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# VERTEX PARTITIONING OF GRAPHS INTO ODD INDUCED SUBGRAPHS

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**Abstract**

A graph $G$ is called an odd (even) graph if for every vertex $v \in V(G)$, $d_G(v)$ is odd (even). Let $G$ be a graph of even order. Scott in 1992 proved that the vertices of every connected graph of even order can be partitioned into some odd induced forests. We denote the minimum number of odd induced subgraphs which partition $V(G)$ by $od(G)$. If all of the subgraphs are forests, then we denote it by $od_F(G)$. In this paper, we show that if $G$ is a connected subcubic graph of even order or $G$ is a connected planar graph of even order, then $od_F(G) \leq 4$. Moreover, we show that for every tree $T$ of even order $od_F(T) \leq 2$ and for every unicyclic graph $G$ of even order $od_F(G) \leq 3$. Also, we prove that if $G$ is claw-free, then $V(G)$ can be partitioned into at most $\Delta(G) - 1$ induced forests and possibly one independent set. Furthermore, we demonstrate that the vertex set of the line graph of a tree can be partitioned into at most two odd induced subgraphs and possibly one independent set.

**Keywords:** odd induced subgraph, subcubic graphs, claw-free graphs, line graphs, independent set.

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## 1. Introduction

All graphs considered in this paper are simple, that is, with no loops or multiple edges. Let $G$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. The number of vertices of $G$ is called the *order* of $G$. Also, $\delta(G)$ and $\Delta(G)$ denote the minimum degree and the maximum degree of $G$, respectively. For a vertex $v \in V(G)$, $d_G(v)$ and $N_G(v)$ denote the degree of $v$ in $G$ and the set of all the neighbors of $v$ in $G$, respectively. Also, for a subset $S \subseteq V(G)$, define $N_G(S) = \bigcup_{v\in S} N_G(v)$.

**Definition.** A graph $G$ is called an *odd* (*even*) *graph* if for every vertex $v \in V(G)$, $d_G(v)$ is odd (even).

For a subset $X \subseteq V(G)$, we denote the induced subgraph on $X$ by $\langle X\rangle$. By $G \setminus H$, we mean the induced subgraph on $V(G) \setminus V(H)$.

**Definition.** An *induced matching* in a graph $G$ is a set of edges, no two of which meet a common vertex or are joined by an edge of $G$.

For two vertex disjoint subgraphs $S$ and $T$ of $G$, $e_G(S,T)$ denotes the number of edges with one end in $S$ and the other one in $T$.

**Definition.** The *line graph* $L(G)$ of a graph $G$, is obtained by associating a vertex with each edge of $G$ and joining two vertices with an edge if and only if the corresponding edges of $G$ have a vertex in common.

**Definition.** A *cubic* graph $G$ is a graph with $\Delta(G) = \delta(G) = 3$, and a *subcubic* graph $G$ is a graph with $\Delta(G) \leq 3$.

**Definition.** A *claw-free* graph is a graph that does not have a claw as an induced subgraph, where a claw is the complete bipartite graph $K_{1,3}$.

**Definition.** A graph $G$ is called *unicyclic* if it is connected and contains exactly one cycle.

**Definition.** An *outerplanar* graph is a planar graph such that all vertices lie on a single face.

Moreover, the complete graph of order $n$ is denoted by $K_n$.

**Definition.** A *cut vertex* is a vertex whose removal from a graph creates more components than the previous graph. Also, a graph is called *$k$-connected* if the minimum number of vertices whose removal would disconnect the graph is at least $k$.

**Definition.** A *block* of a graph is a maximal 2-connected subgraph. The *block decomposition* of a graph is just the set of all blocks of the graph. A *leaf block* of a graph $G$ is a block containing at most one cut vertex of $G$.

**Definition.** A *proper $k$-coloring* of a graph $G$ is an assignment of $k$ colors to the vertices of $G$ such that no two adjacent vertices have the same colors.

**Definition.** The *chromatic number* of $G$ is the minimum number of colors to color the vertices of $G$ such that no two adjacent vertices have the same color and is denoted by $\chi(G)$.

We denote the minimum number of odd induced subgraphs which partition $V(G)$ by $od(G)$. Moreover, we denote the minimum number of odd induced forests which partition $V(G)$ by $od_F(G)$. Clearly, $od(G) \leq od_F(G)$. It was proved that the vertices of every graph $G$ can be partitioned into at most two even induced subgraphs, see [7, Exercise 5.19]. A spanning subgraph $S$ of $G$ is called a *perfect forest* if $S$ is a forest and each tree of $S$ is an odd induced subgraph of $G$. Scott proved that the vertices of every connected graph $G$ of even order can be partitioned into some odd induced subgraphs, see [10] and [11]. The following theorem was proved in [5].

**Theorem A.** *Every connected graph of even order contains a perfect forest.*

In this paper, we show that if $G$ is a connected subcubic graph of even order or $G$ is a connected planar graph of even order, then $od_F(G) \leq 4$. By a computer search, we see that the smallest graph with $od(G) = od_F(G) = 4$ is the graph given in Figure 1. This graph shows that 4 is sharp.

