# ON THE RAMSEY NUMBERS OF NON-STAR TREES VERSUS CONNECTED GRAPHS OF ORDER SIX 

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#### Abstract

This paper completes our studies on the Ramsey number $r\left(T_{n}, G\right)$ for trees $T_{n}$ of order $n$ and connected graphs $G$ of order six. If $\chi(G) \geq 4$, then the values of $r\left(T_{n}, G\right)$ are already known for any tree $T_{n}$. Moreover, $r\left(S_{n}, G\right)$, where $S_{n}$ denotes the star of order $n$, has been investigated in case of $\chi(G) \leq 3$. If $\chi(G)=3$ and $G \neq K_{2,2,2}$, then $r\left(S_{n}, G\right)$ has been determined except for some $G$ and some small $n$. Partial results have been obtained for $r\left(S_{n}, K_{2,2,2}\right)$ and for $r\left(S_{n}, G\right)$ with $\chi(G)=2$. In the present paper we investigate $r\left(T_{n}, G\right)$ for non-star trees $T_{n}$ and $\chi(G) \leq 3$. Especially, $r\left(T_{n}, G\right)$ is completely evaluated for any non-star tree $T_{n}$ if $\chi(G)=3$ where $G \neq K_{2,2,2}$, and $r\left(T_{n}, K_{2,2,2}\right)$ is determined for a class of trees $T_{n}$ with small maximum degree. In case of $\chi(G)=2, r\left(T_{n}, G\right)$ is investigated for $T_{n}=P_{n}$, the path of order $n$, and for $T_{n}=B_{2, n-2}$, the special broom of order $n$ obtained by identifying the centre of a star $S_{3}$ with an end-vertex of a path $P_{n-2}$. Furthermore, the values of $r\left(B_{2, n-2}, S_{m}\right)$ are determined for all $n$ and $m$ with $n \geq m-1$. As a consequence of this paper, $r(F, G)$ is known for all trees $F$ of order at most five and all connected graphs $G$ of order at most six.


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## 1. Introduction

Ramsey number and Ramsey goodness. For graphs $F$ and $G$ the Ramsey number $r(F, G)$ is the smallest integer $p$ such that in every 2-coloring of the edges of $K_{p}$ there is a copy of $F$ in the first color or a copy of $G$ in the second color. The chromatic surplus $s(G)$ is defined to be the smallest number of vertices in a color class under any $\chi(G)$-coloring of the vertices of $G$, where $\chi(G)$ denotes the chromatic number of $G$. It is well-known (see [6] or [7]) that for any connected graph $F$ with $n$ vertices and any graph $G$ with $s(G) \leq n$ the Ramsey number $r(F, G)$ satisfies

$$
\begin{equation*}
r(F, G) \geq(n-1)(\chi(G)-1)+s(G) \tag{1}
\end{equation*}
$$

If equality occurs, then $F$ is said to be $G$-good. Chvátal [3] has proved that every tree $T_{n}$ of order $n$ is $K_{m}$-good, i.e., $r\left(T_{n}, K_{m}\right)=(n-1)(m-1)+1$. Moreover, several classes of non-complete graphs $G$ are known where every tree $T_{n}$ is $G$ good, but there are also graphs $G$ and trees $T_{n}$ such that $r\left(T_{n}, G\right)$ differs even considerably from the lower bound given in (1) - a survey on results for $r\left(T_{n}, G\right)$ can be found in [17].
Our contribution. Faudree, Rousseau and Schelp [7] initiated the systematic study of $r\left(T_{n}, G\right)$ for graphs $G$ of small order $p(G)$ and investigated the case $p(G) \leq 5$. In [11] and [12] we started to extend these investigations to graphs $G$ with $p(G)=6$. Using the result on $r\left(T_{n}, K_{m}\right)$ due to Chvátal and results on $r\left(T_{n}, G\right)$ for nearly complete graphs $G$ due to Chartrand, Gould and Polimeni [2] and Gould and Jacobson [8] it was not difficult to derive that any tree $T_{n}$ with $n \geq 5$ is $G$-good for all graphs $G$ with $p(G)=6$ and $\chi(G) \geq 4$. In [11] our main focus was on $r\left(S_{n}, G\right)$ where $S_{n}$ denotes the star of order $n$ and $G$ is a connected graph of order six with $G \neq K_{2,2,2}$ and $\chi(G) \leq 3$, in [12] we studied $r\left(S_{n}, K_{2,2,2}\right)$. Especially we proved that in case of $\chi(G)=3$ and $G \neq K_{2,2,2}$ the star $S_{n}$ is $G$-good or, in a few cases, $r\left(S_{n}, G\right)$ differs by 1 or 2 from the lower bound (1). In contrast, for $n$ sufficiently large, $r\left(S_{n}, K_{2,2,2}\right)>2 n-2+\left\lfloor\sqrt{n-1}-6(n-1)^{11 / 40}\right\rfloor$, i.e., $r\left(S_{n}, K_{2,2,2}\right)$ differs considerably from the lower bound $2 n$ given in (1).

In this paper we study $r\left(T_{n}, G\right)$ for non-star trees $T_{n}$ and connected graphs $G$ with $p(G)=6$ and $\chi(G) \leq 3$. We prove that every non-star tree $T_{n}$ is $G$-good for every connected graph $G \notin\left\{K_{1,1,4}, K_{2,2,2}\right\}$ with $p(G)=6$ and $\chi(G)=3$. A more general result on $r\left(T_{n}, K_{1,1, m}\right)$ due to Erdős, Faudree, Rousseau and Schelp [6] and our results from [13] show that, except for $n \leq 5$, every non-star tree $T_{n}$ is also $K_{1,1,4}$-good. The case $G=K_{2,2,2}$ remains to a great extent unsolved. We present several $K_{2,2,2}$ - good non-star trees $T_{n}$ with small maximum degree, but the behavior of $r\left(S_{n}, K_{2,2,2}\right)$ implies that non-star trees $T_{n}$ with sufficiently large $n$ and maximum degree close to $n-1$ cannot be $K_{2,2,2^{-} \text {-good. }}$.

To determine $r\left(T_{n}, G\right)$ for every tree $T_{n}$ and all connected graphs $G$ of order six with $\chi(G)=2$, i.e., the star $S_{6}$ and the connected spanning subgraphs of $K_{2,4}$
and $K_{3,3}$, seems to be a hard problem. Partial results on $r\left(S_{n}, G\right)$ were obtained in [11]. In this paper we investigate $r\left(T_{n}, G\right)$ for two non-star trees $T_{n}$, namely $T_{n}=P_{n}$, the path on $n$ vertices, and $T_{n}=B_{2, n-2}$, a special case of a broom $B_{k, n-k}$ defined as a tree of order $n \geq 5$ obtained by identifying the centre of a star $S_{k+1}, k \geq 2$, with an end-vertex of a path $P_{n-k}$. The choice of these two non-star trees is due to the project to evaluate $r(F, G)$ for graphs $F$ of order at most five and graphs $G$ of order six - the only non-star trees on at most five vertices are the paths $P_{n}$ with $4 \leq n \leq 5$ and the broom $B_{2,3}$. Instead of $r\left(T_{n}, S_{6}\right)$ we consider the more general case $r\left(T_{n}, S_{m}\right)$. Parsons [14] has already determined $r\left(P_{n}, S_{m}\right)$ for all $n$ and $m$ by explicit formulas and a recurrence, and we evaluate $r\left(B_{2, n-2}, S_{m}\right)$ for all $n$ and $m$ with $n \geq m-1$. The results in this paper together with the results in [11] and [12] imply that $r(F, G)$ is known for all trees $T_{n}$ of order at most five and all connected graphs $G$ of order six.

Notation and terminology. Some specialized notation and terminology will be used. The vertex set of a graph $G$ is denoted by $V(G)$. We write $G^{\prime} \subseteq G$ if $G^{\prime}$ is a subgraph of $G$ and, for $U \subseteq V\left(K_{n}\right),[U]$ is the subgraph induced by $U$. A coloring of a graph here always means a 2 -coloring of its edges with colors red and green. An $\left(F_{1}, F_{2}\right)$-coloring is a coloring containing neither a red copy of $F_{1}$ nor a green copy of $F_{2}$. Given a coloring of $K_{n}$, we define the $r$-degree $d_{r}(v)$ to be the number of red edges incident to $v \in V\left(K_{n}\right)$. Moreover, $\Delta_{r}=\max _{v \in V\left(K_{n}\right)} d_{r}(v)$. The set of vertices joined red to $v$ is denoted by $N_{r}(v)$. Similarly we define $d_{g}(v), \Delta_{g}$ and $N_{g}(v)$. Furthermore, $[U]_{r}$ and $[U]_{g}$ are the red and the green subgraphs induced by $U$. For disjoint subsets $U_{1}, U_{2} \subseteq V\left(K_{n}\right), q_{r}\left(U_{1}, U_{2}\right)$ denotes the number of red edges between $U_{1}$ and $U_{2}$, and $q_{g}\left(U_{1}, U_{2}\right)$ is defined similarly. The vertex of degree $n-1$ in a star $S_{n}$ with $n \geq 3$ is called the centre of the star. We write $P_{k}=u_{1} u_{2} \cdots u_{k}$ for the path $P_{k}$ with vertices $u_{1}, \ldots, u_{k}$ and edges $u_{i} u_{i+1}$ for $i=1, \ldots, k-1$. Moreover, $\left(u_{1} u_{2} \cdots u_{k}\right)$ means the cycle $C_{k}$ obtained from $P_{k}=u_{1} u_{2} \cdots u_{k}$ by adding the edge $u_{1} u_{k}$, and an edge $u_{i} u_{j}$ is called a diagonal of length $\ell$ of $C_{k}$ if $u_{i}$ and $u_{j}$ are vertices with distance $\ell$ on $C_{k}$. The bristles of a broom $B_{k, n-k}$ are the $k$ edges joining the vertex $v^{*}$ of degree $k+1$ to a vertex of degree 1 and the path $P_{n-k}$ with end-vertex $v^{*}$ is said to be the handle of the broom. The complement $\overline{K_{n}}$ of $K_{n}$ is denoted by $E_{n}$, and for the complete $k$-partite graph $K_{n_{1}, n_{2}, \ldots, n_{k}}=E_{n_{1}}+E_{n_{2}}+\cdots+E_{n_{k}}$ with $V\left(E_{n_{i}}\right)=U_{i}$ we write $U_{1}+U_{2}+\cdots+U_{k}$.

