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ON THE RAMSEY NUMBERS OF NON-STAR TREES VERSUS CONNECTED GRAPHS OF ORDER SIX

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Abstract

This paper completes our studies on the Ramsey number $r(T_n, G)$ for trees T_n of order n and connected graphs G of order six. If $\chi(G) \geq 4$, then the values of $r(T_n, G)$ are already known for any tree T_n. Moreover, $r(S_n, G)$, where S_n denotes the star of order n, has been investigated in case of $\chi(G) \leq 3$. If $\chi(G) = 3$ and $G \neq K_{2,2,2}$, then $r(S_n, G)$ has been determined except for some G and some small n. Partial results have been obtained for $r(S_n, K_{2,2,2})$ and for $r(S_n, G)$ with $\chi(G) = 2$. In the present paper we investigate $r(T_n, G)$ for non-star trees T_n and $\chi(G) \leq 3$. Especially, $r(T_n, G)$ is completely evaluated for any non-star tree T_n if $\chi(G) = 3$ where $G \neq K_{2,2,2}$, and $r(T_n, K_{2,2,2})$ is determined for a class of trees T_n with small maximum degree. In case of $\chi(G) = 2$, $r(T_n, G)$ is investigated for $T_n = P_n$, the path of order n, and for $T_n = B_{2,n-2}$, the special broom of order n obtained by identifying the centre of a star S_3 with an end-vertex of a path P_{n-2}. Furthermore, the values of $r(B_{2,n-2}, S_m)$ are determined for all n and m with $n \geq m - 1$. As a consequence of this paper, $r(F, G)$ is known for all trees F of order at most five and all connected graphs G of order at most six.

Keywords: Ramsey number, Ramsey goodness, tree, star, path, broom, small graph.

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1. Introduction

Ramsey number and Ramsey goodness. For graphs F and G the Ramsey number $r(F,G)$ is the smallest integer p such that in every 2-coloring of the edges of K_p there is a copy of F in the first color or a copy of G in the second color. The chromatic surplus $s(G)$ is defined to be the smallest number of vertices in a color class under any $\chi(G)$-coloring of the vertices of G, where $\chi(G)$ denotes the chromatic number of G. It is well-known (see [6] or [7]) that for any connected graph F with n vertices and any graph G with $s(G) \leq n$ the Ramsey number $r(F,G)$ satisfies

$$r(F,G) \geq (n-1)(\chi(G)-1) + s(G). \tag{1}$$

If equality occurs, then F is said to be G-good. Chvátal [3] has proved that every tree T_n of order n is K_m-good, i.e., $r(T_n, K_m) = (n-1)(m-1)+1$. Moreover, several classes of non-complete graphs G are known where every tree T_n is G-good, but there are also graphs G and trees T_n such that $r(T_n, G)$ differs even considerably from the lower bound given in (1) — a survey on results for $r(T_n, G)$ can be found in [17].

Our contribution. Faudree, Rousseau and Schelp [7] initiated the systematic study of $r(T_n, G)$ for graphs G of small order $p(G)$ and investigated the case $p(G) \leq 5$. In [11] and [12] we started to extend these investigations to graphs G with $p(G) = 6$. Using the result on $r(T_n, K_m)$ due to Chvátal and results on $r(T_n, G)$ for nearly complete graphs G due to Chartrand, Gould and Polimeni [2] and Gould and Jacobson [8] it was not difficult to derive that any tree T_n with $n \geq 5$ is G-good for all graphs G with $p(G) = 6$ and $\chi(G) \geq 4$. In [11] our main focus was on $r(S_n, G)$ where S_n denotes the star of order n and G is a connected graph of order six with $G \neq K_{2,2,2}$ and $\chi(G) \leq 3$, in [12] we studied $r(S_n, K_{2,2,2})$. Especially we proved that in case of $\chi(G) = 3$ and $G \neq K_{2,2,2}$ the star S_n is G-good or, in a few cases, $r(S_n, G)$ differs by 1 or 2 from the lower bound (1). In contrast, for n sufficiently large, $r(S_n, K_{2,2,2}) > 2n-2+\lfloor\sqrt{n-1} - 6(n-1)^{11/40}\rfloor$, i.e., $r(S_n, K_{2,2,2})$ differs considerably from the lower bound $2n$ given in (1).

In this paper we study $r(T_n, G)$ for non-star trees T_n and connected graphs G with $p(G) = 6$ and $\chi(G) \leq 3$. We prove that every non-star tree T_n is G-good for every connected graph $G \notin \{K_{1,1,4}, K_{2,2,2}\}$ with $p(G) = 6$ and $\chi(G) = 3$. A more general result on $r(T_n, K_{1,1,m})$ due to Erdős, Faudree, Rousseau and Schelp [6] and our results from [13] show that, except for $n \leq 5$, every non-star tree T_n is also $K_{1,1,4}$-good. The case $G = K_{2,2,2}$ remains to a great extent unsolved. We present several $K_{2,2,2}$-good non-star trees T_n with small maximum degree, but the behavior of $r(S_n, K_{2,2,2})$ implies that non-star trees T_n with sufficiently large n and maximum degree close to $n-1$ cannot be $K_{2,2,2}$-good.

