $x \in W_{i_j}$. A graph G' is called an expansion of G relative to the sets $W_1, W_2, \ldots, W_n$ if it is obtained from G in the following way.

1. Replace each vertex x of G with a clique with vertices $x_{i_1}, x_{i_2}, \ldots, x_{i_t}$;
2. If an index i_s belongs to both sets $\{i_1, \ldots, i_t\}, \{i'_1, \ldots, i'_l\}$ corresponding to adjacent vertices x and y in G, then let $x_{i_s} y_{i_s} \in E(G')$.

An expansion of G relative to the sets $W_1, W_2, \ldots, W_n$ is called *peripheral* if there exists $i \in [n]$ such that $W_i = V(G)$. The peripheral expansion of G relative to the sets $W_1, W_2, \ldots, W_n$ will be denoted by $\mathrm{pe}(G; W_1, \ldots, W_n)$.

An illustration of an expansion can be seen in Figure 1. In the left-hand side one can see a cycle C_6 (with the vertices a, b, c, d, e and f) and three subsets of vertices $W_1 = \{a, b, c, d, e, f\}$, $W_2 = \{a, b, c\}$ and $W_3 = \{c, d\}$. It is easy to verify that W_1, W_2 and W_3 satisfy the conditions of the definition of expansion. The expansion of the cycle C_6 with respect to the sets W_1, W_2 and W_3 is obtained in the following way. Since a and b both belong to W_1 and W_2, they are each replaced with a clique on two vertices (a_1 and a_2, and b_1 and b_2, respectively). The vertex c belongs to all three sets (W_1, W_2 and W_3) and is therefore replaced by a clique on three vertices (c_1, c_2 and c_3). The vertex d belongs to W_1 and W_3 and is replaced with a clique on two vertices (d_1 and d_3). The vertices e and f both belong to only one vertex set, namely W_1, they are both replaced by e_1 and f_1, respectively. Finally, edges between vertices with the same index are added, if the corresponding vertices from the original graph are adjacent. The resulting expansion is shown in the right-hand side of Figure 1. Note, since $W_1 = V(C_6)$ the depicted expansion is also a peripheral expansion.

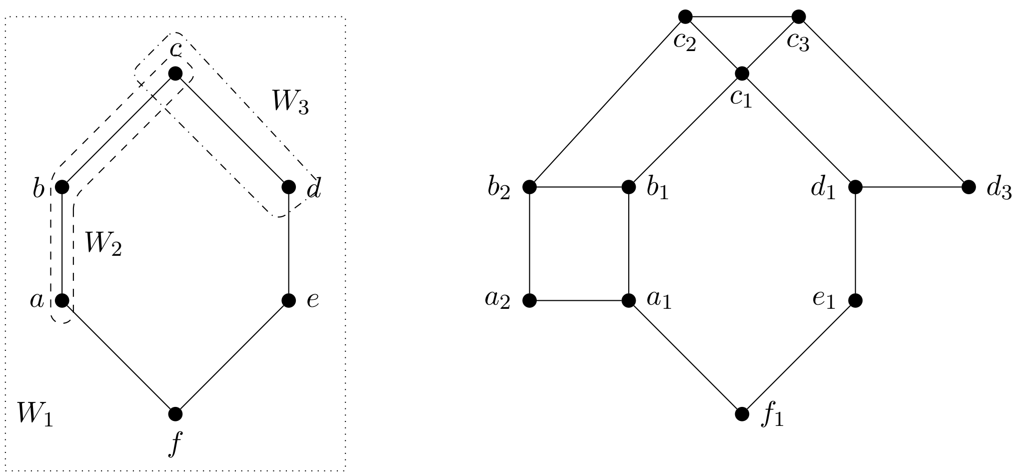

Figure 1. A graph G (left-hand side) and its expansion (right-hand side) with respect to the sets W_1, W_2 and W_3.

Let $G = (V, E)$ be a connected graph and $uv \in E(G)$. We define the following sets.

$$W_{uv} = \{x \in V(G) \mid d(u,x) < d(v,x)\};$$
$$U_{uv} = \{x \in W_{uv} \mid \text{there exists } z \in W_{vu} \text{ such that } xz \in E(G)\};$$
$$F_{uv} = \{xz \in E(G) \mid x \in U_{uv} \wedge z \in U_{vu}\}.$$

With these sets we can define Djoković relation $\sim$ as follows [5]. For $uv, xy \in E(G)$

$$uv \sim xy \text{ if and only if } x \in W_{uv} \wedge y \in W_{vu}.$$

It follows from the definition that F_{uv} is precisely the set of edges from $E(G)$ that are in relation $\sim$ with $uv \in E(G)$. Note also that the relation $\sim$ is reflexive and symmetric but not transitive in general. In [1] Brešar introduced relation $\triangle$ on the edge set of a connected graph as follows.

Definition [1]**.** Let G be a connected graph and $uv, xy \in E(G)$. Then $uv \triangle xy$ if and only if $uv \sim xy$ or there exists a clique with edges $e, f \in E(G)$ such that $xy \sim e$ and $uv \sim f$.