## 2. Non-Star Trees $T_{n}$ and the Graphs $G$ with $\chi(G)=3$

First we consider the graphs $G$ of order six with chromatic number $\chi(G)=3$ and $G \notin\left\{K_{1,1,4}, K_{2,2,2}\right\}$. The following theorem states that for all these graphs $G$ every non-star tree $T_{n}$ is $G$-good.

Theorem 2.1. Let $n \geq 4, T_{n} \neq S_{n}$, and let $G$ be a graph of order six with $\chi(G)=3$ where $G \neq K_{1,1,4}$ and $G \neq K_{2,2,2}$. Then

$$
r\left(T_{n}, G\right)= \begin{cases}2 n-1 & \text { if } G \subseteq K_{1,2,3} \\ 2 n & \text { otherwise }\end{cases}
$$

To prove Theorem 2.1 by induction on $n$ the following properties of trees $T_{n}$ are essential.

Lemma 2.2. (i) If $n \geq 6$ and $T_{n} \notin\left\{S_{n}, B_{n-3,3}\right\}$, then $T_{n}$ contains vertices $v_{1}$ and $v_{2}$ of degree 1 with distance $d\left(v_{1}, v_{2}\right) \geq 3$ such that $T_{n}-\left\{v_{1}, v_{2}\right\}$ is a non-star tree of order $n-2$.
(ii) If $n \geq 5$ and $T_{n} \neq S_{n}$, then $T_{n}$ contains a vertex $v$ of degree 1 such that $T_{n}-\{v\}$ is a non-star tree of order $n-1$.

Proof. Let $P=u_{0} u_{1} \cdots u_{\ell}$ be a path of maximum length $\ell$ in $T_{n}$. Clearly, $d\left(u_{0}\right)=d\left(u_{\ell}\right)=1$. Moreover, $T_{n} \neq S_{n}$ implies $\ell \geq 3$.
(i) Since in a tree any two vertices are connected by a unique path, $d\left(u_{0}, u_{\ell}\right)=$ $\ell \geq 3$. Consider the tree $T^{*}=T_{n}-\left\{u_{0}, u_{\ell}\right\}$ of order $n-2$. Obviously, $T^{*} \neq S_{n-2}$ for $\ell \geq 5$. In case of $\ell=3, T^{*} \neq S_{n-2}$ also holds, since otherwise one of the vertices $u_{1}$ and $u_{2}$ has to be the centre of $S_{n-2}$, and this yields $T_{n}=B_{n-3,3}$, a contradiction. It remains $\ell=4$. Then we are done if $T^{*} \neq S_{n-2}$. In case of $T^{*}=S_{n-2}, u_{2}$ has to be the centre of $S_{n-2}$ and among the $n-3 \geq 3$ vertices of degree 1 in $T^{*}$ adjacent to $u_{2}$ we find a vertex $w$ of degree 1 in $T_{n}$. But then $u_{0}$ and $w$ are vertices of degree 1 with $d\left(u_{0}, w\right) \geq 3$ such that $T_{n}-\left\{u_{0}, w\right\}$ is a non-star tree of order $n-2$.
(ii) Consider the tree $T^{\prime}=T_{n}-\left\{u_{0}\right\}$ of order $n-1$. Clearly, $T^{\prime} \neq S_{n-1}$ for $\ell \geq 4$. It remains $\ell=3$. Then we are done if $T^{\prime} \neq S_{n-1}$. In case of $T^{\prime}=S_{n-1}$, $u_{2}$ has to be the centre of $S_{n-1}$ forcing $T_{n}=B_{n-3,3}$ where $n-3 \geq 2$. But then $T_{n}-\left\{u_{3}\right\}$ is a non-star tree of order $n-1$.

Besides Lemma 2.2 the values of $r\left(T_{n}, P_{3}\right)$ and $r\left(T_{n}, P_{4}\right)$ for $T_{n} \neq S_{n}$ will be used to prove Theorem 2.1. Chvátal and Harary [4] obtained a formula to derive $r\left(G, P_{3}\right)$ for any graph $G$ depending on the edge independence number $\beta_{1}(\bar{G})$ of the complement $\bar{G}$ of $G$.

Theorem 2.3 (Chvátal and Harary [4]). Let $G$ be a graph of order n. Then

$$
r\left(G, P_{3}\right)= \begin{cases}n & \text { if } \bar{G} \text { contains a 1-factor } \\ 2 n-2 \beta_{1}(\bar{G})-1 & \text { otherwise }\end{cases}
$$

For every tree $T_{n} \neq S_{n}, \beta_{1}\left(\overline{T_{n}}\right)=\lfloor n / 2\rfloor$. Applying Theorem 2.3 we obtain the following result.

Corollary 2.4. Let $n \geq 4$ and $T_{n} \neq S_{n}$. Then $r\left(T_{n}, P_{3}\right)=n$.
The next result on $r\left(T_{n}, P_{4}\right)$ was already mentioned without proof by Faudree, Rousseau and Schelp in [7].

Theorem 2.5. Let $n \geq 4$ and $T_{n} \neq S_{n}$. Then $r\left(T_{n}, P_{4}\right)=n+1$.
Proof. Since $\chi\left(P_{4}\right)=2$ and $s\left(P_{4}\right)=2$ we obtain $r\left(T_{n}, P_{4}\right) \geq n+1$ from (1). To prove that $r\left(T_{n}, P_{4}\right) \leq n+1$ we use induction on $n$. It is easy to check that $r\left(T_{n}, P_{4}\right) \leq n+1$ holds for $4 \leq n \leq 5$ if $T_{n} \neq S_{n}$, i.e., $T_{n} \in\left\{P_{4}, P_{5}, B_{2,3}\right\}$ (cf. also [4] and [5]). Now let $n \geq 6$. By the induction hypothesis, $r\left(T_{k}, P_{4}\right) \leq k+1$ for every tree $T_{k} \neq S_{k}$ with $4 \leq k<n$. Suppose that a ( $T_{n}, P_{4}$ )-coloring of $K_{n+1}$ with vertex set $V$ exists for some tree $T_{n} \neq S_{n}$ of order $n$.

Case 1. $K_{3} \subseteq[V]_{g}$. Let $U=\left\{u_{1}, u_{2}, u_{3}\right\}$ be the vertex set of a green $K_{3}$ and $W=V \backslash U$. Since $P_{4} \nsubseteq[V]_{g}$, all edges between $U$ and $W$ have to be red. Thus $K_{n-2,3} \subseteq[V]_{r}$. Since $B_{n-3,3} \subseteq K_{n-2,3}$ and $T_{n} \nsubseteq[V]_{r}$ it follows that $T_{n} \neq B_{n-3,3}$. By Lemma 2.2(i), $T_{n}$ contains two vertices $v_{1}$ and $v_{2}$ of degree 1 with $d\left(v_{1}, v_{2}\right) \geq 3$ such that the tree $T^{*}=T_{n}-\left\{v_{1}, v_{2}\right\}$ of order $n-2$ is not a star. The induction hypothesis yields $r\left(T^{*}, P_{4}\right) \leq n-1$. Consider $V^{\prime}=V \backslash\left\{u_{1}, u_{2}\right\}$. Since $\left|V^{\prime}\right|=n-1$ and $P_{4} \nsubseteq\left[V^{\prime}\right]_{g}$, we obtain that $T^{*} \subseteq\left[V^{\prime}\right]_{r}$. Let $a_{1}$ and $a_{2}$ be the two vertices in $T^{*}$ such that $a_{i}$ is adjacent to $v_{i}$ in $T_{n}$. Since $d\left(v_{1}, v_{2}\right) \geq 3$, $a_{1} \neq a_{2}$. If $\left\{a_{1}, a_{2}\right\} \subseteq W$, then the edges $a_{1} u_{1}$ and $a_{2} u_{2}$ together with $T^{*}$ would yield a red $T_{n}$, a contradiciton. If $a_{1}=u_{3}$ or $a_{2}=u_{3}$, say $a_{1}=u_{3}$, then a vertex $w \in W$ exists where $w \notin V\left(T^{*}\right)$. But then the edges $a_{1} w$ and $a_{2} u_{2}$ together with $T^{*}$ again yield a red $T_{n}$.

Case 2. $K_{3} \nsubseteq[V]_{g}$. Let $v$ be a vertex in $V$ with $d_{g}(v)=\Delta_{g}$. Corollary 2.4 and $T_{n} \nsubseteq[V]_{r}$ force $P_{3} \subseteq[V]_{g}$, and this implies $\Delta_{g} \geq 2$. Let $W=V \backslash\{v\}$. As $K_{3} \nsubseteq[V]_{g}$ and $P_{4} \nsubseteq[V]_{g}$, in $[W]$ every $w \in N_{g}(v)$ is incident to red edges only. By Lemma 2.2(ii), $T_{n}$ must contain a vertex $u$ of degree 1 such that $T^{\prime}=T_{n}-\{u\}$ is a tree of order $n-1$ different from $S_{n-1}$. Let $w \in V\left(T^{\prime}\right)$ be the neighbor of $u$ in $T_{n}$. By the induction hypothesis, $r\left(T^{\prime}, P_{4}\right) \leq n$. Since $|W|=n$ and $P_{4} \nsubseteq[W]_{g}$, a red $T^{\prime}$ occurs in $[W]$. If $w \in N_{r}(v)$, then $T^{\prime}$ together with $v w$ yields a red $T_{n}$, a contradiction. It remains that $w \in N_{g}(v)$. We already know that in [ $W$ ] every $w \in N_{g}(v)$ is incident to red edges only. Since $|W|=n$, there is a vertex $w^{\prime} \in W$ with $w^{\prime} \notin V\left(T^{\prime}\right)$. But then $T^{\prime}$ together with $w w^{\prime}$ yields a red $T_{n}$ and the proof is complete.

