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STRENGTHENING SOME COMPLEXITY RESULTS ON TOUGHNESS OF GRAPHS

Gyula Y. Katona

Department of Computer Science and Information Theory
Budapest University of Technology and Economics, Hungary
and
MTA-ELTE Numerical Analysis and Large Networks Research Group, Hungary

e-mail: katona.gyula@vik.bme.hu

and

Kitti Varga

Department of Computer Science and Information Theory
Budapest University of Technology and Economics, Hungary
and
Alfréd Rényi Institute of Mathematics, Hungary

e-mail: vkitti@cs.bme.hu

Abstract

Let t be a positive real number. A graph is called t-tough if the removal of any vertex set S that disconnects the graph leaves at most $|S|/t$ components. The toughness of a graph is the largest t for which the graph is t-tough.

The main results of this paper are the following. For any positive rational number $t \leq 1$ and for any $k \geq 2$ and $r \geq 6$ integers recognizing t-tough bipartite graphs is coNP-complete (the case $t = 1$ was already known), and this problem remains coNP-complete for k-connected bipartite graphs, and so does the problem of recognizing 1-tough r-regular bipartite graphs. To prove these statements we also deal with other related complexity problems on toughness.

Keywords: toughness, coNP-complete.

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1. Introduction

All graphs considered in this paper are finite, simple and undirected. Let $\omega(G)$ denote the number of components, $\alpha(G)$ the independence number, $\kappa(G)$ the connectivity number and $\delta(G)$ the minimum degree of a graph G. For a vertex v of G the degree of v is denoted by $d(v)$. (Using $\omega(G)$ to denote the number of components may be confusing, however, most of the literature on toughness uses this notation.)

The notion of toughness was introduced by Chvátal [6] to investigate Hamiltonicity.

Definition. Let t be a real number. A graph G is called *t-tough* if $|S| \geq t\omega(G-S)$ holds for any vertex set $S \subseteq V(G)$ that disconnects the graph (i.e., for any $S \subseteq V(G)$ with $\omega(G-S) > 1$). The *toughness* of G, denoted by $\tau(G)$, is the largest t for which G is t-tough, taking $\tau(K_n) = \infty$ for all $n \geq 1$.

We say that a cutset $S \subseteq V(G)$ is a *tough set* if $\omega(G-S) = |S|/\tau(G)$.

Clearly, if a graph is Hamiltonian, then it must be 1-tough. However, not every 1-tough graph contains a Hamiltonian cycle. A well-known counterexample is the Petersen graph. On the other hand, Chvátal conjectured that there exists a constant t_0 such that every t_0-tough graph is Hamiltonian [6]. This conjecture is still open, but it is known that, if exists, t_0 must be at least $9/4$ [5].

The complexity of recognizing t-tough graphs has also been in the interest of research. This paper is motivated by two open problems regarding the complexity of recognizing 1-tough 3-connected bipartite graphs and 1-tough 3-regular bipartite graphs.

Let t be an arbitrary positive rational number and consider the following problem.

***t*-Tough**
Instance: a graph G.
Question: is it true that $\tau(G) \geq t$?

It is easy to see that for any positive rational number t the problem t-Tough is in coNP: a witness is a vertex set S whose removal disconnects the graph and leaves more than $|S|/t$ components. Bauer *et al.* proved that this problem is coNP-complete [1] and the problem 1-Tough remains coNP-complete for at least 3-regular graphs [4].

Theorem 1 [1]**.** *For any positive rational number t the problem t-*Tough *is coNP-complete.*

Theorem 2 [4]**.** *For any fixed integer $r \geq 3$ the problem* 1-Tough *is coNP-complete for r-regular graphs.*

Although the toughness of any bipartite graph (except for the graphs K_1 and K_2) is at most one, the problem 1-Tough does not become easier for bipartite graphs.

Theorem 3 [8]**.** *The problem* 1-Tough *is coNP-complete for bipartite graphs.*

Let t be an arbitrary positive rational number and now consider a variant of the problem t-Tough.

Exact-t-Tough
Instance: a graph G.
Question: is it true that $\tau(G) = t$?

Extremal problems usually seem not to belong to NP $\cup$ coNP, therefore a complexity class called DP was introduced by Papadimitriou and Yannakakis [9].

Definition. A language L is in the class *DP* if there exist two languages $L_1 \in$ NP and $L_2 \in$ coNP such that $L = L_1 \cap L_2$.

A language is called *DP-hard* if all problems in DP can be reduced to it in polynomial time. A language is *DP-complete* if it is in DP and it is DP-hard.

We mention that DP $\neq$ NP $\cap$ coNP if NP $\neq$ coNP. Moreover, NP $\cup$ coNP $\subseteq$ DP. Now we present some related DP-complete problem.

ExactClique
Instance: a graph G and a positive rational number k.
Question: is it true that the largest clique of G has size exactly k?

Theorem 4 [9]**.** *The problem* ExactClique *is DP-complete.*

By taking the complement of the graph, we can obtain ExactIndependenceNumber from ExactClique.

ExactIndependenceNumber
Instance: a graph G and a positive rational number k.
Question: is it true that $\alpha(G) = k$?

Since the clique number of a graph is exactly k if and only if the independence number of its complement is exactly k, it follows from Theorem 4 that the problem ExactIndependenceNumber is also DP-complete.