Note that the relation $\triangle$ is also reflexive and symmetric but it is not necessarily transitive. Brešar proved that the relation $\triangle$ is transitive in partial Hamming graphs [1]. He also proved that each $\triangle$-class is a union of some $\sim$-classes. For edges $ab, cd \in E(G)$ the $\sim$-classes F_{ab} and F_{cd} are in the same $\triangle$-class if and only if there is a clique containing edges $a'b' \in F_{ab}$ and $c'd' \in F_{cd}$.

2. Isometric Daisy Graphs

In [13] a generalization of daisy cubes was defined in the following way.

Definition [13]**.** Let G be a rooted graph with the root r. For $X \subseteq V(G)$ the *daisy graph* $G_r(X)$ *of the graph* G *with respect to* r (*generated by* X) is the subgraph of G where

$$G_r(X) = \langle \{u \in V(G) \mid u \in I_G(r,v) \text{ for some } v \in X\} \rangle .$$

If $H = G_r(X)$ is an isometric subgraph of G we say that H is an *isometric daisy graph* of a graph G with respect to r. Note that it follows from the definition of daisy graphs, that $V(G_r(X)) = \bigcup_{v \in X} I_G(v,r)$. Moreover, if $u \in V(G_r(X))$, then $I_G(u,r) \subseteq V(G_r(X))$. Therefore any convex subgraph H of a rooted graph G with root r, such that H contains r, is a daisy graph of G with respect to r.

In [13] Taranenko presented a sufficient condition for a rooted graph G with the root r in which any daisy graph with respect to r is isometric. He also proved that the mentioned condition is not necessary.

Proposition 1 [13]. *Let G be a rooted graph with the root r. If for any two vertices of G, say u and v, it holds that there exists a pseudo-median of (u, v, r) of size 0, then every daisy graph of G with respect to r is isometric in G.*

We give another sufficient condition for a rooted graph G with respect to the root r in which any daisy graph with respect to r is isometric and prove that both conditions yield a characterization of rooted graphs G satisfying rooted triangle condition in which all daisy graphs with respect to the root are isometric.

Theorem 2. *Let G be a rooted graph with the root r. If for any two vertices of G, say u and v, there exists a pseudo-median of size 1 of the triple of vertices u, v and r, then every daisy graph of G with respect to r is isometric in G.*

Proof. Let H be an arbitrary daisy graph of G with respect to r. Also, let u and v be two arbitrary vertices of H, and let (x, y, z) be a pseudo-median of (u, v, r) of size 1. Hence there exists a shortest u, v-path in G that contains x and y, where $x \in I_G(u, r)$ and $y \in I_G(v, r)$. Thus $I_G(u, x) \subseteq I_G(u, r) \subseteq V(H)$, as H is a daisy graph of G with respect to r and analogously $I_G(v, y) \subseteq I_G(v, r) \subseteq V(H)$. Therefore $d_H(u, x) = d_G(u, x)$ and $d_H(v, y) = d_G(v, y)$. Since x and y lie on a shortest u, v-path we get

$$\begin{aligned} d_G(u, v) &= d_G(u, x) + d_G(x, y) + d_G(y, v) \\ &= d_G(u, x) + d_G(y, v) + 1 = d_H(u, x) + d_H(y, v) + 1 \geq d_H(u, v). \end{aligned}$$

Moreover, H is a subgraph of G and therefore $d_G(u, v) \leq d_H(u, v)$ and consequently H is an isometric subgraph of G. ∎

Definition. A graph G satisfies the triangle condition if for any three vertices $u, v, w \in V(G)$, such that $d(v, w) = 1$ and $d(u, v) = d(u, w) \geq 2$, there exists a vertex $x \in V(G)$ adjacent to v and w with $d(x, u) = d(u, v) - 1$.

Definition. A rooted graph G with the root r satisfies the rooted triangle condition if for any two adjacent vertices $v, w \in V(G)$, such that $d(r, v) = d(r, w) \geq 2$ there exists a vertex $x \in V(G)$ adjacent to v and w with $d(x, r) = d(r, v) - 1$.

Theorem 3. *Let G be a rooted graph with the root r such that G satisfies the rooted triangle condition. If every daisy graph of G with respect to r is isometric in G, then for any $u, v \in V(G)$ there exists a pseudo-median in G of size 0 or 1 for the triple u, v and r.*

Proof. Let u and v be two arbitrary vertices of a rooted graph G with the root r. Let $H = G_r(\{u, v\})$. Hence $V(H) = I_G(u, r) \cup I_G(v, r)$. Since H is an isometric subgraph of G, there exists a shortest u, v-path P in G which is entirely contained in H. Denote $P : u = u_0, u_1, \ldots, u_{k-1}, u_k = v$. As $P \subseteq V(H)$,