With these preparations we can now prove Theorem 2.1.
Proof of Theorem 2.1. By (1), $r\left(T_{n}, G\right) \geq 2 n-1$ for any graph $G$ with $\chi(G)=$ 3. If $G \neq K_{1,1,4}$ and $G \nsubseteq K_{1,2,3}$, then $s(G)=2$, and (1) yields $r\left(T_{n}, G\right) \geq 2 n$. Moreover, $s(G)=2$ and $G \neq K_{2,2,2}$ imply $G \subseteq K_{2,2,2}-e$. Thus, it suffices to
prove $r\left(T_{n}, K_{1,2,3}\right) \leq 2 n-1$ and $r\left(T_{n}, K_{2,2,2}-e\right) \leq 2 n$ for every tree $T_{n} \neq S_{n}$ where $n \geq 4$. We use that the join $E_{2}+P_{4}$ is isomorphic to $K_{2,2,2}-e$ and we write $\left\{v_{1}, v_{2}\right\}+P_{4}$ if $V\left(E_{2}\right)=\left\{v_{1}, v_{2}\right\}$. The proof consists of two parts: in (i) we derive the desired results for $T_{n}=B_{n-3,3}$, and in (ii) we consider the trees $T_{n} \notin\left\{S_{n}, B_{n-3,3}\right\}$.
(i) Let $T_{n}=B_{n-3,3}$ where the degenerated broom $B_{1,3}=P_{4}$ is included. Suppose we have a ( $B_{n-3,3}, K_{1,2,3}$ )-coloring of $K_{2 n-1}$ or a ( $B_{n-3,3}, K_{2,2,2}-e$ )coloring of $K_{2 n}$. Let $V$ denote the vertex sets of the complete graphs.
Claim 2.6. $S_{n-1} \subseteq[V]_{r}$.
Proof. From [11] we know that $r\left(S_{n-1}, G\right) \leq 2 n-1$ if $G=K_{1,2,3}$ or $G=K_{2,2,2}-e$ and $n \geq 5$. Because of $S_{3}=P_{3}, r\left(P_{3}, G\right)=r\left(G, P_{3}\right)$ and Theorem 2.3 this upper bound also holds for $n=4$. Thus, if $K_{1,2,3} \nsubseteq[V]_{g}$ or $K_{2,2,2}-e \nsubseteq[V]_{g}$, then $S_{n-1} \subseteq[V]_{r}$.

Claim 2.7. $S_{n} \nsubseteq[V]_{r}$.
Proof. Assume that $S_{n} \subseteq[V]_{r}$ and let $U$ be the vertex set of a red $S_{n}$ with centre $u_{0}$. Since a red $B_{n-3,3}$ is forbidden, $\left[U \backslash\left\{u_{0}\right\}\right]$ has to be a green $K_{n-1}$. Moreover, all edges between $W=V \backslash U$ and $U \backslash\left\{u_{0}\right\}$ have to be green. This gives a green $K_{6}-K_{3}$ in case of $|V|=2 n-1$, i.e., $|W|=n-1$, contradicting $K_{1,2,3} \nsubseteq[V]_{g}$. In case of $|V|=2 n$, i.e., $|W|=n$, Corollary 2.4 and $B_{n-3,3} \nsubseteq[V]_{r}$ imply that a green $P_{3}$ must occur in [ $W$ ]. This yields a green $K_{6}-e$, a contradiction to $K_{2,2,2}-e \nsubseteq[V]_{g}$.

Now we use Claim 2.6 and consider a red $S_{n-1}$ with vertex set $U$ and centre $u_{0}$. By Claim 2.7 and $B_{n-3,3} \nsubseteq[V]_{r}$, all edges between $U$ and $W=V \backslash U$ have to be green. In case of $|V|=2 n-1$ it follows that $|W|=n$, and Corollary 2.4 together with $B_{n-3,3} \nsubseteq[V]_{r}$ imply that a green $P_{3}=w_{1} w_{2} w_{3}$ occurs in $[W]$. But then $\left\{w_{2}\right\}+\left\{w_{1}, w_{3}\right\}+\left\{u_{0}, u_{1}, u_{2}\right\}$ where $\left\{u_{1}, u_{2}\right\} \subseteq U \backslash\left\{u_{0}\right\}$ is a green $K_{1,2,3}$, a contradiction. In case of $|V|=2 n$ we obtain $|W|=n+1$, and Theorem 2.5 together with $B_{n-3,3} \nsubseteq[V]_{r}$ guarantee a green $P_{4}$ in $[W]$. But this forces $\left\{u_{1}, u_{2}\right\}+P_{4}$ to be a green $K_{2,2,2}-e$, a contradiction, and we are done for $T_{n}=B_{n-3,3}$.
(ii) It remains that $T_{n} \notin\left\{S_{n}, B_{n-3,3}\right\}$. We use induction on $n$ to prove $r\left(T_{n}, K_{1,2,3}\right) \leq 2 n-1$ and $r\left(T_{n}, K_{2,2,2}-e\right) \leq 2 n$ for every tree $T_{n} \notin\left\{S_{n}, B_{n-3,3}\right\}$ with $n \geq 4$.

First we derive the desired results for $4 \leq n \leq 5$. There is only one tree $T_{n} \notin\left\{S_{n}, B_{n-3,3}\right\}$ with $4 \leq n \leq 5$, namely $P_{5}$. To prove $r\left(P_{5}, K_{1,2,3}\right) \leq 9$ and $r\left(P_{5}, K_{2,2,2}-e\right) \leq 10$ assume we have a ( $P_{5}, K_{1,2,3}$ )-coloring of $K_{9}$ or a ( $P_{5}, K_{2,2,2}-e$ )-coloring of $K_{10}$. Let $V$ denote the vertex sets of the complete graphs. Since $P_{4}=B_{1,3}$, by the above result on brooms we already know that $r\left(P_{4}, K_{1,2,3}\right) \leq 7$ and $r\left(P_{4}, K_{2,2,2}-e\right) \leq 8$. Thus, a red $P_{4}=u_{1} u_{2} u_{3} u_{4}$ must occur
in $[V]$, and $P_{5} \nsubseteq[V]_{r}$ forces all edges between $\left\{u_{1}, u_{4}\right\}$ and the vertices in $W=$ $V \backslash\left\{u_{1}, u_{2}, u_{3}, u_{4}\right\}$ to be green. In $K_{9}$ we obtain $|W|=5$, and $r\left(P_{5}, S_{4}\right)=5$ (cf. [5]) guarantees a green $S_{4}$ in $[W]$ with centre $w_{0}$ and vertices $w_{1}, w_{2}, w_{3}$ of degree 1 yielding the green $K_{1,2,3}=\left\{w_{0}\right\}+\left\{u_{1}, u_{4}\right\}+\left\{w_{1}, w_{2}, w_{3}\right\}$, a contradiction. In $K_{10}$ we have $|W|=6$, and $r\left(P_{5}, P_{4}\right)=6$ (see Theorem 2.5) forces a green $P_{4}$ in [ $W$ ]. But then $\left\{u_{1}, u_{4}\right\}+P_{4}$ is a green $K_{2,2,2}-e$, a contradiction.

Now let $n \geq 6$. By the induction hypothesis, $r\left(T_{k}, K_{1,2,3}\right) \leq 2 k-1$ and $r\left(T_{k}, K_{2,2,2}-e\right) \leq 2 k$ for every tree $T_{k} \notin\left\{S_{k}, B_{k-3,3}\right\}$ with $4 \leq k<n$. Suppose we have a ( $T_{n}, K_{1,2,3}$ )-coloring of $K_{2 n-1}$ or a ( $T_{n}, K_{2,2,2}-e$ )-coloring of $K_{2 n}$ for some tree $T_{n}$ where $T_{n} \notin\left\{S_{n}, B_{n-3,3}\right\}$. Again we use $V$ to denote the vertex sets of the complete graphs. By Lemma 2.2(i), $T_{n}$ contains two vertices $v_{1}$ and $v_{2}$ of degree 1 with distance $d\left(v_{1}, v_{2}\right) \geq 3$ such that the tree $T^{*}=T_{n}-\left\{v_{1}, v_{2}\right\}$ of order $n-2$ is not a star. By the induction hypothesis and the above result on brooms, $r\left(T^{*}, K_{1,2,3}\right) \leq 2 n-5$ and $r\left(T^{*}, K_{2,2,2}-e\right) \leq 2 n-4$. Let $a_{1}$ and $a_{2}$ be the two vertices in $T^{*}$ such that $a_{i}$ is adjacent to $v_{i}$ in $T_{n}$, where $1 \leq i \leq 2$. Since $r\left(T_{n}, K_{4}-e\right)=2 n-1$ (see [2]), one of the following two cases must occur.

Case 1. $K_{4} \subseteq[V]_{g}$. Let $U=\left\{u_{1}, u_{2}, u_{3}, u_{4}\right\}$ be the vertex set of a green $K_{4}$ with minimal sum $d_{r}\left(u_{1}\right)+d_{r}\left(u_{2}\right)+d_{r}\left(u_{3}\right)+d_{r}\left(u_{4}\right)$ of $r$-degrees, and let $W=V \backslash U$. Since $|W|=2 n-5$ in case of $K_{2 n-1}$ and $|W|=2 n-4$ in case of $K_{2 n}, T^{*} \subseteq[W]_{r}$. We distinguish two subcases depending on $q_{r}\left(a_{i}, U\right)$.

Case 1.1. $q_{r}\left(a_{1}, U\right) \geq 1$ and $q_{r}\left(a_{2}, U\right) \geq 1$. Then $T_{n} \subseteq[V]_{r}$, except for $q_{r}\left(a_{1}, U\right)=q_{r}\left(a_{2}, U\right)=1$ where $a_{1}$ and $a_{2}$ have the same red neighbor in $U$, say $u_{1}$. But this gives the green $K_{1,2,3}=\left\{u_{2}\right\}+\left\{u_{3}, u_{4}\right\}+\left\{u_{1}, a_{1}, a_{2}\right\}$, a contradiction for $|V|=2 n-1$. In the remaining case $|V|=2 n$ let $W^{\prime}=W \backslash V\left(T^{*}\right)$. Note that $\left|W^{\prime}\right|=n-2$. If $a_{1}$ or $a_{2}$ has a red neighbor in $W^{\prime}$, then again a red $T_{n}$ occurs. Otherwise all $n+1$ vertices in $W^{\prime} \cup\left\{u_{2}, u_{3}, u_{4}\right\}$ are common green neighbors of $a_{1}$ and $a_{2}$, and Theorem 2.5 guarantees a green $P_{4}$ in $\left[W^{\prime} \cup\left\{u_{2}, u_{3}, u_{4}\right\}\right]$. But this forces $\left\{a_{1}, a_{2}\right\}+P_{4}$ to be a green $K_{2,2,2}-e$, a contradiction.