Corollary 5. *The problem* ExactIndependenceNumber *is DP-complete.*

In this paper, first, we prove that Exact-t-Tough is DP-complete for any positive rational number t, moreover, if $t < 1$, then the problem remains DP-complete for bipartite graphs. Note that since the toughness of any bipartite graph (except for K_1 and K_2) is at most 1, the problem Exact-1-Tough-Bipartite is coNP-complete as stated in Theorem 3.

Theorem 6. *For any positive rational number t the problem* Exact-t-Tough *is DP-complete.*

Theorem 7. *For any positive rational number $t < 1$ the problem* Exact-t-Tough *remains DP-complete for bipartite graphs.*

Theorem 8. *For any positive rational number $t \leq 1$ the problem* t-Tough *remains coNP-complete for bipartite graphs.*

Note that Theorem 8 contains Theorem 3 as a special case.

Our constructions used in the proofs of the above three theorems also provide alternative proofs for Theorems 1 and 3. Furthermore, using the same construction as in the proof of Theorem 7, we also prove that t-Tough remains coNP-complete for k-connected bipartite graphs and so does 1-Tough for r-regular bipartite graphs, where $t \leq 1$ is an arbitrary rational number and $k \geq 2$ and $r \geq 6$ are integers. Determining the complexity of recognizing k-connected bipartite graphs and 1-tough 3-regular bipartite graphs was posed as an open problem in [3]. The latter problem remains open along with the problems of recognizing 1-tough 4-regular and 5-regular bipartite graphs.

Theorem 9. *For any fixed integer $k \geq 2$ and positive rational number $t \leq 1$ the problem* t-Tough *remains coNP-complete for k-connected bipartite graphs.*

Theorem 10. *For any fixed integer $r \geq 6$ the problem* 1-Tough *remains coNP-complete for r-regular bipartite graphs.*

In order to prove Theorem 10, we study the problem 1/2-Tough in the class of r-regular graphs. We show that it is coNP-complete if $r \geq 5$ but is in P if $r \leq 4$. (Note that the cases $r = 1$ and $r = 2$ are trivial.)

Theorem 11. *For any fixed integer $r \geq 5$ the problem* 1/2-Tough *remains coNP-complete for r-regular graphs.*

Theorem 12. *For any positive rational number $t < 2/3$ there is a polynomial time algorithm to recognize t-tough* 3*-regular graphs.*

Theorem 13. *There is a polynomial time algorithm to recognize* 1/2*-tough* 4*-regular graphs.*

Note that by Theorem 2, recognizing 1-tough 3-regular graphs is coNP-complete. We remark that the toughness of a 3-regular graph (except for K_4) is at most 3/2 and Jackson and Katerinis gave a characterization of cubic 3/2-tough graphs and these graphs can be recognized in polynomial time [7]. Their characterization uses the concept of inflation, which was introduced by Chvátal in [6], but is not presented here.

Theorem 14 [7]. *A cubic graph G is $3/2$-tough if and only if $G \simeq K_4$, $G \simeq K_2 \times K_3$, or G is the inflation of a 3-connected cubic graph.*

This paper is structured as follows. After proving some useful lemmas in Section 2, we prove Theorem 6 in Section 3. In Section 4 we prove two theorems about bipartite graphs, Theorems 7 and 9. Section 5 is about regular graphs, where we prove Theorems 10–13.

2. Preliminaries

In this section we prove some useful lemmas.

Proposition 15. *Let $G \not\simeq K_1, K_2$ be a $1/2$-tough graph. Then there exists a spanning subgraph H of G for which $\tau(H) = 1/2$.*

Proof. Let H be a spanning subgraph of G so that $\tau(H) \geq 1/2$ and there exists an edge $e \in E(H)$ for which $\tau(H-e) < 1/2$. (Note that since $\tau(G) \geq 1/2$, such a spanning subgraph H can be obtained by repeatedly deleting some edges of G.) Now we show that $\tau(H) \leq 1/2$, which implies that $\tau(H) = 1/2$. Let $e \in E(G)$ be an edge for which $\tau(H-e) < 1/2$.

Case 1. e is a bridge in H. Since G is $1/2$-tough, it is connected. Since $G \not\simeq K_1, K_2$ and G is connected, the graphs G and H have at least three vertices. Hence, at least one of the endpoint of e is a cut-vertex in H, so $\tau(H) \leq 1/2$.

Case 2. e is not a bridge in H. Then there exists a cutset S in $H - e$ for which

$$\omega\big((H-e)-S\big) > 2|S|.$$

Case 2.1. (e is not a bridge in H) and S is a cutset in H. Then

$$\omega(H-S) \leq 2|S|,$$

which is only possible if

$$\omega(H-S) = 2|S| \qquad \text{and} \qquad \omega\big((H-e)-S\big) = 2|S|+1.$$

Therefore, $\tau(H) \leq 1/2$.

Case 2.2. (e is not a bridge in H) and S is not a cutset in H. This is only possible if

$$\omega\big((H-e)-S\big) = 2.$$

Hence

$$2 = \omega\big((H-e)-S\big) > 2|S|,$$

i.e., $|S| < 1$, which means that $S = \emptyset$, so e is a bridge H, which is a contradiction. ■

Proposition 16. *Let $t \le 1$ be a positive rational number and G a t-tough graph. Then*

$$\omega(G - S) \le |S|/t$$

for any proper subset S of $V(G)$.

Proof. If S is a cutset in G, then by the definition of toughness $\omega(G-S) \le |S|/t$ holds.