$u_i \in I_G(u,r) \cup I_G(v,r)$, for any $i \in \{0,1,\ldots,k\}$. If $v \in I_G(u,r)$, then (v,v,v) is a pseudo-median of (u,v,r) of size 0 and the proof is completed. Similarly, (u,u,u) is a pseudo-median of (u,v,r) of size 0 if $u \in I_G(v,r)$, and the proof is also completed in this case. Hence we may assume that $u \notin I_G(v,r)$ and $v \notin I_G(u,r)$. Let $j \in [k]_0$ be the largest index such that $u_j \in I_G(u,r)$. Hence $u_l \in I_G(v,r)$ for any $l \in \{j+1,\ldots,k\}$. If $u_j \in I_G(v,r)$, then $u_j \in I_G(u,r) \cap I_G(v,r)$ and hence (u_j,u_j,u_j) is a pseudo-median of (u,v,r) of size 0. Next, we assume that $u_j \notin I_G(v,r)$. Since $u_{j+1} \notin I_G(u,r)$, $d_G(u_j,r) = d_G(u_{j+1},r) = l$. If $l = 1$, then (u_j,u_{j+1},r) is a pseudo-median of (u,v,r) of size 1. If $l > 1$, then by the rooted triangle condition, there exists $x \in V(G)$ that is adjacent to u_j and u_{j+1} and $x \in I_G(r,u_j) \cap I_G(r,u_{j+1})$. Hence (u_j,u_{j+1},x) is a pseudo-median of (u,v,r) of size 1, which completes the proof. ∎

From the proof of Theorem 3 we get the following.

Corollary 4. *Let G be a rooted graph with the root r such that G satisfies the rooted triangle condition and let $\{u,v\} \subseteq V(G)$. If $H = G_r(\{u,v\})$ is isometric in G, then there exists a pseudo-median in G of size 0 or 1 for the triple u,v and r.*

Proposition 1, Theorem 2 and Theorem 3 give the following characterization of rooted graphs G with the root r satisfying the rooted triangle condition, such that every daisy graphs of G with respect to r is isometric in G.

Corollary 5. *Let G be a rooted graph with the root r such that G satisfies the rooted triangle condition. Every daisy graph of G with respect to r is isometric in G, if and only if for any $u,v \in V(G)$ there exists a pseudo-median of size 0 or 1 of the triple of vertices u,v and r.*

Lemma 6. *If $\mathcal{H}$ is a Hamming graph, then $\mathcal{H}$ satisfies the triangle condition.*

Proof. Let $u = (u_1,\ldots,u_n)$ and $v = (v_1,\ldots,v_n)$ be two adjacent vertices of $\mathcal{H}$ and $w = (w_1,\ldots,w_n) \in V(\mathcal{H})$ such that $d(u,w) = d(v,w) = k$, with $k \geq 2$. Since $uv \in E(\mathcal{H})$, there exists $i \in \{1,\ldots,n\}$ such that $v_i \neq u_i$ and $v_j = u_j$, for all $j \neq i$. Moreover, since $d(w,u) = d(w,v) = k$, it follows that $w_i \neq u_i$ and $w_i \neq v_i$. Let $x = (u_1,\ldots,u_{i-1},w_i,u_{i+1},\ldots,u_n)$. Clearly, $xu \in E(\mathcal{H})$ and $xv \in E(\mathcal{H})$ and $x \in I_{\mathcal{H}}(u,w) \cap I_{\mathcal{H}}(v,w)$. The assertion follows. ∎

Lemma 6 and Corollary 5 imply the following.

Corollary 7. *Let $\mathcal{H}$ be a Hamming graph with the root r. Every daisy graph of $\mathcal{H}$ with respect to r is isometric in $\mathcal{H}$, if and only if for any $u,v \in V(\mathcal{H})$ there exists a pseudo-median of size 0 or 1 for the triple u,v and r.*

The above results refer to rooted graphs G for which all daisy graphs with respect to the root are isometric. Now we chose one daisy graph H of G with respect to the root of G and study when is H isometric in G.

Note that one can easily deduce from the proofs of Proposition 1 and Theorem 2 that if G is a rooted graph with the root r and H a daisy graph of G with respect to r such that for any u and v in H, there exists a pseudo-median of size 0 or 1 of the triple of vertices u, v and r, then H is isometric in G. It is clear that the reverse statement is not necessarily true. For example, let G be the cycle C_6 and u and r two antipodal vertices of $C_6 = u, x_1, x_2, r, y_1, y_2, u$. Then $G_r(\{u\})$ is the whole graph G and thus isometric in G, but there clearly exists a triple of vertices in G, for example (x_1, y_2, r) having no pseudo-median of size 0 or 1 in G.

Problem 8. Let G be a rooted graph with the root r. Characterize daisy graphs of G with respect to r (generated by X) that are isometric in G.

Let G be a rooted graph with the root r. For $X = \{v\} \subseteq V(G)$ the above problem is equivalent to the characterization of intervals $I_G(v, r)$ that are isometric in G.

In the rest of this section we will consider Hamming graphs and study properties of isometric daisy subgraphs. Thus let $\mathcal{H} = \mathcal{H}_{k_1,\ldots,k_n}$ be a Hamming graph with the root $r = 0^n$. Let $G = \mathcal{H}_r(X)$ be a daisy graph of $\mathcal{H}$ with respect to r (generated by X). Note that if $|X| = 1$, then G is a daisy cube. Moreover, if $x = x_1 \cdots x_n$ is the vertex of X, then $G \cong Q_n(\{y_1 \cdots y_n\})$, where $y_i = \min\{x_i, 1\}$, for any $i \in \{1, \ldots, n\}$. For $|X| = 2$ we have the following characterization of isometric daisy graphs of a Hamming graph.