Case 1.2. $q_{r}\left(a_{1}, U\right)=0$ or $q_{r}\left(a_{2}, U\right)=0$, say $q_{r}\left(a_{1}, U\right)=0$. Now let $U^{\prime}=$ $U \cup\left\{a_{1}\right\}$ and $W^{\prime}=V \backslash U^{\prime}$. Note that $\left[U^{\prime}\right]$ is a green $K_{5}$ and that $\left|W^{\prime} \cap V\left(T^{*}\right)\right|=$ $n-3$. If $q_{r}\left(w, U^{\prime}\right) \leq 2$ for some $w \in W^{\prime}$, then we find a green $K_{1,2,3}$ and a green $K_{2,2,2}-e$ in $\left[U^{\prime} \cup\{w\}\right]$, a contradiction. Thus $q_{r}\left(w, U^{\prime}\right) \geq 3$ for every $w \in W^{\prime}$ yielding $q_{r}\left(W^{\prime}, U^{\prime}\right) \geq 3\left|W^{\prime}\right| \geq 3(2 n-6)$. This implies $q_{r}\left(u, W^{\prime}\right)=d_{r}(u) \geq n-2$ for some $u \in U^{\prime}$. In case of $d_{r}\left(a_{1}\right) \leq n-3$ we may assume that $d_{r}\left(u_{4}\right) \geq n-2$. But then the green $K_{4}=\left[\left\{a_{1}, u_{1}, u_{2}, u_{3}\right\}\right]$ would have a smaller sum of $r$-degrees than the green $K_{4}=\left[\left\{u_{1}, u_{2}, u_{3}, u_{4}\right\}\right]$. It remains $d_{r}\left(a_{1}\right) \geq n-2$. This forces $q_{r}\left(a_{1}, W^{\prime}\right) \geq n-2$ and we find a red neighbor $w^{*}$ of $a_{1}$ in $W^{\prime} \backslash V\left(T^{*}\right)$ since $\left|W^{\prime} \cap V\left(T^{*}\right)\right|=n-3$. Moreover, $q_{r}\left(w, U^{\prime}\right) \geq 3$ for every $w \in W^{\prime}$ yields a red neighbor $u^{*}$ of $a_{2}$ in $U$. But then $T^{*}$ together with $w^{*}$ and $u^{*}$ produce a red $T_{n}$, a contradiction.

Case 2. $K_{4}-e \subseteq[V]_{g}$ and $K_{4} \nsubseteq[V]_{g}$. Let $U=\left\{u_{1}, u_{2}, u_{3}, u_{4}\right\}$ be the vertex set of a green $K_{4}-e$ where $u_{1} u_{4}$ is red, and let $W=V \backslash U$. Since $K_{4} \nsubseteq[V]_{g}$, $q_{r}(w, U) \geq 1$ for every $w \in W$. As in Case $1, T^{*} \subseteq[W]_{r}$, and $T_{n} \nsubseteq[V]_{r}$ forces $q_{r}\left(a_{1}, U\right)=q_{r}\left(a_{2}, U\right)=1$. Moreover, $a_{1}$ and $a_{2}$ must have the same red neighbor in $U$, and $K_{4} \nsubseteq[V]_{g}$ implies that $u_{2}$ or $u_{3}$, say $u_{2}$, is the common red neighbor. But then we obtain the green $K_{1,2,3}=\left\{u_{3}\right\}+\left\{u_{1}, u_{4}\right\}+\left\{u_{2}, a_{1}, a_{2}\right\}$, a contradiction for $|V|=2 n-1$. In the remaining case $|V|=2 n$ let $W^{\prime}=$ $W \backslash V\left(T^{*}\right)$. Note that $\left|W^{\prime}\right|=n-2$. If $a_{1}$ or $a_{2}$ has a red neighbor in $W^{\prime}$, then a red $T_{n}$ occurs. Otherwise, the $n+1$ vertices in $W^{\prime} \cup\left\{u_{1}, u_{3}, u_{4}\right\}$ are common green neighbors of $a_{1}$ and $a_{2}$, and Theorem 2.5 guarantees a green $P_{4}$ in $\left[W^{\prime} \cup\left\{u_{1}, u_{3}, u_{4}\right\}\right]$. But this gives a green $K_{2,2,2}-e$ and we are done.

The two graphs $G$ of order six with $\chi(G)=3$ not considered in Theorem 2.1 are $G=K_{1,1,4}$ and $G=K_{2,2,2}$. The values of $r\left(T_{n}, K_{1,1,4}\right)$ for $n \geq 9$ follow from a more general result due to Erdős, Faudree, Rousseau and Schelp [6] who investigated $r\left(T_{n}, B_{m}\right)$ for any tree $T_{n}$ and the book-graph $B_{m}=K_{1,1, m}$.

Theorem 2.8 (Erdős, Faudree, Rousseau and Schelp [6]). If $n \geq 3 m-3$, then

$$
r\left(T_{n}, B_{m}\right)=2 n-1 .
$$

Applying Theorem 2.8 for $B_{4}=K_{1,1,4}$ we obtain $r\left(T_{n}, K_{1,1,4}\right)=2 n-1$ for any tree $T_{n}$ with $n \geq 9$. A result due to Rousseau and Sheehan [18] implies $r\left(P_{n}, K_{1,1,4}\right)=10$ for $4 \leq n \leq 5$ and $r\left(P_{n}, K_{1,1,4}\right)=2 n-1$ for $n \geq 6$. Moreover, in [13] we determined the missing values of $r\left(T_{n}, K_{1,1,4}\right)$ for $n \leq 8$. This proves that any non-star tree $T_{n}$ with $n \geq 6$ is $K_{1,1,4}$-good.

Theorem 2.9. Let $n \geq 4$ and $T_{n} \neq S_{n}$. Then

$$
r\left(T_{n}, K_{1,1,4}\right)= \begin{cases}10 & \text { if } 4 \leq n \leq 5 \\ 2 n-1 & \text { if } n \geq 6\end{cases}
$$

For the remaining graph $G=K_{2,2,2}$ the situation is much more complicated. From (1) we obtain $r\left(T_{n}, K_{2,2,2}\right) \geq 2 n$. On the other hand, for $n$ sufficiently large we know that $r\left(S_{n}, K_{2,2,2}\right)>2 n-2+\left\lfloor\sqrt{n-1}-6(n-1)^{11 / 40}\right\rfloor$ (see [12], note that $\left.K_{2,2,2}=K_{6}-3 K_{2}\right)$ forcing $r\left(T_{n}, K_{2,2,2}\right)>2 n$ also for non-star trees $T_{n}$ with maximum degree close to $n-1$ if $n$ is sufficiently large. Nevertheless, there are non-star trees with small maximum degree where the lower bound $2 n$ is attained. For $T_{n}=P_{n}$ this follows from a more general result due to Gould and Jacobson [8] who proved that any path $P_{n}$ with $n \geq 3$ is ( $\left.K_{2 m}-m K_{2}\right)$-good.
Theorem 2.10 (Gould and Jacobson [8]). If $n \geq 3$ and $m \geq 2$, then

$$
r\left(P_{n}, K_{2 m}-m K_{2}\right)=(n-1)(m-1)+2 .
$$

The following theorem shows that $r\left(T_{n}, K_{2,2,2}\right)=2 n$ also holds for a special class of trees $T_{n}$ with $\Delta\left(T_{n}\right)=3$.

Theorem 2.11. Let $T_{n}$ be a tree of order $n \geq 5$ with $\Delta\left(T_{n}\right)=3$ containing a path $P_{n-1}$. Then

$$
r\left(T_{n}, K_{2,2,2}\right)=2 n
$$

To prove Theorem 2.11 we use a result due to Burr, Erdős, Faudree, Rousseau and Schelp [1] who obtained a formula to determine $r\left(T_{n}, C_{4}\right)$ depending on $r\left(S_{m+1}, C_{4}\right)$ where $m=\Delta\left(T_{n}\right)$.

Theorem 2.12 (Burr, Erdős, Faudree, Rousseau and Schelp [1]). If $T_{n}$ is a tree with $\Delta\left(T_{n}\right)=m$, then $r\left(T_{n}, C_{4}\right)=\max \left\{4, n+1, r\left(S_{m+1}, C_{4}\right)\right\}$.

Thus, $r\left(T_{n}, C_{4}\right)$ is easily evaluated if $r\left(S_{m+1}, C_{4}\right)$ is known, but $r\left(S_{m+1}, C_{4}\right)$ has not yet been completely determined (see Parsons [15] and Wu, Sun, Zhang and Radziszowski [19]).

Proof of Theorem 2.11. It suffices to prove that $r\left(T_{n}, K_{2,2,2}\right) \leq 2 n$. Let $T_{n}$ be a tree with $\Delta\left(T_{n}\right)=3$ containing a path $P_{n-1}$ and suppose we have a $\left(T_{n}, K_{2,2,2}\right)$ coloring of $K_{2 n}$ with vertex set $V$.

Claim 2.13. $\left|N_{g}\left(v_{1}\right) \cap N_{g}\left(v_{2}\right)\right| \leq n$ for any two vertices $v_{1}$ and $v_{2}$.
Proof. Assume that there are vertices $v_{1}$ and $v_{2}$ with $\left|N_{g}\left(v_{1}\right) \cap N_{g}\left(v_{2}\right)\right| \geq n+1$. Since $r\left(S_{4}, C_{4}\right)=6$ (cf. [4]), Theorem 2.12 states $r\left(T_{n}, C_{4}\right)=n+1$. Thus, $T_{n} \nsubseteq[V]_{r}$ forces a green $C_{4}=\left(w_{1} w_{2} w_{3} w_{4}\right)$ in $\left[N_{g}\left(v_{1}\right) \cap N_{g}\left(v_{2}\right)\right]$. But this gives the green $K_{2,2,2}=\left\{v_{1}, v_{2}\right\}+\left\{w_{1}, w_{3}\right\}+\left\{w_{2}, w_{4}\right\}$, a contradiction.