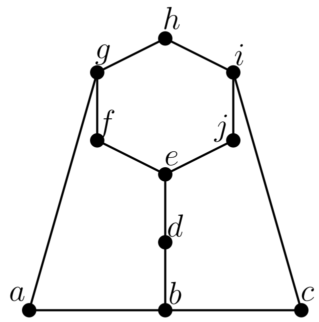


Figure 1. A graph $G$ with $od(G) = od_F(G) = 4$.

Moreover, we prove that for every tree $T$ of even order, $od_F(T) \leq 2$ and for every unicyclic graph $G$ of even order, $od_F(G) \leq 3$. Also we show that for every positive integer $k$, there exists a graph of even order such that $od(G) > k$. Furthermore, we show that if $G$ is a claw-free graph, then $V(G)$ can be partitioned into at most $\Delta(G) - 1$ odd induced forests and possibly one independent set. Besides, we demonstrate that if $T$ is a tree, then $V(L(T))$ can be partitioned into at most two odd induced subgraphs and possibly one independent set.

## 2. Upper Bounds for Odd Induced Number of Some Families of Graphs

Let $G$ be a graph of even order. By Theorem A, there exists a vertex partitioning $\Omega$ in which $V(G)$ can be partitioned into some odd induced trees. Now, define a new graph. Assign a vertex to each odd induced tree and join two vertices if there exists at least one edge between the corresponding odd induced trees. Denote this graph by $H_\Omega(G)$. For instance, consider the graph $G$ shown in Figure 1. There exists a vertex partitioning $\Omega$ as follows.

$$\Omega = \{\{a, b, c, d\}, \{e, f\}, \{g, h\}, \{i, j\}\}.$$

So, $H_\Omega(G)$ is the graph shown in Figure 2.

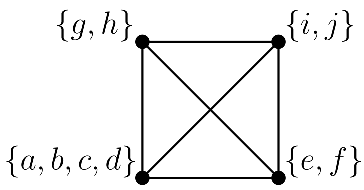


Figure 2. $H_\Omega(G)$.

In this section, we apply Theorem A to prove that the vertices of every tree of even order can be partitioned into at most two odd induced forests. First we start with the following lemma.

**Lemma 1.** *Let $G$ be a graph of even order. Then $od_F(G) \leq \chi(H_\Omega(G))$, where $\Omega$ partitions $V(G)$ into odd induced trees.*

***Proof.*** Consider a proper $\chi(H_\Omega(G))$-coloring of $H_\Omega(G)$. Note that the union of all trees corresponding to vertices of each color class forms an odd induced forest in $G$. So, $od_F(G) \leq \chi(H_\Omega(G))$. ■

**Corollary 2.** *The following statements hold.*

(i) *For every tree $T$ of even order, $od_F(T) \leq 2$.*

(ii) *For every unicyclic graph $G$ of even order, $od_F(G) \leq 3$.*

***Proof.*** (i) Clearly, $H_\Omega(T)$ is a tree, where $\Omega$ partitions $V(G)$ into odd induced forests. Since $\chi(H_\Omega(T)) \leq 2$, Lemma 1 completes the proof.

(ii) It is clear that $H_\Omega(G)$ has at most one cycle and so $\chi(H_\Omega(G)) \leq 3$. Now, by Lemma 1, we are done. ■

**Remark 3.** For every graph $G$ of even order, one can see that $od_F(G) = \min_\Omega(\chi(H_\Omega(G)))$, where $\Omega$ is taken over all odd induced forest partitions of $V(G)$. To see this, by Lemma 1, $od_F(G) \leq \min_\Omega(\chi(H_\Omega(G)))$. Now, consider an odd induced forest partitioning of size $od_F(G)$ and call it by $\Omega$. This yields that $od_F(G) \geq \min_\Omega(\chi(H_\Omega(G)))$ and so, $od_F(G) = \min_\Omega(\chi(H_\Omega(G)))$.

The next result shows that the minimum number of odd induced forests, which partition a planar graph does not exceed 4.

**Theorem 4.** *If $G$ is a connected planar graph of even order, then $od_F(G) \leq 4$.*

***Proof.*** Let $\Omega$ be a partition of $V(G)$ into odd induced forests. Consider $H_\Omega(G)$. We claim that $H_\Omega(G)$ is planar. To see this, suppose that $F$ is one of the odd induced forests which partition $V(G)$. By contraction of $|E(F)|$ edges, $F$ leads to one independent set. If we do the same for every odd induced forest and join two vertices if there exists at least one edge between corresponding forests, this graph is indeed $H_\Omega(G)$. Notice that the contraction of a planar graph in an edge is planar, see [12]. So, $H_\Omega(G)$ is planar. By Four-Color Theorem, see [3], $\chi(H_\Omega(G)) \leq 4$ and so, Lemma 1 completes the proof. ■

**Remark 5.** If $G$ is a connected outerplanar graph of even order, then $od_F(G) \leq 3$. First, notice that for every outerplanar graph $G$, $H_\Omega(G)$ is an outerplanar graph. Now, apply the method used in the proof of Theorem 4 to obtain $\chi(H_\Omega(G)) \leq 3$, see [4].