Theorem 9. *Let $\mathcal{H} = \mathcal{H}_{k_1,\ldots,k_n}$ be a Hamming graph with the root 0^n and let $G = \mathcal{H}_{0^n}(X)$ be a daisy graph of $\mathcal{H}$ generated by the set $X = \{x, y\}$ of cardinality 2. Then G is an isometric subgraph of $\mathcal{H}$ if and only if there exists a pseudo-median of $(x, y, 0^n)$ of size 0 or 1 in G.*

Proof. Let $G = \mathcal{H}_{0^n}(\{x, y\})$. Denote $x = x_1 \cdots x_n$, $y = y_1 \cdots y_n$ and $r = 0^n = r_1 \cdots r_n$.

Suppose first, G is an isometric subgraph of $\mathcal{H}$. By Lemma 6, the graph $\mathcal{H}$ satisfies the triangle condition and consequently also the rooted triangle condition. Using the same line of thought as in the proof of Theorem 3 one can easily check that there exists a pseudo-median of $(x, y, 0^n)$ of size 0 or 1 in G.

For the converse suppose that there is a pseudo-median of size 0 or 1 of $(x, y, 0^n)$ in G. Since the size of the pseudo-median in a Hamming graph is the number of coordinates in which x, y and r are all distinct, there is at most one coordinate in which x, y and r are all pairwise distinct. To simplify, permute factors of $\mathcal{H}$ such that x has the first $i - 1$ coordinates equal to 0 and all other coordinates different from 0 (i.e., $i - 1$ is the number of coordinates of x that are equal to 0), and if there exists a coordinate in which x, y and r are pairwise distinct, let this be the i^{th} coordinate. Since (x, y, r) has a pseudo-median of size 0 or 1, $y_j \in \{x_j, 0\}$, for any valid index $j > i$.

Let $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_n$ be two arbitrary vertices of G. Note, $V(G) = I_{\mathcal{H}}(x, 0^n) \cup I_{\mathcal{H}}(y, 0^n)$. We will prove that there exists a u, v-path in G with $d_G(u, v) = H(u, v) = d_{\mathcal{H}}(u, v)$.

Suppose first that $u, v \in I_{\mathcal{H}}(x, 0^n)$ (the case when $u, v \in I_{\mathcal{H}}(y, 0^n)$ is proved in a similar way). Then $u_j = v_j = 0$, for any $j < i$, and for any $j \geq i$, it holds that $u_j \in \{x_j, 0\}$ and $v_j \in \{x_j, 0\}$. We construct u, v-path of length $H(u, v)$ in G in the following way. Start in u and continue with $u^{(1)}$ which is obtained from u by replacing the first coordinate of u, say u_j, in which u and v differ, with v_j. Since $v_j \neq u_j$ and $u, v \in I_{\mathcal{H}}(x, 0^n)$, $\{v_j, u_j\} = \{x_j, 0\}$ and consequently $u^{(1)} \in I_{\mathcal{H}}(x, 0^n) \subseteq V(G)$. We continue in the same way step by step, such that at the step k we replace the first coordinate of $u^{(k)}$, say $u_j^{(k)}$, in which $u^{(k)}$ and v differ, with v_j. Since all the vertices $u^{(k)}$, for any valid k, are contained in $V(G)$ and the constructed path P is of length $H(u, v)$, P is a u, v-path of G of length $d_{\mathcal{H}}(u, v)$.

Finally, let $u \in I_{\mathcal{H}}(x, 0^n)$ and $v \in I_{\mathcal{H}}(y, 0^n) \setminus I_{\mathcal{H}}(x, 0^n)$.

Let I_D be the set of indices in which u and v differ. We will also use the following sets. The set $I_M = \{i' \in I_D \mid u_{i'} \neq 0 \wedge v_{i'} \neq 0\}$, this is an empty set, if (x, y, r) has a pseudo-median of size 0, otherwise it contains the index i. Let $I_u = \{i' \in I_D \mid u_{i'} = 0\}$ and $I_v = \{i' \in I_D \mid v_{i'} = 0\}$. Note that I_M, I_u and I_v form a partition of I_D.

We construct a u, v-path in the following way. The first part of the path is constructed by using all the indices from the set $I_v = \{i_1, i_2, \ldots, i_{|I_v|}\}$. Let $u^{(0)} = u$ be the first vertex of this path. The next vertex of the path, $u^{(1)}$, is obtained from $u^{(0)}$ by replacing the coordinate $u_{i_1}^{(0)}$ with 0. The vertex $u^{(2)}$, is obtained from $u^{(1)}$ by replacing the coordinate $u_{i_2}^{(1)}$ with 0. Assume we have already obtained the vertex $u^{(j)}$, then we obtain the vertex $u^{(j+1)}$ from $u^{(j)}$ by replacing the coordinate $u_{i_{j+1}}^{(j)}$ with 0. We do this for every index in I_v, so the last vertex we obtain is $u^{(|I_v|)}$. It is easy to see, that these vertices indeed form a path (two consecutive vertices differ in exactly one coordinate). Since we only change coordinates to 0, it is also clear that every vertex constructed so far belongs to $I_{\mathcal{H}}(u, 0^n) \subseteq I_{\mathcal{H}}(x, 0^n) \subseteq V(G)$.

If I_M is not an empty set, we form the next vertex in our path, say $v^{(0)}$, from $u^{(|I_v|)}$ by replacing the coordinate $u_i^{(|I_v|)}$ to v_i. Again, since $v^{(0)}$ and v differ only in indices of the set I_u and the values of coordinates at those indices in $v^{(0)}$ is 0, it is clear that $v^{(0)} \in I_{\mathcal{H}}(v, 0^n) \subseteq I_{\mathcal{H}}(y, 0^n) \subseteq V(G)$. If I_M is an empty set, we denote the vertex $u^{(|I_v|)}$ by $v^{(0)}$.