If S is not a cutset in G, then $\omega(G - S) = 1$ since $S \neq V(G)$. On the other hand, $|S|/t \ge 1$ since $S \neq \emptyset$ and $t \le 1$. Therefore, $\omega(G - S) \le |S|/t$ holds in this case as well. ■

As is clear from its proof, the above proposition holds even if S is not a cutset. However, it does not hold if $t > 1$ and S is not a cutset: if $t > 1$, then the graph cannot contain a cut-vertex; therefore $\omega(G - S) = 1$ for any subset S with $|S| = 1$, while $|S|/t = 1/t < 1$.

Proposition 17. *Let G be a connected noncomplete graph on n vertices. Then $\tau(G)$ is a positive rational number, and if $\tau(G) = a/b$, where a, b are relatively prime positive integers, then $1 \le a, b \le n - 1$.*

Proof. By definition,

$$\tau(G) = \min_{\substack{S \subseteq V(G) \\ \omega(G-S) \ge 2}} \frac{|S|}{\omega(G - S)}$$

for a noncomplete graph G. Since G is connected and noncomplete, $1 \le |S| \le n-2$ for every $S \subseteq V(G)$ with $\omega(G-S) \ge 2$. Obviously, $\omega(G-S) \ge 2$ and since G is connected, $\omega(G - S) \le n - 1$. ■

The following is a trivial consequence of Proposition 17.

Corollary 18. *Let G and H be two connected noncomplete graphs on n vertices. If $\tau(G) \neq \tau(H)$, then*

$$\bigl|\tau(G) - \tau(H)\bigr| > \frac{1}{n^2}.$$

Claim 19. *For any positive rational number t the problem* Exact-t-Tough *belongs to DP.*

Proof. For any positive rational number t,

$$\begin{aligned} &\textsc{Exact-}t\textsc{-Tough} = \{G \text{ graph} \mid \tau(G) = t\} \\ &= \{G \text{ graph} \mid \tau(G) \ge t\} \cap \{G \text{ graph} \mid \tau(G) \le t\}. \end{aligned}$$

Let

$$L_1 = \{G \text{ graph} \mid \tau(G) \le t\}$$

and

$$L_2 = \{G \text{ graph} \mid \tau(G) \geq t\}.$$

Notice that $L_2 = t$-Tough and it is known to be in coNP: a witness is a vertex set $S \subseteq V(G)$ whose removal disconnects G and leaves more than $|S|/t$ components.

Now we show that $L_1 \in$ NP, i.e., we can express L_1 in the form

$$L_1 = \{G \text{ graph} \mid \tau(G) < t + \varepsilon\},$$

which is the complement of a language belonging to coNP. Let G be an arbitrary graph on n vertices. If G is disconnected, then $\tau(G) = 0$, and if G is complete, then $\tau(G) = \infty$, so in both cases $\tau(G) \leq t$ if and only if $\tau(G) < t + \varepsilon$ for any positive ε. If G is connected and noncomplete, then from Corollary 18 it follows that $\tau(G) \leq t$ if and only if $\tau(G) < t + 1/n^2$. Therefore,

$$L_1 = \{G \text{ graph} \mid \tau(G) \leq t\} = \left\{G \text{ graph} \;\middle|\; \tau(G) < t + \frac{1}{|V(G)|^2}\right\},$$

so $L_1 \in$ NP. Hence, we can conclude that Exact-t-Tough $= L_1 \cap L_2 \in$ DP. □

For any positive rational number t let Exact-t-Tough-Bipartite denote the problem of determining whether a given bipartite graph has toughness t. Since the toughness of a bipartite graph is at most 1 (except for the graphs K_1 and K_2), we can conclude the following.

Corollary 20. *For any positive rational number $t \leq 1$ the problem* Exact-t-Tough-Bipartite *belongs to DP. Moreover,* Exact-1-Tough-Bipartite *belongs to coNP.*

3. The Complexity of Determining the Toughness of General Graphs, Proof of Theorem 6

***Proof of Theorem* 6.** In Claim 19 we already proved that Exact-t-Tough $\in$ *DP*. To prove Exact-t-Tough is DP-hard, we reduce ExactIndependenceNumber (which is DP-complete by Corollary 5) to it.

Let G be an arbitrary connected graph on the vertices $v_1, \ldots, v_n$ and let a, b be positive integers such that $t = a/b$. Let k be a positive integer and let G_k be the following graph. For all $i \in [n]$ let

$$V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,a}\},$$

and let

$$V = \bigcup_{i=1}^{n} V_i, \quad U = \bigcup_{i=1}^{n} \bigcup_{j=1}^{b} u_{i,j}, \quad U' = \left\{u'_1, \ldots, u'_{(b-1)k}\right\}, \quad W = \{w_1, \ldots, w_{ak}\},$$

$$V(G_k) = V \cup U \cup U' \cup W.$$

For all $i \in [n]$ place a clique on V_i. For all $i_1, i_2 \in [n]$ if $v_{i_1}v_{i_2} \in E(G)$, then place a complete bipartite graph on $(V_{i_1}; V_{i_2})$. For all $i \in [n]$ and $j \in [b]$ connect $u_{i,j}$ to every vertex of V_i. Place a clique on W and connect every vertex of W to every vertex of $V \cup U \cup U'$, see Figure 1.