Now, we want to determine $od(G^*)$, where $G^*$ is a graph obtained by replacing each edge with a path of length 2.

**Theorem 6.** *If $G$ is a connected graph of even (odd) order with even (odd) number of edges, then $od(G^*) = od_F(G^*) = \chi(G)$.*

***Proof.*** Obviously, $G^*$ is a graph of even order. Now, by Theorem A, $V(G^*)$ can be partitioned into some odd induced subgraphs. In the partition of $V(G^*)$ into odd induced subgraphs, note that if $u, v \in V(G)$ and $uv \in E(G)$, then $u$ and $v$ do not belong to the same odd induced subgraphs of $V(G^*)$, because otherwise the vertex adjacent to $u$ and $v$ has degree zero or two in the odd induced subgraph, a contradiction. Clearly, every connected odd induced subgraph is a star whose center belongs to $V(G)$. Thus, $od_F(G^*) = od(G^*)$. Let $\Omega$ be the partitioning of $V(G^*)$ into odd induced forests, say $F_1, \ldots, F_{od_F(G^*)}$. Then color all vertices of $V(G)$ which are contained in $F_i$ by color $i$, for $i = 1, \ldots, od_F(G^*)$. Since the colored vertices of $G$ are independent, we conclude that $od_F(G^*) \geq \chi(G)$. Now, we show that $od_F(G^*) \leq \chi(G)$. Consider a proper $\chi(G)$-coloring of $G$. For each $i$, $1 \leq i \leq \chi(G)$, we define $A_i$ to be all vertices of $G^*$ which belong to a star in the partition $\Omega$ whose center has color $i$ in $G$. Obviously, $\{A_1, \ldots, A_{\chi(G)}\}$ is an odd induced forest partitioning of $V(G^*)$. Therefore, $od_F(G^*) \leq \chi(G)$. So, $\chi(G) \leq od(G^*) = od_F(G^*) \leq \chi(G)$ and the proof is complete. ■

**Remark 7.** If $n = 0, 3 \pmod 4$, then $od(K_n^*) = n$. Thus, for every positive integer $k$, there exists a graph of even order that $od(G) > k$.

## 3. Vertex Partitioning of Subcubic Graphs into Odd Induced Forests

In this section, we would like to investigate $od_F(G)$ for a subcubic graph $G$.

**Theorem 8.** *Let $G$ be a connected subcubic graph of even order. Then $od_F(G) \leq 4$.*

***Proof.*** The proof is by induction on $|V(G)|$. Clearly, the assertion holds for $|V(G)| = 2$. Now, two cases may be occurred.

*Case* 1. Suppose that $G$ has a cut vertex $u$. Since $G \setminus \{u\}$ is a graph of odd order, so it has at least one connected component of odd order. Call this component by $H_1$ and let $H_1' = \langle V(H_1) \cup \{u\} \rangle$ and $H_2 = G \setminus H_1'$. By the induction hypothesis, $od_F(H_1') \leq 4$. Suppose that $V(H_1')$ is partitioned into odd induced forests $\{O_1, \ldots, O_l\}$, where $l \leq 4$ and $u \in V(O_1)$. We define $OD(H_1') = \{O_1, \ldots, O_l, \emptyset, \ldots, \emptyset\}$, in which the number of $\emptyset$ is $4 - l$. Now, add a new vertex $v$ to $\langle V(H_2) \cup \{u\} \rangle$ and join $v$ to $u$. Call the new graph by $H_2'$. Clearly, $H_2'$ has even order. Two cases can be considered.

(i) $H_2' \not\simeq G$. Then by the induction hypothesis, $od_F(H_2') \leq 4$. Let $\{O_1', \ldots, O_t'\}$, $t \leq 4$ be the set of odd induced forests which partition $V(H_2')$ such that $u, v \in V(O_1')$. We define $OD(H_2') = \{O_1', \ldots, O_4'\}$, where $O_i'$ may be empty for some $i$. Now, since $d_{O_1}(u) + d_{O_1' \setminus \{v\}}(u)$ is odd, it is easy to see that $\{O_1 \cup (O_1' \setminus \{v\}), O_2 \cup O_2', O_3 \cup O_3', O_4 \cup O_4'\}$, after removing the empty sets, is a partitioning of $V(G)$ into odd induced forests and we are done.