We continue with the construction of our u, v-path by using all the indices from the set $I_u = \{j_1, j_2, \ldots, j_{|I_u|}\}$. The next vertex of the path, $v^{(1)}$, is obtained from $v^{(0)}$ by replacing the coordinate $v^{(0)}_{j_1}$ with v_{j_1}. The vertex $v^{(2)}$, is obtained from $v^{(1)}$ by replacing the coordinate $v^{(1)}_{j_2}$ with v_{j_2}. Assume we have already obtained the vertex $v^{(k)}$, then we obtain the vertex $v^{(k+1)}$ from $v^{(k)}$ by replacing the coordinate $v^{(k)}_{j_{k+1}}$ with $v_{j_{k+1}}$. We do this for every index in I_u, so the last vertex we obtain is $v^{(|I_u|)}$. It is easy to see, that these vertices indeed form a path (two consecutive vertices differ in exactly one coordinate). Since we only change coordinates, say at index j', from 0 to $v_{j'}$, it is also clear that every vertex constructed in this part of the path belongs to $I_{\mathcal{H}}(v, 0^n) \subseteq I_{\mathcal{H}}(y, 0^n) \subseteq V(G)$. Note, that the vertex $v^{(|I_u|)}$ is actually the vertex v. The fact, that the sets I_M, I_u and I_v form a partition of I_D implies that the length of the constructed path is $H(u, v)$. This concludes our proof. ■

In section 3 we give a constructive characterization of isometric daisy graphs of a Hamming graph. The above characterization of isometric daisy graphs of a Hamming graph generated by a set of cardinality at most 2, rises the question about a non-constructive characterization of isometric daisy graphs of a Hamming graph generated by a set of cardinality at least 3. Note, this is a specific case of Problem 8.

3. Characterization of Isometric Daisy Hamming Graphs

Let G' be a daisy graph of a Hamming graph $\mathcal{H}' = \mathcal{H}_{k_1,\ldots,k_{n-1}}$ with respect to 0^{n-1}. Let G be a peripheral expansion of G' relative to $W'_0 = V(G'), W'_1, \ldots, W'_k$. If for any $i \in \{1, \ldots, k\}$, the graph $\langle W'_i \rangle_{\mathcal{H}'}$ is a daisy graph of $\mathcal{H}'$ with respect to 0^{n-1}, then the peripheral expansion $\operatorname{pe}\big(G'; W'_0, \ldots, W'_k\big)$ is called *daisy peripheral* expansion of G' relative to $W'_0, \ldots, W'_k$.

In this section we prove that isometric daisy graphs of a Hamming graph are precisely the graphs that can be obtained from K_1 by a sequence of daisy peripheral expansions.

Theorem 10. *Let $\mathcal{H} = \mathcal{H}_{k_1,\ldots,k_n}$ be a Hamming graph with the root 0^n. If G is an isometric daisy graph of $\mathcal{H}$ with respect to the root 0^n, then the daisy peripheral expansion of G relative to the sets $V(G) = W_0, \ldots, W_l$, is an isometric daisy graph of $\mathcal{H}' = K_{l+1} \Box \mathcal{H}$ with respect to 0^{n+1}.*

Proof. Let G' be the daisy peripheral expansion of G relative to $W_0, W_1, \ldots, W_l$. Therefore, G' consists of a disjoint union of a copy of $G = \langle W_0 \rangle$ and a copy of $\langle W_i \rangle$, for any $i \in \{1, \ldots, l\}$. We define the labels of the vertices of G' as follows. Prepend i to each vertex of G' corresponding to the copy of $\langle W_i \rangle$, for all

$i \in \{0, \dots, l\}$. Hence the labels of the vertices of G' are vectors of length $n+1$ and the first coordinate is an integer from $\{0, \dots, l\}$.

First, we prove that two vertices of G' are adjacent if and only if the corresponding vectors differ in exactly one position. Since G' is the expansion of G relative to $W_0, \dots, W_l$, it follows from the definition of expansion that two vertices $u' = u_1 \cdots u_n u_{n+1}$ and $v' = v_1 \cdots v_n v_{n+1}$ of G' are adjacent in G' if and only if $u = u_2 \cdots u_{n+1}$ and $v = v_2 \cdots v_{n+1}$ are adjacent in G and both belong to the same set W_i, or if $u = u_2 \cdots u_{n+1} = v = v_2 \cdots v_{n+1}$ and u belongs to two different sets W_{u_1} and W_{v_1}. The last condition directly implies that u' and v' differ in exactly one coordinate, namely the first coordinate. If $u = u_2 \cdots u_{n+1}$ and $v = v_2 \cdots v_{n+1}$ are adjacent in G and contained in the same set W_i, then u and v differ in exactly one coordinate. But then, since they are both in W_i, $u_1 = v_1 = i$ and hence u' and v' differ in exactly one coordinate. Hence G' is an induced subgraph of $\mathcal{H}' = K_{l+1} \Box \mathcal{H}$.