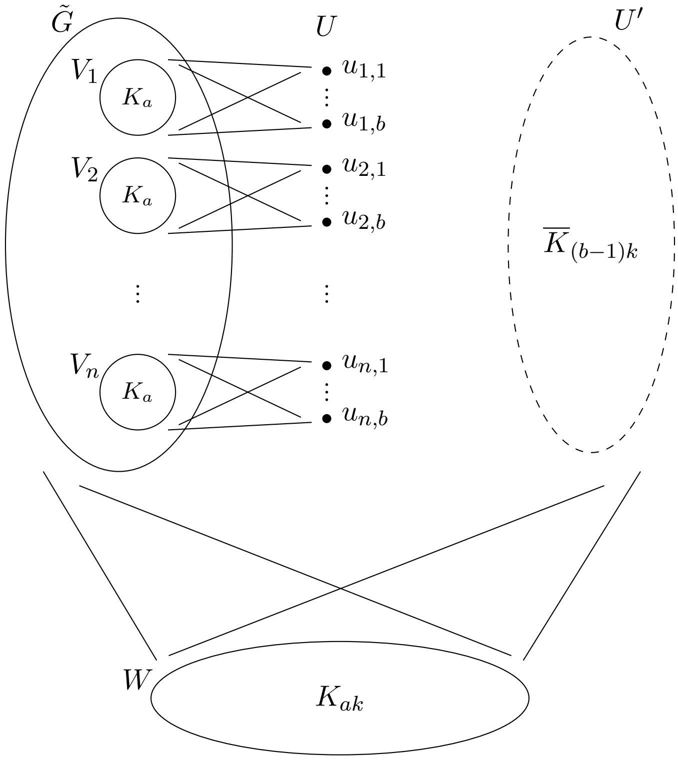

Figure 1. The graph G_k.

Obviously, G_k can be constructed from G in polynomial time. Now we show that $\alpha(G) = k$ if and only if $\tau(G_k) = t = a/b$, i.e.,

– if $\alpha(G) > k$, then
$$\frac{|S|}{\omega(G_k - S)} > t$$
for any cutset S of G_k;

– if $\alpha(G) < k$, then there exists a cutset S_0 of G_k such that
$$\frac{|S_0|}{\omega(G_k - S_0)} < t;$$

– if $\alpha(G) = k$, then
$$\frac{|S|}{\omega(G_k - S)} > t$$

for any cutset S of G_k and there exists a cutset S_0 of G_k such that

$$\frac{|S_0|}{\omega(G_k - S_0)} < t.$$

Let $S \subseteq V(G_k)$ be an arbitrary cutset of G_k. Since S is a cutset, it must contain W. Let

$$I = \{i \in [n] \mid V_i \subseteq S\}.$$

After the removal of W, the removal of any vertex of $U \cup U'$ or the removal of only a proper subset of V_i for any $i \in [n]$ does not disconnect anything in the graph. So consider the cutset

$$S' = S \setminus \left[(U \cup U') \cup \left(\bigcup_{i \notin I} V_i \right) \right].$$

In $G_k - S'$ there are two types of components: isolated vertices from $U \cup U'$ and components containing at least one vertex from V. There are at most $\alpha(G)$ components of the second type since picking a vertex from each such component forms an independent set of $G[V]$. On the other hand, there are exactly $b|I| + |U'| = b|I| + (b-1)k$ components of the first type. So

$$|S| \geq |S'| = \sum_{i \in I} |V_i| + |W| = a|I| + ak = a\big(|I| + k\big)$$

and

$$\omega(G_k - S) = \omega(G_k - S') \leq \alpha(G) + b|I| + (b-1)k = b\big(|I| + k\big) + \big(\alpha(G) - k\big).$$

Therefore,

$$\frac{|S|}{\omega(G_k - S)} \geq \frac{|S'|}{\omega(G_k - S')} \geq \frac{a\big(|I| + k\big)}{b\big(|I| + k\big) + \big(\alpha(G) - k\big)}.$$

Let $\{v_j \in V(G) \mid j \in J\}$ be an independent set of size $\alpha(G)$ in the graph G for some $J \subseteq [n]$, and consider another cutset

$$S_0 = \left(\bigcup_{i \notin J} V_i \right) \cup W$$

in G_k. Then

$$|S_0| = a\big(n - \alpha(G)\big) + ak = a\big(n - \alpha(G) + k\big)$$

and (similarly as before)

$$\omega(G_k - S_0) = \alpha(G) + b\bigl(n - \alpha(G)\bigr) + (b-1)k = b\bigl(n - \alpha(G) + k\bigr) + \bigl(\alpha(G) - k\bigr),$$

so

$$\frac{|S_0|}{\omega(G_k - S_0)} = \frac{a\bigl(n - \alpha(G) + k\bigr)}{b\bigl(n - \alpha(G) + k\bigr) + \bigl(\alpha(G) - k\bigr)}.$$

Case 1. $\alpha(G) < k$. Then

$$\frac{|S|}{\omega(G_k - S)} \geq \frac{a\bigl(|I| + k\bigr)}{b\bigl(|I| + k\bigr) + \bigl(\alpha(G) - k\bigr)} > \frac{a\bigl(|I| + k\bigr)}{b\bigl(|I| + k\bigr)} = \frac{a}{b} = t$$

holds for every cutset S of G_k, which implies that $\tau(G_k) > t$.