By Theorem 2.10 and $K_{2,2,2}=K_{6}-3 K_{2}$, a red $P_{n-1}=u_{1} u_{2} \cdots u_{n-1}$ must occur. First let $n$ be odd or, in case of $n$ even, let $T_{n}$ not be isomorphic to the tree obtained from $u_{1} u_{2} \cdots u_{n-1}$ by joining a vertex $w \in W=V \backslash\left\{u_{1}, \ldots, u_{n-1}\right\}$ to $u_{n / 2}$. Then $T_{n} \nsubseteq[V]_{r}$ implies that there is some $i$ with $1 \leq i \leq\lfloor(n-1) / 2\rfloor-1$ such that $u_{1+i}$ and $u_{n-1-i}$ are joined green to all $n+1$ vertices in $W$, a contradiction to Claim 2.13. Consider now the remaining case for $n$ even. Since $T_{n} \nsubseteq[V]_{r}$, all edges from $u_{n / 2}$ to $W$ have to be green, and then Claim 2.13 forces at least one red edge from every $u_{i}$ with $i \neq n / 2$ to $W$. Moreover, two independent red edges between $\left\{u_{1}, u_{n / 2-1}\right\}$ and $W$ would yield a red $T_{n}$. Thus we may assume that $u_{1}$ and $u_{n / 2-1}$ have a common red neighbor $w^{*} \in W$ and that all edges between $\left\{u_{1}, u_{n / 2-1}\right\}$ and $W \backslash\left\{w^{*}\right\}$ are green. Then $T_{n} \nsubseteq[V]_{r}$ forces $u_{1} u_{n-1}$ to be green. Furthermore, by Claim 2.13, the edges $u_{n / 2-1} u_{n-1}$ and $u_{n / 2} u_{n-1}$ have to be red. Remind that a red edge $u_{n-1} w$ with $w \in W$ must occur. But then the red path $P_{n-1}=u_{1} \cdots u_{n / 2-1} u_{n-1} u_{n / 2} \cdots u_{n-2}$ together with the red edge $u_{n-1} w$ yields the forbidden red $T_{n}$, a contradiction, and we are done.

## 3. Trees $T_{n} \in\left\{P_{n}, B_{2, n-2}\right\}$ and the Graphs $G$ with $\chi(G)=2$

It seems to be out of reach to determine the exact value of $r\left(T_{n}, G\right)$ for every tree $T_{n}$ and all connected bipartite graphs $G$ of order six, i.e., the star $S_{6}=K_{1,5}$ and the connected spanning subgraphs of $K_{2,4}$ and $K_{3,3}$. Burr, Erdős, Faudree, Rousseau and Schelp [1] derived upper bounds for $r\left(T_{n}, K_{2,4}\right)$ and $r\left(T_{n}, K_{3,3}\right)$. They proved that for all sufficiently large $n$,

$$
r\left(T_{n}, K_{2,4}\right)<n+3 n^{1 / 2}
$$

Moreover they showed that there exists a constant $c$ such that for every tree $T_{n}$ with maximum degree $\Delta\left(T_{n}\right)=m$,

$$
r\left(T_{n}, K_{3,3}\right) \leq \max \left\{n+\left\lceil c n^{1 / 3}\right\rceil, r\left(S_{m+1}, K_{3,3}\right)\right\}
$$

and

$$
r\left(S_{m+1}, K_{3,3}\right)<m+3 m^{2 / 3}
$$

for $m$ sufficiently large. Lower bounds can be obtained from $r\left(T_{n}, C_{4}\right)$ since $C_{4} \subseteq K_{2,4}$ and $C_{4} \subseteq K_{3,3}$. In [1] it was proved that for all sufficiently large $n$,

$$
r\left(S_{m+1}, C_{4}\right)>m+\left\lfloor m^{1 / 2}-6 m^{11 / 40}\right\rfloor .
$$

This together with Theorem 2.12 implies that $r\left(T_{n}, K_{2,4}\right)$ and $r\left(T_{n}, K_{3,3}\right)$ differ considerably from the lower bound (1) if $n$ is sufficiently large and $\Delta\left(T_{n}\right)=m$ is close to $n-1$. Clearly, the same holds for $r\left(T_{n}, G\right)$ if $G$ is any bipartite graph with $C_{4} \subseteq G$. Here we restrict ourselves to study $r\left(T_{n}, G\right)$ for two trees with small maximum degree, namely $T_{n} \in\left\{P_{n}, B_{2, n-2}\right\}$. The choice of these two trees is essentielly due to our project to determine $r\left(T_{n}, G\right)$ for every connected graph of order six and all trees of order at most five - the only non-star trees on at most five vertices are the paths $P_{4}$ and $P_{5}$ and the broom $B_{2,3}$. Our results show that, except for some small $n$, the trees $T_{n} \in\left\{P_{n}, B_{2, n-2}\right\}$ are $G$-good for any connected bipartite graph $G$ of order $p(G)=6$, i.e., $r\left(T_{n}, G\right)$ attains the general lower bound from (1). Instead of $r\left(T_{n}, S_{6}\right)$ here we consider the more general case $r\left(T_{n}, S_{m}\right)$. We start by improving the lower bound (1) for $T_{n} \in\left\{P_{n}, B_{2, n-2}\right\}$ and any connected bipartite graph $G$ in case of small $n$.

Lemma 3.1. Let $G \subseteq K_{m_{1}, m_{2}}$ be a connected graph of order $m=m_{1}+m_{2}$ where $1 \leq m_{1} \leq m_{2}$. Then $r\left(P_{n}, G\right) \geq m-1+\lfloor n / 2\rfloor$ for $n \geq 2$ and $r\left(B_{2, n-2}, G\right) \geq$ $m-1+\lfloor(n-1) / 2\rfloor$ for $n \geq 5$.

Proof. From (1) it follows that $r\left(G, T_{n}\right) \geq m-1+s\left(T_{n}\right)$. Due to $r(F, G)=$ $r(G, F), s\left(P_{n}\right)=\lfloor n / 2\rfloor$ for $n \geq 2$ and $s\left(B_{2, n-2}\right)=\lfloor(n-1) / 2\rfloor$ for $n \geq 5$ we obtain the desired results.

If $G$ is a connected spanning subgraph of $K_{m_{1}, m_{2}}$ with $1 \leq m_{1} \leq m_{2}$, then $s(G)=m_{1}$, and the general lower bound (1) implies $r\left(T_{n}, G\right) \geq n+m_{1}-1$ for any tree $T_{n}$. Hence the general lower bound is improved by the lower bounds from Lemma 3.1 for $T_{n}=P_{n}$ if $n \leq 2 m_{2}-2$ and for $T_{n}=B_{2, n-2}$ if $n \leq 2 m_{2}-3$. The following lemma shows that in case of $T_{n}=B_{2, n-2}$ the general lower bound can also be improved for $n=2 m_{2}-2$ or $n=2 m_{2}$ and certain graphs $G \subseteq K_{m_{1}, m_{2}}$.

Lemma 3.2. Let $n \geq 6$ be even and let $m_{1} \leq m_{2}$. Then $r\left(B_{2, n-2}, G\right) \geq n+m_{1}$ if $m_{1} \geq 1, n=2 m_{2}$ and $G=K_{m_{1}, m_{2}}$ or if $m_{1} \geq 2, n=2 m_{2}-2$ and $G \in$ $\left\{K_{m_{1}, m_{2}}-e, K_{m_{1}, m_{2}}-2 K_{2}\right\}$. Moreover, $r\left(B_{2,3}, K_{m_{1}, m_{2}}\right) \geq m_{1}+m_{2}+2$.
Proof. For $n=2 m_{2}$, the coloring of $K_{n+m_{1}-1}$ with $[V]_{g}=2 K_{m_{2}}+\overline{K_{m_{1}-1}}$ contains no red $B_{2, n-2}$ and no green $K_{m_{1}, m_{2}}$. For $n=2 m_{2}-2$, the coloring of $K_{n+m_{1}-1}$ with $[V]_{g}=2 K_{m_{2}-1}+\overline{K_{m_{1}-1}}$ contains no red $B_{2, n-2}$ and no green $K_{m_{1}, m_{2}}-2 K_{2}$. Moreover, the coloring of $K_{m_{1}+m_{2}+1}$ with $[V]_{r}=C_{m_{1}+m_{2}+1}$ contains no red $B_{2,3}$ and no green $K_{m_{1}, m_{2}}$.

Now we consider $r\left(T_{n}, S_{m}\right)$. Parsons [14] has already determined the exact value of $r\left(P_{n}, S_{m}\right)$ by explicit formulas and a recurrence.

Theorem 3.3 (Parsons [14]). Let $n \geq 4$ and $m \geq 4$. Then

$$
r\left(P_{n}, S_{m}\right)= \begin{cases}2 m-3 & \text { if } m-1 \leq n<2 m-3 \\ n & \text { if } n \geq 2 m-3\end{cases}
$$

and $r\left(P_{n}, S_{m}\right)=\max \left\{r\left(P_{n-1}, S_{m}\right), r\left(P_{n}, S_{m-(n-1)}\right)+n-1\right\}$ if $n<m-1$.
Remark. For $n \geq 4$ and $m=5$ only $r\left(P_{4}, S_{6}\right)$ is not explicitely given by Theorem 3.3. Applying the recurrence and Theorem 2.3 we derive $r\left(P_{4}, S_{6}\right)=7$.

We use the result of Parsons to completely determine the exact values of $r\left(B_{2, n-2}, S_{m}\right)$ if $n \geq m-1$.
Theorem 3.4. Let $n \geq 5$ and $m \geq 4$. Then $r\left(B_{2,3}, S_{4}\right)=6$ and

$$
r\left(B_{2, n-2}, S_{m}\right)= \begin{cases}2 m-3 & \text { if } m-1 \leq n \leq 2 m-3 \text { and } m \geq 5, \\ n+1 & \text { if } n=2 m-2, \\ n & \text { if } n \geq 2 m-1 .\end{cases}
$$

To prove Theorem 3.4 the straightforward statements of the following lemma will be used.