(ii) $H_2' \simeq G$. Now, if $H_2$ is connected, then by the induction hypothesis, $od_F(H_2) \leq 4$. Let $OD(H_2) = \{\hat{O}_1, \ldots, \hat{O}_4\}$. Now, since $G$ is a subcubic graph, with no loss of generality, one may assume that $e_G(\langle u, v' \rangle, \hat{O}_1) = 0$, where $v'$ is the corresponding vertex $v$ in $G$. This implies that $\{\langle \hat{O}_1 \cup \{u, v'\} \rangle, \hat{O}_2, \ldots, \hat{O}_4\}$, after removing the empty sets, is a partitioning of $V(G)$ into odd induced forests. Now, assume that $H_2$ is not connected. Clearly, since $G$ is subcubic, we have $d_G(u) = 3$ and $H_2 = W_1 \cup W_2$, where $W_1$ and $W_2$ are two connected components of $H_2$. Let $w_1 \in V(W_1)$, $w_2 \in V(W_2)$ and $uw_1, uw_2 \in E(G)$. Since

$H_2$ is of even order, the orders of $W_1$ and $W_2$ have the same parity. Now, if $|V(W_1)| = |V(W_2)| = 0 \pmod 2$, then by the induction hypothesis, there are odd induced forest partitioning of $V(W_1)$ and $V(W_2)$, say $OD(W_1) = \{O_1 \ldots, O_4\}$ and $OD(W_2) = \{O'_1 \ldots, O'_4\}$, where $w_1 \in V(O_1)$ and $w_2 \in V(O'_1)$. Then clearly, $\{O_1 \cup O'_1, \langle O_2 \cup O'_2 \cup \{u, v'\}\rangle, O_3 \cup O'_3, O_4 \cup O'_4\}$, after removing the empty sets, is a partitioning of $V(G)$ into odd induced forests. Now, assume that $|W_1| = |W_2| = 1 \pmod 2$. Add two new vertices $s_1$ and $s_2$ to $W_1$ and $W_2$, join $s_1$ to $w_1$ and $s_2$ to $w_2$ and call the resultant graphs by $G_1$ and $G_2$, respectively. By the induction hypothesis, there are odd induced forests which partition $V(G_1)$ and $V(G_2)$, say $OD(G_1) = \{O_1 \ldots, O_4\}$ and $OD(G_2) = \{O'_1 \ldots, O'_4\}$ such that $w_1, s_1 \in V(O_1)$ and $w_2, s_2 \in V(O'_1)$. Thus, $OD(G) = \{(O_1 \setminus \{s_1\}) \cup (O'_1 \setminus \{s'_1\}) \cup \{u, v'\}, O_2 \cup O'_2, O_3 \cup O'_3, O_4 \cup O'_4\}$, after removing the empty sets, is a partitioning of $V(G)$ into odd induced forests and we are done.

*Case* 2. Suppose that $G$ is 2-connected. First, let us assume that G is cubic. Since $G$ is 2-connected, by Petersen Theorem [9], $G$ has a perfect matching. Form a new graph $H$ by assigning a vertex to each edge of this perfect matching and joining two vertices if there exists at least one edge between the corresponding edges. Since $G$ is cubic, $\Delta(H) \leq 4$. Now, connectivity of $G$ implies that $H$ is connected as well. If $H \not\simeq K_5$, then by Brooks' Theorem, see [13, p. 197], one can properly color the vertices of $H$ with 4 colors. Then, the induced subgraph corresponding to every color class of $V(H)$ forms an induced perfect matching in $G$, so $od_F(G) \leq 4$. Thus, we can assume that $H \simeq K_5$. Since every cubic graph of order 10 has 15 edges, there is exactly one edge between any pair of the edges of the perfect matching. If $G$ is claw-free, consider a triangle with vertices $u, v$ and $w$. Since $G$ is a claw-free graph of even order, every edge is contained in a perfect matching, see [6]. Therefore, $uv$ can be extended to a perfect matching. Now, consider that edge in perfect matching which is incident with $w$. Since there are at least two edges between this edge and $uv$, we obtain a contradiction. Now, assume that $G$ is not claw-free. So, $G$ contains an induced subgraph $K_{1,3}$. Let $L = G \setminus V(K_{1,3})$. Clearly, $L$ is a connected graph of order 6. By Theorem A, $V(L)$ can be partitioned into at most three odd induced forests. Thus, $od_F(G) \leq 4$, as desired.

Now, assume that $G$ is not cubic. By [13, p. 208], for every 2-connected graph $G$ and for every vertex $u \in V(G)$, there exists $v \in N_G(u)$ such that $G \setminus \{u, v\}$ is connected. Then we can assume that there exist two adjacent vertices $u$ and $v$, such that $G \setminus \{u, v\}$ is connected and $d_G(u) = 2$. Set $H' = G \setminus \{u, v\}$. By the induction hypothesis, there is an odd induced forest partitioning of $V(H')$ such that $OD(H') = \{O_1, \ldots, O_4\}$. It is easy to see that $e_G(O_1 \cup \cdots \cup O_4, \langle u, v\rangle) \leq 3$. Thus with no loss of generality, one can assume that $e_G(O_1, \langle u, v\rangle) = 0$ and so $OD(G) = \{O_1 \cup \{u, v\}, \ldots, O_4\}$. By removing the empty sets, we have an odd induced forest partitioning of $V(G)$ and the proof is complete. ■

Now, we propose the following conjecture.