In the second step we prove that G' is a daisy graph of $\mathcal{H}'$ with respect to 0^{n+1}. Let $v' = v_0 v_1 \cdots v_n \in V(G')$ and let $x' = x_0 \cdots x_n \in I_{\mathcal{H}'}(v', 0^{n+1})$. Hence $x_i \in \{0, v_i\}$, for any $i \in \{0, \dots, n\}$. Since $v' = v_0 v_1 \cdots v_n$, it follows that $v = v_1 \cdots v_n \in W_{v_0}$. We know that the graph $\langle W_{v_0} \rangle$ is a daisy graph of $\mathcal{H}$ with respect to 0^n and $x = x_1 \cdots x_n \in I_{\mathcal{H}}(v, 0^n)$, therefore $x \in V(\langle W_{v_0} \rangle)$. Hence if $x' = 0 x_1 \cdots x_n$, then x' is in the copy of G in G'. If $x' = v_0 x_1 \cdots x_n$, then x' is in the copy of $\langle W_{v_0} \rangle$ in G'. In both cases we deduce that $x' \in V(G')$, which completes this part of the proof.

It remains to prove that G' is an isometric subgraph of $\mathcal{H}'$. Let $u' = u_0 \cdots u_n$ and $v' = v_0 \cdots v_n$ be two arbitrary vertices of G'.

If $u_0 = v_0$, then $u = u_1 \cdots u_n \in W_{u_0}$ and $v = v_1 \cdots v_n \in W_{u_0}$. Since G' is an expansion of G, relative to $W_0, \dots, W_l$, the definition of expansion implies that $\langle W_{u_0} \rangle$ is isometric in G. As G is isometric in $\mathcal{H}$,

$$d_{\langle W_{u_0} \rangle}(u, v) = d_G(u, v) = d_{\mathcal{H}}(u, v) = H(u, v).$$

Hence

$$d_{G'}(u', v') = d_{\langle W_{u_0} \rangle}(u, v) = H(u, v) = H(u', v') = d_{\mathcal{H}'}(u', v'),$$

where the penultimate equality holds because $u_0 = v_0$.

Finally, consider the case where $u_0 \neq v_0$. Hence $u = u_1 \cdots u_n \in W_{u_0}$ and $v = v_1 \cdots v_n \in W_{v_0}$. Since $\langle W_{u_0} \cup W_{v_0} \rangle$ is isometric in G (by the definition of expansion), there exists a shortest u, v-path $P : u = u^0, u^1, \dots, u^k = v$ in G (note that each u^i is a vertex in G and hence has the form $u^i = u^i_1 \cdots u^i_n$) which is entirely contained in $\langle W_{u_0} \cup W_{v_0} \rangle$. Since G is isometric in $\mathcal{H}$, we get

$$d_{\langle W_{u_0} \cup W_{v_0} \rangle}(u, v) = d_G(u, v) = d_{\mathcal{H}}(u, v) = H(u, v).$$

Let $i \in \{0, \ldots, k\}$ be the smallest index such that $u^i \in W_{v_0}$. Since there are no edges between $W_{u_0} \setminus W_{v_0}$ and $W_{v_0} \setminus W_{u_0}$, $u^i \in W_{u_0}$. Then the path $u' = u^{0'}, u^{1'}, \ldots, u^{i'}, v^{i'}, \ldots, v^{k'} = v'$, where $u^{l'} = u_0 u^l$, for any $l \in \{0, \ldots, i\}$ and $v^{l'} = v_0 u^l$, for any $l \in \{i, \ldots, k\}$, is a u', v'-path in G'. Hence

$$d_{G'}(u', v') \leq d_G(u, v) + 1 = H(u, v) + 1 = H(u', v') = d_{\mathcal{H}'}(u', v'),$$

where the penultimate equality holds because $u_0 \neq v_0$. Since G' is a subgraph of $\mathcal{H}'$, the assertion follows. ■

Let G be an isometric daisy graph of a Hamming graph $\mathcal{H} = \mathcal{H}_{k_1,\ldots,k_n}$ with respect to 0^n, where $\mathcal{H}$ is the smallest possible. We introduce the following terminology which will be used throughout this section. For any $j \in [n]$ we define the following sets.

$$\begin{aligned}
W_i^j &= \{u = u_1 \cdots u_n \in V(G) \mid u_j = i\}, \text{ for any } i \in [k_j]_0;\\
U_i^j &= \left\{x \in W_i^j \mid \exists y \in W_0^j : xy \in E(G)\right\}, \text{ for any } i \in [k_j]_0;\\
U_{0i}^j &= \left\{x \in W_0^j \mid \exists y \in W_i^j : xy \in E(G)\right\}, \text{ for any } i \in \{1, \ldots, k_j - 1\};\\
U_0^j &= \bigcup_{i=1}^{k_j-1} U_{0i}^j.
\end{aligned}$$

Also, for any $j \in [n]$ and any $i \in [k_j]_0$ denote by e_i^j the vertex of the Hamming graph $\mathcal{H}$ labeled by $0^{j-1} i 0^{n-j}$.