Lemma 3.5. Let $n \geq 5$ and let $\chi$ be a coloring of a complete graph with vertex set $V$ and $P_{n}=u_{1} \cdots u_{n} \subseteq[V]_{r}$, but $B_{2, n-2} \nsubseteq[V]_{r}$. Then $u_{1} u_{3}, u_{1} u_{n-1}, u_{2} u_{n}$ and $u_{n-2} u_{n}$ have to be green. Furthermore, if $n \geq 7$ and $u_{1} u_{i}$ is red for some $i$ with $5 \leq i \leq n-2$, then $u_{i-2} u_{n}$ has to be green.

Proof of Theorem 3.4. In [5] it was already shown that $r\left(B_{2,3}, S_{4}\right)=6$. From (1) we obtain $r\left(B_{2, n-2}, S_{m}\right) \geq n$. Lemma 3.2 yields $r\left(B_{2, n-2}, S_{m}\right) \geq n+1$ if $n=2 m-2$. Moreover, the coloring of $K_{2 m-2}$ with $[V]_{r}=2 K_{m-2}$ shows $r\left(B_{2, n-2}, S_{m}\right) \geq 2 m-3$ for $n \geq m-1$. Thus, to establish the results from Theorem 3.4 it suffices to prove that $r\left(B_{2, n-2}, S_{m}\right) \leq n$ for $n \geq 2 m-1$ with $m \geq 4$ and for $n=2 m-3$ with $m \geq 5$ by using the monotonicity property $r\left(B_{2, n-2}, S_{m}\right) \leq r\left(B_{2, n^{\prime}-2}, S_{m}\right)$ for $n<n^{\prime}$. To obtain the desired upper bounds suppose that we have a $\left(B_{2, n-2}, S_{m}\right)$-coloring of $K_{n}$ with vertex set $V$ where $n \geq 2 m-1$ with $m \geq 4$ or $n=2 m-3$ with $m \geq 5$. Then, by Theorem 3.3 , a red $P_{n}=u_{1} u_{2} \cdots u_{n}$ occurs. Note that $S_{m} \nsubseteq[V]_{g}$ forces $\Delta_{g} \leq m-2$. By Lemma $3.5, u_{1} u_{3}, u_{1} u_{n-1}, u_{2} u_{n}$ and $u_{n-2} u_{n}$ have to be green.

Case 1. $n \geq 2 m-1$ where $m \geq 4$. Since $d_{g}\left(u_{1}\right) \leq m-2, q_{g}\left(u_{1},\left\{u_{5}, \ldots\right.\right.$, $\left.\left.u_{n-2}\right\}\right) \leq m-4$. Thus, $q_{r}\left(u_{1},\left\{u_{5}, \ldots, u_{n-2}\right\}\right) \geq n-6-(m-4) \geq 2 m-1-m-2=$ $m-3$, and Lemma 3.5 implies $q_{g}\left(u_{n},\left\{u_{3}, \ldots, u_{n-4}\right\}\right) \geq m-3$. But this yields $d_{g}\left(u_{n}\right) \geq m-1$, a contradiction.

Case 2. $n=2 m-3$ where $m \geq 5$. We distinguish two subcases depending on the color of $u_{1} u_{n}$.

Case 2.1. $u_{1} u_{n}$ is green. Then $q_{g}\left(u_{1},\left\{u_{5}, \ldots, u_{n-2}\right\}\right) \leq m-5$ because $d_{g}\left(u_{1}\right) \leq m-2$. This forces $q_{r}\left(u_{1},\left\{u_{5}, \ldots, u_{n-2}\right\}\right) \geq n-6-(m-5)=$ $2 m-3-m-1=m-4$, and Lemma 3.5 yields $q_{g}\left(u_{n},\left\{u_{3}, \ldots, u_{n-4}\right\}\right) \geq m-4$. Again we obtain $d_{g}\left(u_{n}\right) \geq m-1$, a contradiction.

Case 2.2. $u_{1} u_{n}$ is red, i.e., $C_{n}=\left(u_{1} u_{2} \cdots u_{n}\right)$ is a red cycle. The remaining edges are the diagonals $u_{i} u_{i+\ell}$ of length $\ell$ with $\ell=2, \ldots, m-2$ and $i=1, \ldots, n$, where the indices should be read modulo $n$. To finish Case 2.2 we use the following properties of the diagonals of $C_{n}$.

Claim 3.6. If a diagonal $u_{i} u_{i+\ell}$ of length $\ell$ with $2 \leq \ell \leq m-2$ and $1 \leq i \leq n$ is red, then also $u_{i+1} u_{i+\ell+1}$ has to be red.

Proof. If $u_{i} u_{i+\ell}$ is red and $u_{i+1} u_{i+\ell+1}$ is green, then the end-vertices of the red $P_{n}=u_{i+\ell+1} u_{i+\ell+2} \cdots u_{n} u_{1} \cdots u_{i} u_{i+\ell} u_{i+\ell-1} \cdots u_{i+1}$ are joined green, a situation already considered in Case 2.1.

Claim 3.7. For $2 \leq \ell \leq m-2$, all diagonals of length $\ell$ must have the same color.

Proof. This is an immediate consequence of Claim 3.6.
Claim 3.8. If the diagonals of length $\ell$ with $2 \leq \ell \leq m-3$ are red, then the diagonals of length $\ell+1$ have to be green.

Proof. Assume that the diagonals of length $\ell+1$ are also red. Using the diagonal $u_{1} u_{\ell+1}$ of length $\ell$, the diagonal $u_{2} u_{\ell+3}$ of lenght $\ell+1$ and edges from the red $C_{n}$ we obtain the red $B_{2, n-2}$ with bristles $u_{1} u_{\ell+1}, u_{\ell+1} u_{\ell+2}$ and handle $u_{\ell+1} u_{\ell} \cdots u_{2} u_{\ell+3} \cdots u_{n}$, a contradiction.

Claim 3.9. The diagonals of length $\ell$ with $2 \leq \ell \leq 3$ and, for $m \geq 6$, also the diagonals of lenght $\ell=4$ have to be green.

Proof. Assume that for some $\ell$ with $2 \leq \ell \leq 4$ the diagonals of length $\ell$ are red. If $\ell=2$, then $u_{1} u_{3}$ together with edges of the red $C_{n}$ give the red $B_{2, n-2}$ with bristles $u_{1} u_{3}, u_{2} u_{3}$ and handle $u_{3} u_{4} \cdots u_{n}$, a contradiction. If $\ell=3$, then the red $B_{2, n-2}$ with bristles $u_{1} u_{2}, u_{1} u_{n}$ and handle $u_{1} u_{4} u_{3} u_{6} \cdots u_{n-4} u_{n-1} u_{n-2}$ would occur. If $\ell=4$ and $m \geq 6$, then the diagonals $u_{1} u_{5}$ and $u_{3} u_{7}$ together with edges from the red $C_{n}$ would yield the red $B_{2, n-2}$ with bristles $u_{2} u_{3}, u_{3} u_{4}$ and handle $u_{3} u_{7} u_{6} u_{5} u_{1} u_{n} u_{n-1} \cdots u_{8}$.

Now we finish Case 2.2 by deriving a contradiction to $\Delta_{g} \leq m-2$. Note that for $2 \leq \ell \leq m-2$ every $u_{i}$ is incident to two diagonals of lenght $\ell$. Thus, Claim 3.9 yields the desired contradiction for $5 \leq m \leq 7$. In the remaining case $m \geq 8$ we additionally have to consider the diagonals of length $\ell \geq 5$. There are $m-6$ different diagonal lengths $\ell$ with $5 \leq \ell \leq m-2$ and Claim 3.8 implies that at least $\lfloor(m-6) / 2\rfloor$ of them belong to green diagonals. Hence $d_{g}\left(u_{i}\right) \geq 6+2\lfloor(m-6) / 2\rfloor \geq m-1$, a contradiction, and we are done.

In the following two theorems $r\left(P_{n}, G\right)$ and $r\left(B_{2, n-2}, G\right)$ are determined for any connected spanning subgraph $G$ of $K_{2,4}$.

Theorem 3.10. Let $n \geq 4$ and let $G$ be a connected graph of order six where $G \subseteq K_{2,4}$. Then

$$
r\left(P_{n}, G\right)= \begin{cases}7 & \text { if } 4 \leq n \leq 5 \\ 8 & \text { if } n=6 \\ n+1 & \text { otherwise }\end{cases}
$$

Proof. From (1) we obtain $r\left(P_{n}, G\right) \geq n+1$. Moreover, Lemma 3.1 implies $r\left(P_{n}, G\right) \geq 7$ for $4 \leq n \leq 5$ and $r\left(P_{6}, G\right) \geq 8$. To establish equality it suffices to show $r\left(P_{5}, K_{2,4}\right) \leq 7$ and $r\left(P_{n}, K_{2,4}\right) \leq n+1$ for $n \geq 7$. Consider any coloring of $K_{7}$ not containing a red $P_{5}$ and any coloring of $K_{n+1}, n \geq 7$, not containing a red $P_{n}$. We have to prove that a green $K_{2,4}$ occurs. Let $P_{k}=u_{1} \cdots u_{k}$ be a red path of maximum length, $U=\left\{u_{1}, \ldots, u_{k}\right\}$ and $W=V \backslash U$ where $V$ denotes the vertex sets of the complete graphs. If $k=1$, then only green edges occur and we find a green $K_{2,4}$. Now let $k \geq 2$. The maximality of $k$ forces that $u_{1}$ and $u_{k}$ are joined green to all vertices in $W$. This yields a green $K_{2,4}$ if $|W| \geq 4$. It remains $|W|=3$ in case of $K_{7}$ and $2 \leq|W| \leq 3$ in case of $K_{n+1}, n \geq 7$.

Case 1. $|W|=3$. Then $k=n-1=4$ in case of $K_{7}$ and $k=n-2 \geq 5$ in case of $K_{n+1}, n \geq 7$. Let $W=\left\{w_{1}, w_{2}, w_{3}\right\}$. Only green edges between $W$ and $\left\{u_{2}, u_{k-1}\right\}$ imply a green $K_{2,4}$. Otherwise we may assume that $u_{2} w_{1}$ is red. Since $P_{k+1} \nsubseteq[V]_{r}, w_{1}$ has to be joined green to $w_{2}, w_{3}$ and $u_{3}$. Furthermore, $u_{1} u_{3}$ and $u_{1} u_{k}$ have to be green, and we obtain the green $K_{2,4}=\left\{u_{1}, w_{1}\right\}+\left\{w_{2}, w_{3}, u_{3}, u_{k}\right\}$.