**Conjecture 9.** *Let $G$ be a connected graph of even order. Then, $od_F(G) \leq \Delta(G) + 1$.*

**Theorem 10.** *Let $G$ be a subcubic graph. Then $V(G)$ can be partitioned into at most three odd induced subgraphs and possibly one independent set.*

***Proof.*** We know that $V(G)$ can be partitioned into two induced subgraphs, one odd and one even, see [7]. Let us denote the odd induced subgraph by $O_1$. Also, the even induced subgraph is a disjoint union of cycles and an independent set, say $S$. It is not hard to see that the vertices of all cycles can be partitioned into two induced matchings and possibly one independent set. We show them by $O_2, O_3$ and $S'$, respectively. Obviously, $O_1, O_2, O_3$ and $S \cup S'$ are the desired partitioning of $V(G)$ and the proof is complete. ■

We close this section with the following conjecture.

**Conjecture 11.** *The vertices of every graph $G$ can be partitioned into at most $\Delta(G) - 1$ odd induced subgraphs and possibly one independent set.*

## 4. Vertex Partitioning of Claw-Free Graphs into Odd Induced Forests and One Independent Set

In this section, we focus on claw-free graphs.

**Theorem 12.** *If $G$ is a cubic claw-free graph, then $V(G)$ can be partitioned into two induced matchings and possibly one independent set.*

***Proof.*** Let $H$ and $L$ partition $V(G)$ such that $e_G(H, L)$ is maximum among all possible partitions of $V(G)$ into two parts. This implies that $\Delta(H) \leq 1$ and $\Delta(L) \leq 1$. Since $\Delta(H) \leq 1$, $H$ is the union of two subgraphs $H_0$ and $H_1$ such that $H_0$ is an independent set and $H_1$ is 1-regular. Similarly, define $L_0$ and $L_1$. Let $S = \langle H_0 \cup L_0 \rangle$. Since $G$ is claw-free and $e_G(H_0, H_1) = e_G(L_0, L_1) = 0$, one can easily see that $S$ is the union of isolated edges and some isolated vertices. Among all partitions of $V(G)$ into $(H_0, H_1, L_0, L_1)$, consider that partition such that $|E(S)|$ is minimum. By switching some vertices between three sets $L_1$, $H_1$ and $S$ and updating them, we want to remove all edges in $S$ such that the remaining would be just isolated vertices. Note that if $S$ includes only isolated vertices, then $H_1$, $L_1$ and $S$ would be the desired partition. Now, assume that there exists an edge $uv \in E(S)$. Without loss of generality, suppose that $u \in V(H_0)$ and $v \in V(L_0)$. Since $G$ is a cubic claw-free graph, $u$ is adjacent to two vertices $x, y \in V(L_1)$ such that $xy \in E(L_1)$. Now, two cases may occur.

First, suppose that $|N_{H_0}(y)| = 1$. Then, add $u$ to $L_1 \setminus \{y\}$ and $y$ to $L_0$.

Now, assume that $|N_{H_0}(y)| = 2$ and $y$ is adjacent to $u'$, $u' \in V(H_0)$. Add $u$ to $L_1 \setminus \{y\}$ and $\{y, u'\}$ to $H_1$. It is not hard to see that this procedure will not add any new edge to $S$. This leads to a new $S$ which has at least one less edge. By removing each edge in $S$ step by step, we find the desired partition. ■

Now, we generalize the previous result to subcubic claw-free graphs.

**Theorem 13.** *If $G$ is a subcubic claw-free graph, then $V(G)$ can be partitioned into at most two induced matchings and possibly one independent set.*

***Proof.*** Clearly, we can assume that $G$ is connected. If $\Delta(G) \leq 2$, then $G$ is a disjoint union of paths and cycles and since the vertices of each path and each cycle can be partitioned into at most two induced matchings and one independent set, we are done. So, we can assume that there is at least one vertex $v$ of degree 3. Since $G$ is claw-free, at least two neighbors of $v$, say $u$ and $w$, are adjacent. Now, we prove the assertion by induction on $|V(G)|$. The assertion holds for $|V(G)| = 2$. Let $X = \{u, v, w\}$ and $G' = G \setminus X$. Then, $G'$ is a claw-free graph with $\Delta(G') \leq 3$ and $V(G')$ can be partitioned into at most two induced matchings $A, B$ and one independent set $S$ (Note that maybe $A, B$ or $S$ do not exist). First we prove the theorem if $u$ or $w$, say $u$, has degree 2. With no loss of generality, we can suppose that $N_G(v) \cap V(A) = \emptyset$. If $N_G(w) \cap V(A) = \emptyset$, then $A \cup \{vw\}, B$ and $S \cup \{u\}$ are the desired partition. Moreover, if $N_G(w) \cap V(A) \neq \emptyset$, then $A \cup \{u, v\}, B$ and $S \cup \{w\}$ are the desired partition. Thus, we can suppose that $d_G(u) = d_G(v) = d_G(w) = 3$. Now, four cases may occur.

*Case* 1. $V(A) \cap N_G(X) \neq \emptyset$, $V(B) \cap N_G(X) \neq \emptyset$ and $V(S) \cap N_G(X) \neq \emptyset$. Without loss of generality, suppose that $N_G(u) \in V(A)$, $N_G(v) \in V(B)$ and $N_G(w) \in V(S)$. So, $A \cup \{vw\}$, $B$ and $S \cup \{u\}$ is the desired partition of $V(G)$.