Case 2. $\alpha(G) = k$. Then

$$\frac{|S|}{\omega(G_k - S)} \geq \frac{a\bigl(|I| + k\bigr)}{b\bigl(|I| + k\bigr) + \bigl(\alpha(G) - k\bigr)} = \frac{a\bigl(|I| + k\bigr)}{b\bigl(|I| + k\bigr)} = \frac{a}{b} = t$$

holds for every cutset S of G_k, which implies that $\tau(G_k) \geq t$.

On the other hand,

$$\tau(G_k) \leq \frac{|S_0|}{\omega(G_k - S_0)} = \frac{a\bigl(n - \alpha(G) + k\bigr)}{b\bigl(n - \alpha(G) + k\bigr) + \bigl(\alpha(G) - k\bigr)} = \frac{an}{bn} = \frac{a}{b} = t.$$

Hence, $\tau(G_k) = t$.

Case 3. $\alpha(G) > k$. Then

$$\begin{aligned}\tau(G_k) &\leq \frac{|S_0|}{\omega(G_k - S_0)} = \frac{a\bigl(n - \alpha(G) + k\bigr)}{b\bigl(n - \alpha(G) + k\bigr) + \bigl(\alpha(G) - k\bigr)} \\ &< \frac{a\bigl(n - \alpha(G) + k\bigr)}{b\bigl(n - \alpha(G) + k\bigr)} = \frac{a}{b} = t.\end{aligned}$$

This means that $\alpha(G) = k$ if and only if $\tau(G_k) = t = a/b$. ■

The construction we used here is a slight modification of the one that Bauer *et al.* used in [2] for proving that for any rational number $t \geq 1$ recognizing t-tough graphs is coNP-complete; in their proof a variant of IndependenceNumber is reduced to the complement of t-Tough.

Since in our proof $\alpha(G) > k$ if and only if $\tau(G_k) < t$, we can reduce IndependenceNumber to the complement of t-Tough, therefore providing another proof of Theorem 1.

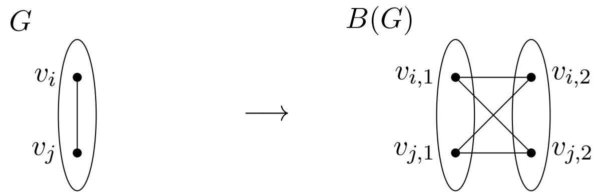

Figure 2. The construction of the bipartite graph $B(G)$.

4. The Complexity of Determining the Toughness of Bipartite Graphs, Proofs of Theorems 7 and 9

Let G be an arbitrary connected graph on the vertices $v_1, \ldots, v_n$ and let $B(G)$ be the following bipartite graph. Let

$$V\big(B(G)\big) = \big\{v_{i,1}, v_{i,2} \mid i \in [n]\big\}$$

and for all $i, j \in [n]$ if $v_i v_j \in E(G)$, then connect $v_{i,1}$ to $v_{j,2}$ and $v_{i,2}$ to $v_{j,1}$. Also for all $i \in [n]$ connect $v_{i,1}$ to $v_{i,2}$, see Figure 2.

To prove Theorems 7 and 9, first we show how the toughness of $B(G)$ can be computed from the toughness of G.

Claim 21. *Let G be an arbitrary connected graph. Then $\tau(B(G)) = \min(2\tau(G), 1)$.*

Proof. Let G be an arbitrary graph on the vertices $v_1, \ldots, v_n$ with $\tau(G) = t$.

Case 1. $t \leq 1/2$. Let $G' = B(G)$ and let $S_0 \subseteq V(G)$ be an arbitrary tough set in G. (Note that since $\tau(G) \leq 1/2$, the graph G is noncomplete, therefore it has a tough set.) Consider the vertex set

$$S_0' = \big\{v_{i,1}, v_{i,2} \mid v_i \in S_0\big\}.$$

Clearly, S_0' is a cutset in G' and

$$\omega\big(G' - S_0'\big) = \omega(G - S_0) = \frac{|S_0|}{t} = \frac{|S_0'|}{2t},$$

so $\tau(G') \leq 2t$.

Now we prove that $\tau(G') \geq 2t$, i.e.,

$$\omega(G' - S') \leq \frac{|S'|}{2t}$$

holds for any cutset S' of G'. Therefore, let S' be an arbitrary cutset in G' and let

$$S_1' = \big\{v_{i,1} \in S' \mid v_{i,2} \notin S'\big\} \cup \big\{v_{i,2} \in S' \mid v_{i,1} \notin S'\big\}$$

and

$$S_2' = S' \setminus S_1'.$$

Consider the components of $G' - S'$ which contain either both or none of the vertices $v_{i,1}, v_{i,2}$ for any $i \in [n]$. These components of $G' - S'$ are also components of $G' - S_2'$, so (similarly as before) the number of these components is at most $|S_2'|/2t$. The number of the remaining components — so in which there is at least one vertex without its pair — can be at most $|S_1'|$, because the pair of the vertex mentioned before must be in S_1'. Since $t \leq 1/2$,

$$\omega(G' - S') \leq \frac{|S_2'|}{2t} + |S_1'| \leq \frac{|S_2'|}{2t} + \frac{|S_1'|}{2t} = \frac{|S'|}{2t},$$

which implies that $\tau(G') \geq 2t$.