Case 2. $|W|=2$ in case of $K_{n+1}, n \geq 7$. This implies $k=n-1$. Let $W=\left\{w_{1}, w_{2}\right\}$. If $K_{2,4} \nsubseteq[V]_{g}$, then at most one vertex from $\left\{u_{2}, \ldots, u_{n-2}\right\}$ is joined green to $w_{1}$ and to $w_{2}$. Therefore we may assume that every vertex in $\left\{u_{2}, \ldots, u_{\lfloor(n-1) / 2\rfloor}\right\}$ is joined red to $w_{1}$ or to $w_{2}$. Note that $\lfloor(n-1) / 2\rfloor \geq 3$. Since $P_{k+1} \nsubseteq[V]_{r}$, a common red neighbor of $u_{2}$ and $u_{3}$ in $W$ is forbidden. Thus, we may assume that $u_{2} w_{1}$ and $u_{3} w_{2}$ are red. Then $P_{k+1} \nsubseteq[V]_{r}$ forces $w_{1} w_{2}, w_{1} u_{3}, w_{1} u_{4}, u_{1} u_{3}, u_{1} u_{4}$ and $u_{1} u_{n-1}$ to be green, and this yields the green $K_{2,4}=\left\{u_{1}, w_{1}\right\}+\left\{w_{2}, u_{3}, u_{4}, u_{n-1}\right\}$.

Theorem 3.11. Let $n \geq 5$ and let $G$ be a connected graph of order six where $G \subseteq K_{2,4}$. Then, if $G \neq K_{2,4}$,

$$
r\left(B_{2, n-2}, G\right)= \begin{cases}7 & \text { if } n=5 \\ 8 & \text { if } n=6 \text { and } K_{2,4}-2 K_{2} \subseteq G \\ n+1 & \text { otherwise }\end{cases}
$$

and

$$
r\left(B_{2, n-2}, K_{2,4}\right)= \begin{cases}8 & \text { if } n \leq 7 \\ 10 & \text { if } n=8 \\ n+1 & \text { otherwise }\end{cases}
$$

Proof. From (1) we obtain $r\left(B_{2, n-2}, G\right) \geq n+1$, and Lemma 3.1 yields $r\left(B_{2,3}, G\right)$ $\geq 7$. Lemma 3.2 gives $r\left(B_{2,3}, K_{2,4}\right) \geq 8, r\left(B_{2,4}, K_{2,4}-2 K_{2}\right) \geq 8$ and $r\left(B_{2,6}, K_{2,4}\right)$ $\geq 10$. To establish equality it suffices to show that $r\left(B_{2,3}, K_{2,4}-e\right) \leq 7$, $r\left(B_{2,4}, G^{*}\right) \leq 7$ for $G^{*}$ obtained from $K_{2,4}$ by deleting two edges incident to the same vertex of degree 4 and $r\left(B_{2, n-2}, G\right) \leq n+1$ for $G=K_{2,4}$ if $n=7$ or $n \geq 9$ and for $G=K_{2,4}-e$ if $n=8$.

To verify that $r\left(B_{2,3}, K_{2,4}-e\right) \leq 7$ and $r\left(B_{2,4}, G^{*}\right) \leq 7$ consider any coloring of $K_{7}$ with vertex set $V$. If a green $K_{2,4}$ occurs, then we are done. Otherwise, by Theorem 3.10, a red $P_{5}=u_{1} \cdots u_{5}$ must occur. Let $U=\left\{u_{1}, \ldots, u_{5}\right\}$ and let $W=V \backslash U=\left\{w_{1}, w_{2}\right\}$. Assume first that $B_{2,3} \nsubseteq[V]_{r}$. Then all edges between $W$ and $\left\{u_{2}, u_{3}, u_{4}\right\}$ have to be green. Moreover, at least one edge from $u_{1}$ to $W$ must be green yielding a green $K_{2,4}-e$. Suppose now that $B_{2,4} \nsubseteq[V]_{r}$. Then all edges between $W$ and $\left\{u_{2}, u_{4}\right\}$ have to be green. If $w_{1}$ or $w_{2}$ is joined green to both $u_{1}$ and $u_{5}$, then a green $G^{*}$ occurs. Neither $u_{1}$ nor $u_{5}$ can be joined red to $w_{1}$ and to $w_{2}$ since $B_{2,4} \nsubseteq[V]_{r}$. Thus we may assume that $u_{1} w_{1}$ and $u_{5} w_{2}$ are
green and that $u_{1} w_{2}$ and $u_{5} w_{1}$ are red. But then $B_{2,4} \nsubseteq[V]_{r}$ forces $u_{3} w_{1}$ to be green, and we obtain a green $G^{*}$.

To prove that $r\left(B_{2, n-2}, G\right) \leq n+1$ for $G=K_{2,4}$ if $n=7$ or $n \geq 9$ and for $G=K_{2,4}-e$ if $n=8$ consider any coloring of $K_{n+1}, n \geq 7$, not containing a red $B_{2, n-2}$. Let $V=V\left(K_{n+1}\right)$. We have to show that a green $K_{2,4}-e$ occurs in case of $n=8$ and a green $K_{2,4}$ otherwise.

Case 1. There is a red cycle $C_{k}=\left(u_{1} \cdots u_{k}\right)$ of length $k=n$ or $k=n+1$. Let $U=\left\{u_{1}, \ldots, u_{k}\right\}$. We consider two subcases depending on $k$.

Case 1.1. $k=n$. Then $B_{2, n-2} \nsubseteq[V]_{r}$ implies that all edges between $U$ and the vertex $w \in V \backslash U$ are green. By Theorem 3.4, $r\left(B_{2, n-2}, S_{5}\right)=n$ if $n=7$ or $n \geq 9$, and $r\left(B_{2, n-2}, S_{4}\right)=n$ if $n=8$. This yields a green $S_{5}$ in [U] for $n=7$ and for $n \geq 9$ and a green $S_{4}$ in $[U]$ for $n=8$. Together with the green edges incident to $w$ we obtain a green $K_{2,4}$ and a green $K_{2,4}-e$, respectively.

Case 1.2. $k=n+1$. Then $B_{2, n-2} \nsubseteq[V]_{r}$ forces all diagonals of length $\ell \leq 3$ to be green. If, in addition, all diagonals of length $\ell=4$ are green, then $\left\{u_{1}, u_{2}\right\}+\left\{u_{4}, u_{5}, u_{n-1}, u_{n}\right\}$ is a green $K_{2,4}$. The remaining case is that at least one diagonal of length 4 , say $u_{1} u_{5}$, is red. Any red diagonal of length $\ell \geq 4$ incident to $u_{3}$ yields a red $B_{2, n-2}$ with bristles $u_{2} u_{3}$ and $u_{3} u_{4}$, a contradiction. Otherwise all diagonals of length $\ell \geq 2$ incident to $u_{3}$ are green. Thus, $u_{1}, u_{6}$ and $u_{7}$ are common green neighbors of $u_{3}$ and $u_{4}$. If $u_{4} u_{n+1}$ is also green, then $\left\{u_{3}, u_{4}\right\}+\left\{u_{1}, u_{6}, u_{7}, u_{n+1}\right\}$ is a green $K_{2,4}$. On the other hand, if $u_{4} u_{n+1}$ is red, then all diagonals incident to $u_{2}$ have to be green since $B_{2, n-2} \nsubseteq[V]_{r}$. But then $u_{2}$ and $u_{3}$ have at least four common green neighbors and again a green $K_{2,4}$ occurs.

Case 2. Every red cycle has length at most $n-1$. If $K_{2,4} \subseteq[V]_{g}$, then we are done. Otherwise, by Theorem 3.10, a red $P_{n}=u_{1} \cdots u_{n}$ occurs. Let $U=\left\{u_{1}, \ldots, u_{n}\right\}$ and let $w$ be the vertex in $V \backslash U$. Since $B_{2, n-2} \nsubseteq[V]_{r}$, the edges $w u_{2}, w u_{3}, w u_{n-2}$ and $w u_{n-1}$ have to be green. By Lemma 3.5, the edges $u_{1} u_{3}, u_{1} u_{n-1}, u_{2} u_{n}$ and $u_{n-2} u_{n}$ are green. Moreover, $C_{n} \nsubseteq[V]_{r}$ forces $u_{1} u_{n}$ to be green, and $C_{n+1} \nsubseteq[V]_{r}$ implies that at least one of the edges $w u_{1}$ and $w u_{n}$, say $w u_{n}$, is green. To avoid a green $K_{2,4}=\left\{u_{1}, w\right\}+\left\{u_{2}, u_{n-2}, u_{n-1}, u_{n}\right\}, u_{1} u_{n-2}$ has to be red. Then, by Lemma 3.5, $u_{n-4} u_{n}$ must be green. Furthermore, $C_{n} \nsubseteq[V]_{r}$ implies that $u_{n-3} u_{n-1}$ is green, and $B_{2, n-2} \nsubseteq[V]_{r}$ forces $u_{n-3} u_{n}$ to be green. If $n=7$, then $w u_{1}, w u_{4}$ and $u_{3} u_{6}$ have to be red as otherwise $\left\{w, u_{7}\right\}+\left\{u_{1}, u_{2}, u_{3}, u_{4}, u_{5}\right\}$ contains a green $K_{2,4}$ or $\left\{u_{6}, u_{7}\right\}+\left\{u_{1}, u_{3}, u_{4}, w\right\}$ is a green $K_{2,4}$. But this yields a red $C_{n}$, a contradiction. If $n \geq 8$, then $w u_{n-4}$ has to be green if no red $B_{2, n-2}$ with bristles $u_{n-4} u_{n-3}$ and $u_{n-4} w$ shall occur. Hence $\left\{w, u_{n}\right\}+\left\{u_{1}, u_{2}, u_{3}, u_{n-4}, u_{n-2}\right\}$ contains a green $K_{2,4}$ or $u_{3} u_{n}$ and $w u_{1}$ are red. But then we obtain a red $B_{2, n-2}$ with bristles $u_{1} u_{2}$ and $u_{1} w$, a contradiction.