*Case* 2. $V(S) \cap N_G(X) = \emptyset$. Two cases can be considered. Without loss of generality, suppose that $N_G(u) \in V(A)$ and $N_G(v), N_G(w) \in V(B)$ or $N_G(u), N_G(v), N_G(w) \in V(B)$. Obviously, in both cases $A \cup \{vw\}$, $B$ and $S \cup \{u\}$ are the desired partition of $V(G)$.

*Case* 3. $V(A) \cap N_G(X) \neq \emptyset$ and $V(B) \cap N_G(X) = \emptyset$. Without loss of generality, suppose that $N_G(u) \in V(A)$. Then one can see that $A$, $B \cup \{vw\}$ and $S \cup \{u\}$ is the desired partition of $V(G)$.

*Case* 4. $(V(A) \cup V(B)) \cap N_G(X) = \emptyset$. Let call the neighbors of $u, v, w$ in $S$, by $u', v', w'$, respectively. Two cases may occur. First, assume that at least two neighbors of $X$ are the same and without loss of generality, suppose that $u' = v'$. Since $d_G(u') \leq 3$, $|N_A(u') \cup N_B(u')| \leq 1$. So, we can assume that $N_A(u') = \emptyset$. It is obvious that $A \cup \{uu'\}$, $B \cup \{vw\}$ and $S \setminus \{u'\}$ partition $V(G)$. Thus suppose that $\{u', v', w'\}$ are different. Now, if one of the $\{u', v', w'\}$, say $u'$, has degree two in $G$, then by the same method used in the previous argument we are done. Thus one may assume that $d_G(u') = d_G(v') = d_G(w') = 3$. Therefore we can assume that every vertex of each triangle in $G$ has degree 3. Also, the neighbors of vertices of any triangle in $G$ are independent. Clearly, $G$ is cubic and by Theorem 12, the proof is complete. ■

**Theorem 14.** *If $G$ is a claw-free graph with $\Delta(G) \geq 3$, then $V(G)$ can be partitioned into at most $\Delta(G)-1$ induced matchings and possibly one independent set.*

***Proof.*** If $\Delta(G) = 3$, then by Theorem 13, the assertion is trivial. So, assume that $\Delta(G) \geq 4$. It was proved that if there are non-negative integers $k_1, \ldots, k_m$ such that

$$\sum_{i=1}^{m} k_i \geq \Delta(G) + 1 - m,$$

then $V(G)$ can be partitioned into $m$ induced subgraphs, each of which has maximum degree at most $k_i$, see [8]. Now, define $k_1 = 3$ and $k_2 = \Delta(G) - 4$. Since $k_1 + k_2 = \Delta(G) - 1 \geq \Delta(G) + 1 - 2$, $V(G)$ can be partitioned into two induced subgraphs $H_1$ and $H_2$ such that $\Delta(H_1) \leq 3$ and $\Delta(H_2) \leq \Delta(G) - 4$. Moreover, since $G$ is claw-free, $H_1$ and $H_2$ are both claw-free. Now, we prove the theorem by induction on $\Delta(G)$. Since $\Delta(H_1) \leq 3$, $V(H_1)$ can be partitioned into at most two induced matchings $O_1$, $O_2$ and one independent set $S$. Now, by induction hypothesis $V(H_2)$ can be partitioned into at most $\Delta(H_2)-1 \leq \Delta(G)-5$ induced matchings $O'_1, \ldots, O'_{\Delta(H_2)-1}$ and one independent set $S'$. Let $K = \langle S \cup S' \rangle$. Since $S$ and $S'$ are independent sets, $K$ is a bipartite graph and moreover since $G$ is claw-free, $\Delta(K) \leq 2$. Consequently, $K$ is a disjoint union of some paths and even cycles. Notice that the vertices of every path and cycle can be partitioned into at most two odd induced matchings and one independent set. Thus $V(K)$ can be partitioned into at most two induced matchings and one independent set. Call the induced matchings of $K$ by $\tilde{O}_1$, $\tilde{O}_2$, and the independent set by $\tilde{S}$. It is obvious that $O_1, O_2, O'_1, \ldots, O'_{\Delta(H_2)-1}, \tilde{O}_1, \tilde{O}_2$ and $\tilde{S}$ partition $V(G)$ and the proof is complete. ■

## 5. Vertex Partitioning of Line Graphs into Odd Induced Subgraphs and Independent Sets

In this section, we study the partitioning of the vertices of line graphs into odd induced subgraphs and independent sets.