Hence,

$$\tau(G') = 2t = 2\tau(G) = \min\big(2\tau(G), 1\big).$$

Case 2. $t > 1/2$. By Proposition 15, there exists a spanning subgraph H with $\tau(H) = 1/2$. Then $B(H)$ is a spanning subgraph of $B(G)$, so

$$\tau\big(B(G)\big) \geq \tau\big(B(H)\big),$$

and as we saw in Case 1,

$$\tau\big(B(H)\big) = 2\tau(H) = 1.$$

Since $B(G)$ is a bipartite graph, $\tau\big(B(H)\big) \leq 1$. Hence,

$$\tau\big(B(G)\big) = 1 = \min\big(2\tau(G), 1\big).$$

□

***Proof of Theorem* 7 *and alternative proof of Theorem* 3.** In Corollary 20 we already proved that if $t \leq 1$, then Exact-t-Tough-Bipartite $\in$ DP, moreover, (Exact-)1-Tough-Bipartite $\in$ coNP.

Now we reduce the DP-complete problem Exact-$t/2$-Tough to Exact-t-Tough-Bipartite if $t < 1$, and the coNP-complete problem $1/2$-Tough to (Exact-)1-Tough-Bipartite.

Let $t < 1$ be a positive rational number and let G be an arbitrary connected graph. By Claim 21,

- $\tau\big(B(G)\big) = t$ if and only if $\tau(G) = t/2$, and
- $\tau\big(B(G)\big) = 1$ if and only if $\tau(G) \geq 1/2$,

thus the statement of the theorem follows. ■

***Proof of Theorem* 8.** Since in the above proof $\tau\big(B(G)\big) \geq t$ if and only if $\tau(G) \geq t/2$ for any positive rational number $t \leq 1$, we can reduce $t/2$-Tough to t-Tough-Bipartite, so the statement of the theorem follows. ■

Note that the case $t = 1$ was already proved by Kratsch *et al.* in [8]. In their proof the vertices $v_{i,1}$ and $v_{i,2}$ are not connected by an edge, but by a path with two inner vertices. With that construction the original graph is 1-tough if and only if the obtained bipartite graph is exactly 1-tough. However, due to the inner vertices of the paths mentioned before, the constructed bipartite graph has a lot of vertices of degree 2, so these graphs are neither regular (except for cycles) nor 3-connected.

To deal with the problem of determining the complexity of recognizing 3-connected bipartite graphs, we only need one more proposition.

Proposition 22. *Let G be an arbitrary graph. Then $\kappa\big(B(G)\big) \geq \kappa(G)$.*

Proof. Let S be an arbitrary cutset in $B(G)$. We need to show that $|S| \geq \kappa(G)$. Let

$$W = \big\{v_{i,1}, v_{i,2} \;\big|\; \{v_{i,1}, v_{i,2}\} \cap S = \emptyset\big\}.$$

Case 1. The vertices of W belong to at least two components of $B(G) - S$. Then

$$S' = \big\{v_j \in V(G) \mid v_{j,1}, v_{j,2} \notin W\big\}$$

is a cutset in G. Its removal from G disconnects the corresponding vertices of W that belong to different components of $B(G) - S$. Obviously,

$$|S| \geq |S'| \geq \kappa(G).$$

Case 2. All vertices of W belong to one component of $B(G) - S$. Since S is a cutset in $B(G)$, there exists a component L for which $L \cap W = \emptyset$. We can assume that $v_{i,1} \in L$ for some $i \in [n]$. Then $v_{i,2} \in S$ since $L \cap W = \emptyset$. Also, for every $j \in [n]$, if $v_i v_j \in E(G)$, then either $v_{j,2} \in S$ or $v_{j,2} \in L$, and in the latter case $v_{j,1} \in S$ holds since $L \cap W = \emptyset$. Therefore,

$$|S| \geq d(v_{i,1}) = d(v_i) + 1 \geq \delta(G) + 1 > \kappa(G).$$

Hence, $\kappa\big(B(G)\big) \geq \kappa(G)$. ■

***Proof of Theorem* 9.** Let $k \geq 2$ be an integer and $t \leq 1$ positive rational number. Applying the proof of Theorem 8 for k-connected bipartite graphs, the statement of theorem follows from Proposition 22. ■

5. On the Toughness of Regular Graphs, Proofs of Theorems 10, 11, and 12

For any positive rational number t and positive integer r let t-Tough-r-Regular denote the problem of determining whether a given r-regular graph is t-tough, and let t-Tough-r-Regular-Bipartite denote the same problem for bipartite graphs.

For any odd number $r \geq 5$ let H_r be the complement of the graph whose vertex set is

$$V = \left\{w, u_1, \ldots, u_{r+1}\right\}$$

and whose edge set is

$$E = \left(\bigcup_{i=1}^{\frac{r-1}{2}} \{u_i, u_{r-i+2}\} \right) \cup \left\{w, u_{(r+1)/2}\right\} \cup \left\{w, u_{(r+3)/2}\right\}.$$

For any even number $r \geq 6$ let H_r be a bipartite graph with color classes

$$A = \left\{w_a, a_1, \ldots, a_{r-1}\right\} \qquad \text{and} \qquad B = \left\{w_b, b_1, \ldots, b_{r-1}\right\},$$

which can be obtained from the complete bipartite graph by removing the edge $\{w_a, w_b\}$. (See the graphs $\overline{H}_5$, H_5 and H_6 in Figure 3.)