Finally we determine $r\left(P_{n}, G\right)$ and $r\left(B_{2, n-2}, G\right)$ for all connected spanning subgraphs $G$ of $K_{3,3}$.

Theorem 3.12. Let $n \geq 4$ and let $G$ be a connected graph of order six where $G \subseteq K_{3,3}$. Then

$$
r\left(P_{n}, G\right)= \begin{cases}7 & \text { if } n=4 \\ n+2 & \text { otherwise }\end{cases}
$$

Proof. By (1), $r\left(P_{n}, G\right) \geq n+2$. Moreover, Lemma 3.1 yields $r\left(P_{4}, G\right) \geq 7$. To establish equality it suffices to prove that $r\left(P_{n}, K_{3,3}\right) \leq n+2$ for $n \geq 5$. Consider any coloring of $K_{n+2}, n \geq 5$, not containing a red $P_{n}$. We have to show that a green $K_{3,3}$ occurs. Let $P_{k}=u_{1} \cdots u_{k}$ be a red path of maximum length, $U=\left\{u_{1}, \ldots, u_{k}\right\}$ and $W=V \backslash U=\left\{w_{1}, w_{2}, \ldots, w_{n+2-k}\right\}$. In case of $k \leq 2$ either at most one red edge occurs or any two red edges are independent. This yields a green $K_{6}-2 K_{2}$ containing a green $K_{3,3}$. Now let $k \geq 3$. All edges between $\left\{u_{1}, u_{k}\right\}$ and $W$ have to be green. Since $k \leq n-1,|W|=n+2-k \geq 3$. If $q_{g}\left(u_{i}, W\right) \geq 3$ for some $i$ with $2 \leq i \leq k-1$, then a green $K_{3,3}$ occurs. Otherwise $q_{r}\left(u_{i}, W\right) \geq 1$ for every $i$ with $2 \leq i \leq k-1$, and we may assume that $u_{2} w_{1}$ is red. Since $P_{k+1} \nsubseteq[V]_{r}$, all edges incident to $w_{1}$ in $[W]$ have to be green. This produces a green $K_{3,3}$ if $|W| \geq 4$. The remaining case is $|W|=3$ which implies $k=n-1 \geq 4$. Again we apply $P_{k+1} \nsubseteq[V]_{r}$. Thus, $u_{1} u_{k}, u_{1} u_{3}$ and $u_{3} w_{1}$ must be green. Moreover we may assume that $u_{3} w_{2}$ is red, and this forces $u_{2} w_{2}$ and $u_{2} u_{k}$ to be green. If $u_{2} w_{3}$ is green, then we obtain the green $K_{3,3}=\left\{u_{1}, u_{2}, w_{1}\right\}+\left\{u_{k}, w_{2}, w_{3}\right\}$. If $u_{2} w_{3}$ is red, then $w_{2} w_{3}$ and $u_{3} w_{3}$ have to be green yielding the green $K_{3,3}=\left\{u_{1}, w_{1}, w_{3}\right\}+\left\{u_{3}, u_{k}, w_{2}\right\}$, and the proof is complete.

Theorem 3.13. Let $n \geq 5$ and let $G$ be a connected graph of order six where $G \subseteq K_{3,3}$. Then

$$
r\left(B_{2, n-2}, G\right)= \begin{cases}n+3 & \text { if } G=K_{3,3} \text { and } 5 \leq n \leq 6 \\ n+2 & \text { otherwise } .\end{cases}
$$

Proof. By (1), $r\left(B_{2, n-2}, G\right) \geq n+2$. Moreover, Lemma 3.2 yields $r\left(B_{2,3}, K_{3,3}\right) \geq$ 8 and $r\left(B_{2,4}, K_{3,3}\right) \geq 9$. To establish equality we prove $r\left(B_{2, n-2}, K_{3,3}\right) \leq n+3$ as well as $r\left(B_{2, n-2}, K_{3,3}-e\right) \leq n+2$ for $n \geq 5$ and $r\left(B_{2, n-2}, K_{3,3}\right) \leq n+2$ for $n \geq 7$. Consider any coloring of $K_{m}$ with $n+2 \leq m \leq n+3$ and $n \geq 5$ not containing a red $B_{2, n-2}$. Let $V=V\left(K_{m}\right)$. If a green $K_{3,3}$ occurs, then we are done. Otherwise Theorem 3.12 guarantees a red $P_{n}=u_{1} \cdots u_{n}$. Let $U=\left\{u_{1}, \ldots, u_{n}\right\}$. $B_{2, n-2} \nsubseteq$ $[V]_{r}$ forces only green edges between $\left\{u_{2}, u_{3}, u_{n-2}, u_{n-1}\right\}$ and $W=V \backslash U$. Hence $K_{3,3} \subseteq[V]_{g}$ in case of $m=n+3$, i.e., $|W|=3$, a contradiction. It remains $m=n+2$. Let $W=\left\{w_{1}, w_{2}\right\}$. By Lemma 3.5, $u_{1} u_{3}, u_{1} u_{n-1}, u_{2} u_{n}$ and $u_{n-2} u_{n}$
have to be green. Thus we find a green $K_{3,3}-e$ in $\left\{u_{1}, w_{1}, w_{2}\right\}+\left\{u_{2}, u_{3}, u_{n-1}\right\}$ proving that $r\left(B_{2, n-2}, K_{3,3}-e\right) \leq n+2$ if $n \geq 5$. Now let $n \geq 7$. To avoid that $\left\{u_{1}, w_{1}, w_{2}\right\}+\left\{u_{3}, u_{n-2}, u_{n-1}\right\}$ or $\left\{w_{1}, w_{2}, u_{n}\right\}+\left\{u_{2}, u_{3}, u_{n-2}\right\}$ is a green $K_{3,3}, u_{1} u_{n-2}$ and $u_{3} u_{n}$ have to be red, and then $B_{2, n-2} \nsubseteq[V]_{r}$ implies that $u_{1} w_{1}$ and $u_{1} w_{2}$ are green. This forces $u_{1} u_{n}$ to be red as otherwise $\left\{u_{1}, u_{2}, u_{n-2}\right\}+$ $\left\{w_{1}, w_{2}, u_{n}\right\}$ is a green $K_{3,3}$. Consequently, since $B_{2, n-2} \nsubseteq[V]_{r}$, all edges between $U$ and $W$ have to be green. By Theorem 3.4, $r\left(B_{2, n-2}, S_{4}\right)=n$ for $n \geq 7$. But this implies a green $S_{4}$ in $[U]$ yielding a green $K_{3,3}$ together with $w_{1}$ and $w_{2}$, a contradiction, and we are done.

## 4. Concluding Remarks

Summarizing Theorems 2.1, 2.9, the results from [11] concerning non-bipartite graphs $G$ and the results from $[13]$ for $r\left(S_{n}, K_{1,1,4}\right)$, we see that $r\left(T_{n}, G\right)$ has been determined for any tree $T_{n}$ and all connected graphs $G \neq K_{2,2,2}$ of order six with $\chi(G) \geq 3$, except for $T_{n}=S_{n}$ in case of some small $n$ and some $G$ where $\chi(G)=3$. The exact values of $r\left(S_{n}, G\right)$ are still missing in the following cases (the numbering of $G$ corresponds to the numbering of the 112 connected graphs of order six used in [11]): $G=G_{100}=K_{1,2,3}$ with $n \in\{7,9,11\}, G=G_{94}=E_{2}+\left(E_{1} \cup P_{3}\right)$ with $n=7, G=G_{92}=K_{3,3}+e$ with $6 \leq n \leq 12, G=G_{78}=E_{2}+\left(E_{2} \cup K_{2}\right)$ with $6 \leq n \leq 8, G=G_{60}$ and $G=G_{79}$ (the two graphs obtained from $K_{1,1,3}$ by joining an additional vertex to one or two of the three vertices of degree 2) with $n=6$. In all these cases we know that the value of $r\left(S_{n}, G\right)$ differs by at most 2 from the lower bound given in (1). By a detailed case analysis, perhaps assisted by computer algorithms, it should be possible to determine the missing exact values.

To achieve significant progress in evaluating $r\left(T_{n}, K_{2,2,2}\right)$ seems to be difficult, especially for trees $T_{n}$ with maximum degree $\Delta\left(T_{n}\right)$ close to $n-1$, where we know that, for $n$ sufficiently large, $r\left(T_{n}, K_{2,2,2}\right)$ differs considerably from the lower bound $2 n$ obtained from (1). In contrast, for some trees with small maximum degree as $P_{n}$ and a special class of trees with $\Delta\left(T_{n}\right)=3, r\left(T_{n}, K_{2,2,2}\right)$ attains the bound $2 n$ (see Theorems 2.10 and 2.11). It seems to be promising to study $r\left(T_{n}, K_{2,2,2}\right)$ for further trees with small maximum degree, in particular it would be desirable to obtain a characterization of all $K_{2,2,2}$ good trees $T_{n}$.

As already explained, it seems to be extremely difficult to evaluate $r\left(T_{n}, G\right)$ for trees $T_{n}$ with maximum degree $\Delta\left(T_{n}\right)$ close to $n-1$ and all connected bipartite graphs $G$ of order six, i.e., all connected spanning subgraphs of $K_{m_{1}, m_{2}}$ with $1 \leq m_{1} \leq m_{2}$ and $m_{1}+m_{2}=6$. If $\Delta\left(T_{n}\right)$ is small, then the situation is entirely different. For $T_{n} \in\left\{P_{n}, B_{2, n-2}\right\}$ we have shown that, except for small $n, T_{n}$ is $G$-good for any connected bipartite graph $G$ of order six, and there might be
other trees $T_{n}$ with small maximum degree where the general lower bound (1) is attained. Especially, by Theorems 3.3, 3.10 and 3.12, $P_{n}$ is $G$-good if and only if $n \geq 2 m_{2}-1$. This improves in a very special case a general result due to Pokrovskiy and Sudakov [16] who recently have shown that $P_{n}$ is $G$-good for any graph $G$ on $p(G)$ vertices if $n \geq 4 p(G)$.

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