**Theorem 15.** *If $G$ is a graph, then $V(L(G))$ can be partitioned into at most $\left\lceil \frac{3\Delta(G)+2}{5} \right\rceil$ induced matchings and possibly $\left\lceil \frac{3\Delta(G)+2}{5} \right\rceil$ independent sets.*

***Proof.*** In [2], it is proved that the edges of every graph $G$ can be partitioned into at most $\left\lceil \frac{3\Delta(G)+2}{5} \right\rceil$ forests such that each component is a path. Denote these forests by $F_1, \ldots, F_{\lceil \frac{3\Delta(G)+2}{5} \rceil}$. Now, consider $L(F_1), \ldots, L\left(F_{\lceil \frac{3\Delta(G)+2}{5} \rceil}\right)$. Each $L(F_i)$ is an induced subgraph of $L(G)$ which consists of paths. Since the vertices of every induced path can be partitioned into at most one induced matching and one independent set, $V(L(G))$ can be partitioned into at most $\left\lceil \frac{3\Delta(G)+2}{5} \right\rceil$ induced matchings and $\left\lceil \frac{3\Delta(G)+2}{5} \right\rceil$ independent sets, as desired. ■

The following conjecture is due to [1].

**Conjecture 16.** *The edges of every graph $G$ can be partitioned into $\left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ forests such that each component is a path.*

**Remark 17.** We note that if Conjecture 16 holds, then for every graph $G$, $V(L(G))$ can be partitioned into at most $\left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ induced matchings and $\left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ independent sets.

Now, we focus on the line graph of trees. We prove the following theorem.

**Theorem 18.** *If $T$ is a tree, then $V(L(T))$ can be partitioned into at most two odd induced subgraphs and possibly one independent set.*

***Proof.*** The proof is by induction on $|V(L(T))|$. Obviously, the assertion holds for $|V(L(T))| \leq 2$. Now, consider the block decomposition of $G = L(T)$. It is not hard to see that since $T$ is a tree, each block of $G$ is a complete graph. Now, if $G$ is 2-connected, then $G$ is a complete graph and we are done. Thus, we can assume that $G$ has at least one cut vertex. Moreover, since $G$ is claw-free, every cut vertex of $G$ is included in at most two blocks. Suppose that $B$ is a leaf block of $G$ containing the cut vertex $w$. Note that $B \setminus \{w\} = K_r$, for some positive integer $r$. Now, two cases may occur.

First, suppose that $r \geq 2$. Since $r \geq 2$, there are at least two vertices $u$ and $v$, such that $uv \in E(B \setminus \{w\})$. Remove $u$ and $v$. Clearly, $G \setminus \{u, v\}$ is the line graph of a tree. So, by induction hypothesis, $V(G \setminus \{u, v\})$ can be partitioned into at most two odd induced subgraphs and one independent set. If there is no edge with one endpoint in $\{u, v\}$ and another endpoint in one of the odd induced subgraphs of $G \setminus \{u, v\}$, then we add $uv$ to that odd induced subgraph and we are done. So, we can assume that there is at least one edge between each odd induced subgraph of $G \setminus \{u, v\}$ and $\{u, v\}$. If there exists exactly one odd induced subgraph of $G \setminus \{u, v\}$, then by considering the odd induced subgraph $\{u, v\}$, we are done. Thus assume that there are two odd induced subgraphs for $G \setminus \{u, v\}$. Since $w$ can be contained in at most one of the odd induced subgraphs of $G \setminus \{u, v\}$, there is at least an odd induced subgraph $O_1$ which does not contain $w$. Now, clearly there is an odd component in $O_1$, call $K$, such that $N_{O_1}(u) = N_{O_1}(v) = V(K)$. Thus, $\langle K \cup \{u, v\}\rangle$ is a complete graph of even order and we are done.

Therefore, one may assume that all leaf blocks of $G$ are $K_2$. If we remove all pendant vertices, the remaining graph $S$, will also be the line graph of a tree and the remaining blocks form the block decomposition of $S$. Let $B'$ be a leaf block of $S$. Obviously, $B'$ is a block in the block decomposition of $G$ and all blocks in $G$ that have a common vertex with $B'$ are $K_2$ except at most one block. Now, suppose that $B$ is the leaf block of $G$, such that $V(B) \cap V(B') = \{t\}$. If $G = B \cup B'$, then clearly we are done. Therefore $B'$ contains a cut vertex of $G$, say $z \neq t$. Two cases may be considered.

First, assume that $d_{B'\setminus\{z\}}(t)$ is odd. Define $H$ to be $B' \setminus \{z\}$ union all leaf blocks containing a vertex in $B' \setminus \{z\}$. By induction hypothesis, every connected component of $G \setminus H$ can be partitioned into at most two odd induced subgraphs $O_1, O_2$ and one independent set $I$. Since $N_{G\setminus H}(t) = \{z\}$, there is one odd induced subgraph, say $O_1$, such that $e_G(t, O_1) = 0$. Now, add $B'\setminus\{z\}$ to $O_1$ and call it $O_1'$. Also add the remaining vertices of $H$ to $I$ and call it by $I'$. Clearly, $\{O_1', O_2, I'\}$ is the desired partition of $G$. So, we can assume that $d_{B'\setminus\{z\}}(t)$ is even. Two cases may occur.