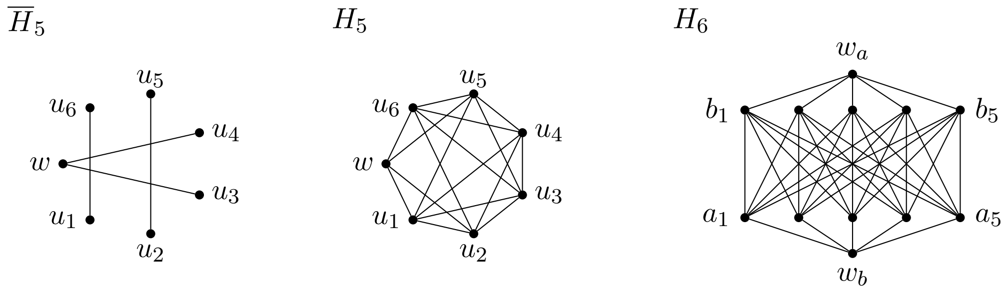

Figure 3. The graphs $\overline{H}_5$, H_5 and H_6.

Claim 23. *For any integer $r \geq 5$, $\tau(H_r) \geq 1$.*

Proof. There is a Hamiltonian cycle in H_r, namely

$$w u_1 u_2 \cdots u_{r+1} w$$

if r is odd, and

$$w_a b_1 a_1 w_b a_2 b_2 a_3 b_3 \cdots a_{r-1} b_{r-1} w_a$$

if r is even, so $\tau(H_r) \geq 1$. □

Lemma 24. *For any fixed odd number $r \geq 5$ the problem $1/2$-Tough is coNP-complete for r-regular graphs.*

Proof. Obviously, 1/2-Tough-r-Regular $\in$ coNP. To prove that it is coNP-hard we reduce 1-Tough-$(r-1)$-Regular (which is coNP-complete by Theorem 2) to it.

Let G be an arbitrary connected $(r-1)$-regular graph on the vertices $v_1, \dots, v_n$ and let G' be defined as follows. For all $i \in [n]$ let

$$V_i = \{w^i, u_1^i, \dots, u_{r+1}^i\}$$

and place the graph H_r on the vertices of V_i and also connect v_i to w^i, see Figure 4. It is easy to see that G' is r-regular and can be constructed from G in polynomial time. Now we prove that G is 1-tough if and only if G' is 1/2-tough.

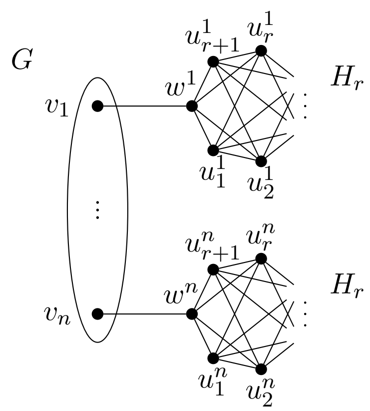

Figure 4. The graph G' constructed in the proof of Lemma 24.

If G is not 1-tough, then there exists a cutset $S \subsetneq V(G)$ satisfying $\omega(G-S) > |S|$. Then S is also a cutset in G' and

$$\omega(G'-S) = \omega(G-S) + |S| > 2|S|,$$

so $\tau(G') < 1/2$.

Now assume that G is 1-tough. Let $S \subseteq V(G')$ be an arbitrary cutset in G', and let $S_0 = V(G) \cap S$ and $S_i = V_i \cap S$ for all $i \in [n]$. Using these notations it is clear that

$$S = S_0 \cup \left(\bigcup_{i=1}^{n} S_i \right)$$

and

$$\omega(G'-S) \le \omega\big(G - S_0\big) + |S_0| + \sum_{i=1}^{n} \omega(H_r^i - S_i),$$

where H_r^i denotes the i-th copy of H_r, i.e., the graph on the vertex set V_i for all $i \in [n]$. Since G is 1-tough and by Claim 23, so is H_r, it follows from Proposition 16 that

$$\omega(G - S_0) \le |S_0|$$

and

$$\omega(H_r^i - S_i) \le |S_i|.$$

Therefore,

$$\omega(G' - S) \le |S_0| + |S_0| + \sum_{i=1}^{n} |S_i| \le 2|S|,$$

so $\tau(G') \ge 1/2$. ■

Lemma 25. *For any fixed even number $r \ge 6$ the problem* 1/2-Tough *is coNP-complete for r-regular graphs.*

Proof. Obviously, 1/2-Tough-r-Regular $\in$ coNP. To prove that it is coNP-hard we reduce 1-Tough-$(r-2)$-Regular (which is coNP-complete by Theorem 2) to it.

Let G be an arbitrary connected $(r-2)$-regular graph on the vertices $v_1, \dots, v_n$ and let G' be defined as follows. For all $i \in [n]$ let

$$A_i = \left\{w_a^i, a_1^i, \dots, a_{r-1}^i\right\}, \qquad B_i = \left\{w_b^i, b_1^i, \dots, b_{r-1}^i\right\}$$

and place the graph H_r on the color classes A_i and B_i and also connect v_i to w_a^i and w_b^i, see Figure 5. It is easy to see that G' is r-regular and can be constructed from G in polynomial time.

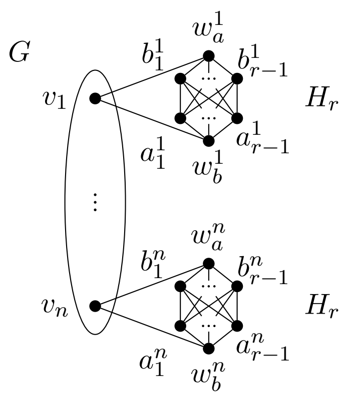

Figure 5. The graph G' constructed in the proof of Lemma 25.