Lemma 11. *Let G be an isometric daisy graph of a Hamming graph $\mathcal{H} = \mathcal{H}_{k_1,\ldots,k_n}$ with respect to 0^n, where $\mathcal{H}$ is the smallest possible. For any $j \in [n]$ and any $i \in [k_j]_0$, if $W_i^j \neq \emptyset$, then there exists $uv \in E(G)$ such that $W_i^j = W_{uv}$.*

Proof. Let $j \in [n]$ and $i \in [k_j]_0$ be arbitrary, with $W_i^j \neq \emptyset$, and $x = x_1 \cdots x_n \in W_i^j$. Hence $x_j = i$. Since G is a daisy graph of $\mathcal{H}$ with respect to 0^n and $x' = 0^{j-1} i 0^{n-j} \in I_{\mathcal{H}}(0^n, x)$, it follows that $x' \in V(G)$. Since $x'_j = i$, $x' \in W_i^j$. Then $W_{x'0^n}$ contains exactly all the vertices of G, that are closer to x' than 0^n, i.e., all vertices of G with j-th coordinate equal to i. Hence $W_{x'0^n} = W_i^j$. ■

For the edge uv of a partial Hamming graph, the sets W_{uv} have many nice properties [1, 4, 15]. Since our graph G is a partial Hamming graph, it follows from Lemma 11 that the sets W_i^j also have these properties.

Lemma 12. *Let G be an isometric daisy graph of a Hamming graph $\mathcal{H} = \mathcal{H}_{k_1,\ldots,k_n}$ with respect to 0^n, where $\mathcal{H}$ is the smallest possible. For any $\triangle$-class F of G, there exists an edge $f \in F$ with 0^n as an endpoint.*

Proof. Let F be an arbitrary $\triangle$-class of G and $uv \in F$, where $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_n$. Hence, $u_i \neq v_i$, for some $i \in [n]$, and $u_j = v_j$, for any $j \in [n] \setminus \{i\}$.

First, suppose that one of u_i and v_i equals 0, say u_i. It follows that $0^n \in W_{uv}$. Since $e^i_{v_i} \in I_{\mathcal{H}}(v, 0^n)$ and G is a daisy graph of $\mathcal{H}$ with respect to 0^n, it follows that $e^i_{v_i} \in V(G)$. Since the i^{th} coordinate of $e^i_{v_i}$ is v_i, the vertex $e^i_{v_i} \in W_{vu}$. Hence, $0^n e^i_{v_i} \sim uv$ and therefore $0^n e^i_{v_i} \in F$.

Finally, suppose neither u_i nor v_i equals 0. Since $x = u_1 \cdots u_{i-1} 0 u_{i+1} \cdots u_n \in I_{\mathcal{H}}(u, 0^n)$ and G is a daisy graph of $\mathcal{H}$ with respect to 0^n, the vertex $x \in V(G)$. Note that u, v and x induce K_3 in G. Hence, $vx \bigtriangleup uv$ and consequently the edge vx belongs to F. Now, consider the vertex $e^i_{v_i}$, which belongs to $I_{\mathcal{H}}(v, 0^n)$ and therefore is a vertex of G. Similarly to the first case, we deduce that $e^i_{v_i} \in W_{vx}$. Clearly, $0^n \in W_{xv}$ and $0^n e^i_{v_i}$ is an edge of G. It follows that $0^n e^i_{v_i} \sim xv$ and therefore $0^n e^i_{v_i} \in F$. ∎

From the definition of the relation $\bigtriangleup$ it follows that the $\bigtriangleup$-class F_j generated by the edge $0^n e^j_i$, for some $i \neq 0$, contains exactly all edges between U^j_k and U^j_l, for any $0 \leq k < l \leq k_j - 1$. Thus using Lemma 12 we deduce the following.

Corollary 13. *Let G be an isometric daisy graph of a Hamming graph $\mathcal{H} = \mathcal{H}_{k_1,\ldots,k_n}$ with respect to 0^n, where $\mathcal{H}$ is the smallest possible. There are exactly n $\bigtriangleup$-classes $F_1, \ldots, F_n$ of $E(G)$, where for any $j \in [n]$ the $\bigtriangleup$-class F_j is generated by the edge $0^n e^j_i$, for some $0 < i \leq k_j - 1$.*

Let G be an isometric daisy graph of a Hamming graph $\mathcal{H} = \mathcal{H}_{k_1,\ldots,k_n}$ with respect to 0^n, where $\mathcal{H}$ is the smallest possible. Let $j \in [n]$ and $i \in [k_j]_0$. A subgraph $\langle W^j_i \rangle$ of a graph G is called *peripheral* if $U^j_i = W^j_i$. The $\bigtriangleup$-class F generated by the edge $0^n e^j_l$, for some $0 < l \leq k_j - 1$, of the graph G is called *peripheral* if $U^j_{l'} = W^j_{l'}$, for any $l' \in \{1, \ldots, k_j - 1\}$.