(i) Every vertex of $B'\setminus\{z\}$ is adjacent to a leaf block. Let $H$ be that subgraph of $G$ as defined before. Clearly, $H$ is an odd graph. By induction hypothesis, $G \setminus H$ can be partitioned into at most two odd induced subgraphs $O_1, O_2$ and one independent set $I$. With no loss of generality, assume that $z \in V(O_1)$. Then clearly, $\{O_1, O_2 \cup H, I\}$ is the desired partition of $G$ and we are done.

(ii) There is a vertex $x \in B' \setminus \{z\}$ which is not contained in a leaf block. Let $H$ be the union of $B' \setminus \{z, x\}$ and all leaf blocks adjacent to $B' \setminus \{z, x\}$. Clearly, $B' \setminus \{z, x\}$ is an odd graph. By the induction hypothesis, $V(G \setminus H)$ can be partitioned into at most two odd induced subgraphs $O_1, O_2$ and one independent set $I$. Note that since $x$ is a pendant vertex in $G \setminus H$, $\{x, z\}$ appears in at most one of the odd induced subgraphs of $G \setminus H$. Suppose that $O_1$ is the odd induced subgraph of $G \setminus H$ such that there is no edge between $O_1$ and $H$. Now, denote $O_1' = O_1 \cup (B' \setminus \{z, x\})$. Also, add the remaining vertices $H \setminus B'$ to $I$ and denote the resulting set by $I'$. It is not hard to see that $\{O_1', O_2, I'\}$ is the desired partition and the proof is complete. ∎

**Remark 19.** Obviously, the previous result holds if we replace the tree with a forest.

In the sequel, we translate the vertex partitioning problem of a graph into odd or even induced subgraphs to linear algebraic language. If $V(G) = \{v_1, \ldots, v_n\}$, then the *adjacency matrix* of $G$ is a symmetric $n \times n$ matrix $A$, such that $a_{ij} = 1$ if and only if $v_i$ and $v_j$ are adjacent, otherwise $a_{ij} = 0$. Now, using a linear algebraic method, we would like to present an algorithm in order to partition an even or odd graph into two even subgraphs, two odd subgraphs, or one even subgraph and one odd subgraph.

**Remark 20.** Let $G$ be a graph with the adjacency matrix $A$. Table 1 shows how the solutions of the system of linear equations over integers modulo 2 give us the possible vertex partitioning of $G$ into odd or even induced subgraphs. Let $V_0 = \{v_i \,|\, x_i = 0\}$ and $V_1 = \{v_i \,|\, x_i = 1\}$, where $x$ is defined in Table 1. In the following table, $\mathbf{j}$ denotes a vector whose all entries are 1.

| Graph Type | | Odd | Even |
|---|---|---|---|
| $Ax = 0$ | $V_0$ | Odd | Even |
| | $V_1$ | Even | Even |
| $Ax = x$ | $V_0$ | Odd | Even |
| | $V_1$ | Odd | Odd |
| $Ax = \mathbf{j}$ | $V_0$ | Even | Odd |
| | $V_1$ | Odd | Odd |
| $(A+I)x = \mathbf{j}$ | $V_0$ | Even | Odd |
| | $V_1$ | Even | Even |

Table 1

For instance, let $G$ be an odd graph of order $n$ and $z$ be a solution of $Ax = \mathbf{j}$ (mod 2). Assume that for $i = 1, \ldots, n$, $A_i$ denotes the $i$-th column of $A$. Since $Az = \mathbf{j}$, we conclude that $\sum_{i=1}^{n} z_i A_i = \mathbf{j}$, where $z^T = [z_1, \ldots, z_n]$. Consider those entries of $z$ which are 1. Thus we have $\sum_{v_i \in V_1} A_i = \mathbf{j}$. This means that every vertex in $V(G)$, has an odd number of neighbors in $V_1$. Hence, $\langle V_1 \rangle$ is an odd subgraph. Since each vertex of $G$ has odd degree, any vertex of $G$ has even number of neighbors in $V_0$. This yields that $\langle V_0 \rangle$ is an even subgraph. Other cases appeared in Table 1 are similarly discussed.

Now, we close the paper with the following result.

**Theorem 21.** *For every forest $F$, $V(F)$ can be partitioned into one odd induced forest and possibly one independent set.*

***Proof.*** It is well-known that the vertices of each graph can be partitioned into two induced subgraphs, which one is odd and one is even, see [7]. We prove this result using linear algebraic methods. Let $V(G) = \{v_1, \ldots, v_n\}$ and $A$ be the adjacency matrix of $G$. Define an $n \times n$ matrix $B = [b_{ij}]$, $b_{ij} = a_{ij}$ if $i \neq j$ and $b_{ii} = 1 + d(v_i)$ (mod 2). Consider the equation $Bx = [b_{11}, \ldots, b_{nn}]^T$. By [7, p. 44], this equation has at least one solution (mod 2). It is not hard to see that $V_0$ forms an odd induced subgraph and $V_1$ forms an even induced subgraph. Since $F$ is a forest, $V_1$ is an independent set and we are done. ■

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**Authors' Contributions**

All authors contributed equally to this work. All authors read and approved the final manuscript.

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