Similarly as in the proof of Lemma 24, it can be shown that G is 1-tough if and only if G' is 1/2-tough. ■

***Proof of Theorem* 11.** The theorem directly follows from Lemmas 24 and 25. ■

Using this result, we can prove Theorem 10.

***Proof of Theorem* 10.** Obviously, 1-Tough-r-Regular-Bipartite $\in$ coNP. To prove that it is coNP-hard we reduce 1/2-Tough-$(r-1)$-Regular (which is coNP-complete by Theorem 11) to it.

Let G be an arbitrary connected $(r-1)$-regular graph and let $B(G)$ denote the bipartite graph defined at the beginning of Section 4. Then $B(G)$ is r-regular and by Claim 21, the graph G is 1/2-tough if and only if $B(G)$ is 1-tough. ■

For any $r \in \{3, 4, 5\}$ the problem of determining the complexity of 1-Tough-r-Regular-Bipartite remains open. The reason why our construction does not work in these cases is that we can decide in polynomial time whether an at most 4 regular graph is 1/2-tough, which we prove in the rest of this paper.

Lemma 26. *For any connected* 3*-regular graph G, the following are equivalent.*

(1) *There is a cut-vertex in G.*

(2) $\tau(G) \leq 1/2$.

(3) $\tau(G) < 2/3$.

Proof.

(1) $\Longrightarrow$ (2) : Trivial.

(2) $\Longrightarrow$ (3) : Trivial.

(3) $\Longrightarrow$ (1) : If $\tau(G) < 2/3$, then there exists a cutset $S \subseteq V(G)$ satisfying

$$\omega(G-S) > \frac{3}{2}|S|.$$

Hence there must exist a component of $G-S$ that has exactly one neighbor in S: since G is connected, every component has at least one neighbor in S, and if every component of $G-S$ had at least two neighbors in S, then the number of edges going into S would be at least $2\omega(G-S) > 3|S|$, which would contradict the 3-regularity of G. Obviously, this neighbor in S is a cut-vertex in G. ■

***Proof of Theorem* 12.** Let G be an arbitrary connected 3-regular graph. First check whether G contains a cut-vertex. By Lemma 26, if it does not, then $\tau(G) \geq 2/3$, but if it does, then $\tau(G) \leq 1/2$. We prove that in the latter case either $\tau(G) = 1/3$ or $\tau(G) = 1/2$, and we can also decide in polynomial time which one holds.

Since G is 3-regular, $\omega(G-S) \leq 3|S|$ holds for any cutset S of G, so $\tau(G) \geq 1/3$. Now we show that if $\tau(G) < 1/2$, then $\tau(G) \leq 1/3$.

So assume that $\tau(G) < 1/2$ and let S be a tough set of G and let $k = \omega(G-S)$. Then $k > 2|S|$. Contract the components of $G - S$ into single vertices $u_1, \dots, u_k$ while keeping the multiple edges and let H denote the obtained multigraph. Since G is connected, $d(u_i) \geq 1$ holds for any $i \in [k]$, so

$$k = \big|\{i \in [k] : d(u_i) = 1\}\big| + \big|\{i \in [k] : d(u_i) \geq 2\}\big|.$$

Since G is 3-regular,

$$\begin{aligned} 3|S| &\geq \sum_{i=1}^{k} d(u_i) \geq \big|\{i \in [k] : d(u_i) = 1\}\big| + 2 \cdot \big|\{i \in [k] : d(u_i) \geq 2\}\big| \\ &= k + \big|\{i \in [k] : d(u_i) \geq 2\}\big| > 2|S| + \big|\{i \in [k] : d(u_i) \geq 2\}\big|, \end{aligned}$$

so

$$|S| > \big|\{i \in [k] : d(u_i) \geq 2\}\big|.$$

Therefore,

$$\big|\{i \in [k] : d(u_i) = 1\}\big| = k - \big|\{i \in [k] : d(u_i) \geq 2\}\big| > 2|S| - |S| = |S|,$$

which means that there exists a vertex in S having at least two neighbors in $\{u_1, \dots, u_k\}$ of degree 1. Then the removal of this vertex leaves at least three components (and note that since G is 3-regular, it cannot leave more than three components), so $\tau(G) \leq 1/3$.

From this it also follows that $\tau(G) = 1/3$ if and only if there exists a cut-vertex whose removal leaves three components.

To summarize, it can be decided in polynomial time whether a connected 3-regular graph is 2/3-tough, and if it is not, then its toughness is either 1/3 or 1/2. In both cases the graph contains at least one cut-vertex, and if the removal of any of them leaves (at least) three components, then the toughness of the graph is 1/3, otherwise it is 1/2. ■

Claim 27. *The toughness of any connected* 4*-regular graph is at least* 1/2*.*

Proof. Let G be a connected 4-regular graph and let S be an arbitrary cutset in G and L be a component of $G - S$. Since every vertex has degree 4 in G, the number of edges between S and L is even (more precisely, it is equal to the sum of the degrees in G of the vertices of L minus two times the number of edges induced by L). Since G is connected, the number of these edges is at least two. On the other hand, since G is 4-regular, there are at most $4|S|$ edges between S and L. Therefore $\omega(G - S) \leq 2|S|$, which means that $\tau(G) \geq 1/2$. □

***Proof of Theorem* 13.** It directly follows from Claim 27. ■

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