Lemma 14. *If G is an isometric daisy graph of a Hamming graph $\mathcal{H} = \mathcal{H}_{k_1,\ldots,k_n}$ with respect to 0^n, where $\mathcal{H}$ is the smallest possible, then every $\bigtriangleup$-class F of the graph G is peripheral.*

Proof. Let F be an arbitrarily chosen $\bigtriangleup$-class of G, such that $0^n e^j_l \in F$. Let $i \in \{1, \ldots, k_j - 1\}$ be arbitrary. To prove the assertion, we will show that any vertex of W^j_i has a neighbour in W^j_0 (which means $W^j_i = U^j_i$). Take any $x = x_1 \cdots x_n \in W^j_i$, hence $x_j = i$. Now, consider $x' = x_1 \cdots x_{j-1} 0 x_{j+1} \cdots x_n$. Note, that $x' \in I_{\mathcal{H}}(0^n, x) \subseteq V(G)$ and therefore $x' \in W^j_0$. Since $xx' \in E(G)$, the assertion follows. ∎

Lemma 15. *Let G be an isometric daisy graph of a Hamming graph $\mathcal{H} = \mathcal{H}_{k_1,\ldots,k_n}$ with respect to 0^n, where $\mathcal{H}$ is the smallest possible. For every $j \in [n]$ and any $i \in [k_j]_0$ the subgraph $\langle W^j_i \rangle$ of the graph G is a daisy graph of $\mathcal{H}' = \mathcal{H}_{k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_n}$ with respect to 0^{n-1}.*

Proof. Define $X^j_i = \{x_1 \cdots x_{j-1} x_{j+1} \cdots x_n \mid x_1 \cdots x_n \in W^j_i\}$. Let $r : W^j_i \to X^j_i$

be the projection defined by $r : x_1 \cdots x_n \mapsto x_1 \cdots x_{j-1}x_{j+1} \cdots x_n$, which is clearly bijection between W_i^j and X_i^j.

Let $u = u_1 \cdots u_{n-1} \in X_i^j$ be arbitrary and $w \in I_{\mathcal{H}'}(0^{n-1}, u)$. We claim that $w \in X_i^j$. Since $u \in X_i^j$, it follows from the definition of X_i^j that $u' = u_1 \cdots u_{j-1} i u_j \cdots u_{n-1} \in W_i^j$. Since $w \in I_{\mathcal{H}'}(0^{n-1}, u)$, it follows that $w_l = u_l$ or $w_l = 0$, for all $1 \le l \le n-1$. Let $w' = w_1 \cdots w_{j-1} i w_j \cdots w_{n-1}$. Since $w' \in I_{\mathcal{H}}(0^n, u')$, it follows that $w' \in V(G)$ and as the i^{th} coordinate of w' is i, the vertex w' belongs to W_i^j. By the definition of X_i^j, $w \in X_i^j$. Therefore $\langle X_i^j \rangle_{\mathcal{H}'}$ is a daisy graph of $\mathcal{H}'$ with respect to 0^{n-1}. Since $\langle W_i^j \rangle_{\mathcal{H}} \cong \langle X_i^j \rangle_{\mathcal{H}'}$, the assertion follows. ■

In [1] the contraction of a partial Hamming graph G was defined in the following way. Let $uv \in E(G)$ and let $\triangle$-class with respect to $uv \in E(G)$, denote it by $\triangle_{uv}$, be the union of k distinct $\sim$-classes $F_{x_i x_j}$. A graph G' is a contraction of a partial Hamming graph G with respect to the edge $uv \in E(G)$ if each clique induced by edges belonging to $\triangle_{uv}$ is contracted to a single vertex. For all $i \in [k]$, let W_i' be the set of vertices in G' that corresponds to $W_{x_i} = \{w \in V(G) \mid d(w, x_i) < d(w, x_j),$ for any $j \neq i\}$. Brešar proved that the expansion of G' relative to $W_1', \ldots, W_k'$ is exactly the graph G [1].

Theorem 16. *Let G be an isometric daisy graph of a graph $\mathcal{H} = \mathcal{H}_{k_1,\ldots,k_n}$ with respect to 0^n, where $\mathcal{H}$ is the smallest possible. Then there exists a daisy graph $G' \subseteq G$ such that G can be obtained from G' by a daisy peripheral expansion.*

Proof. Let F be an arbitrary $\triangle$-class of the graph G. By Corollary 13 there exist $j \in [n]$ and $i \in \{1, \ldots, k_j - 1\}$ such that F is generated by the edge $0^n e_i^j$. Let the graph G' be obtained from the graph G by a contraction with respect to the edge $0^n e_i^j$. For any $l \in [k_j]_0$, denote by X_l the set of vertices in G' that corresponds to W_l^j in G. By the definition of a contraction, the graph G is the expansion of G' relative to sets $X_0, \ldots, X_{k_j-1}$. By Lemma 14, it follows that F is a peripheral $\triangle$-class. Using the fact that F is generated by the edge $0^n e_i^j$, it follows from the definition of peripheral classes, that $U_{i'}^j = W_{i'}^j$, for any $i' \in \{1, \ldots, k_j - 1\}$ (every vertex of $W_{i'}^j$ has a neighbour in W_0^j). Since $\bigcup_{i=0}^{k_j-i} X_i = V(G)$ (definition of expansion) we obtain that $X_0 = V(G')$. By Lemma 15, it follows that the subgraphs $\langle X_i \rangle_{G'}$ are daisy graphs which proves that G is obtained from G' by daisy peripheral expansion. ■

From Theorem 10 and Theorem 16 we immediately obtain the following characterization.

Theorem 17. *A graph G is an isometric daisy graph of a graph $\mathcal{H} = \mathcal{H}_{k_1,\ldots,k_n}$ with respect to 0^n, where $\mathcal{H}$ is the smallest possible, if and only if it can be obtained from the one vertex graph by a sequence of daisy peripheral expansions.*

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