To determine $r(T_n, G)$ for every tree T_n and all connected graphs G of order six with $\chi(G) = 2$, i.e., the star S_6 and the connected spanning subgraphs of $K_{2,4}$

and $K_{3,3}$, seems to be a hard problem. Partial results on $r(S_n, G)$ were obtained in [11]. In this paper we investigate $r(T_n, G)$ for two non-star trees T_n, namely $T_n = P_n$, the path on n vertices, and $T_n = B_{2,n-2}$, a special case of a broom $B_{k,n-k}$ defined as a tree of order $n \geq 5$ obtained by identifying the centre of a star S_{k+1}, $k \geq 2$, with an end-vertex of a path P_{n-k}. The choice of these two non-star trees is due to the project to evaluate $r(F, G)$ for graphs F of order at most five and graphs G of order six — the only non-star trees on at most five vertices are the paths P_n with $4 \leq n \leq 5$ and the broom $B_{2,3}$. Instead of $r(T_n, S_6)$ we consider the more general case $r(T_n, S_m)$. Parsons [14] has already determined $r(P_n, S_m)$ for all n and m by explicit formulas and a recurrence, and we evaluate $r(B_{2,n-2}, S_m)$ for all n and m with $n \geq m - 1$. The results in this paper together with the results in [11] and [12] imply that $r(F, G)$ is known for all trees T_n of order at most five and all connected graphs G of order six.

Notation and terminology. Some specialized notation and terminology will be used. The vertex set of a graph G is denoted by $V(G)$. We write $G' \subseteq G$ if G' is a subgraph of G and, for $U \subseteq V(K_n)$, $[U]$ is the subgraph induced by U. A coloring of a graph here always means a 2-coloring of its edges with colors red and green. An (F_1, F_2)-coloring is a coloring containing neither a red copy of F_1 nor a green copy of F_2. Given a coloring of K_n, we define the r-degree $d_r(v)$ to be the number of red edges incident to $v \in V(K_n)$. Moreover, $\Delta_r = \max_{v \in V(K_n)} d_r(v)$. The set of vertices joined red to v is denoted by $N_r(v)$. Similarly we define $d_g(v)$, Δ_g and $N_g(v)$. Furthermore, $[U]_r$ and $[U]_g$ are the red and the green subgraphs induced by U. For disjoint subsets $U_1, U_2 \subseteq V(K_n)$, $q_r(U_1, U_2)$ denotes the number of red edges between U_1 and U_2, and $q_g(U_1, U_2)$ is defined similarly. The vertex of degree $n - 1$ in a star S_n with $n \geq 3$ is called the centre of the star. We write $P_k = u_1 u_2 \cdots u_k$ for the path P_k with vertices $u_1, \ldots, u_k$ and edges $u_i u_{i+1}$ for $i = 1, \ldots, k - 1$. Moreover, $(u_1 u_2 \cdots u_k)$ means the cycle C_k obtained from $P_k = u_1 u_2 \cdots u_k$ by adding the edge $u_1 u_k$, and an edge $u_i u_j$ is called a diagonal of length ℓ of C_k if u_i and u_j are vertices with distance ℓ on C_k. The bristles of a broom $B_{k,n-k}$ are the k edges joining the vertex v^* of degree $k + 1$ to a vertex of degree 1 and the path P_{n-k} with end-vertex v^* is said to be the handle of the broom. The complement $\overline{K_n}$ of K_n is denoted by E_n, and for the complete k-partite graph $K_{n_1, n_2, \ldots, n_k} = E_{n_1} + E_{n_2} + \cdots + E_{n_k}$ with $V(E_{n_i}) = U_i$ we write $U_1 + U_2 + \cdots + U_k$.

2. Non-Star Trees T_n and the Graphs G with $\chi(G) = 3$

First we consider the graphs G of order six with chromatic number $\chi(G) = 3$ and $G \notin \{K_{1,1,4}, K_{2,2,2}\}$. The following theorem states that for all these graphs G every non-star tree T_n is G-good.

Theorem 2.1. *Let $n \geq 4$, $T_n \neq S_n$, and let G be a graph of order six with $\chi(G) = 3$ where $G \neq K_{1,1,4}$ and $G \neq K_{2,2,2}$. Then*

$$r(T_n, G) = \begin{cases} 2n-1 & \text{if } G \subseteq K_{1,2,3}, \\ 2n & \text{otherwise.} \end{cases}$$

To prove Theorem 2.1 by induction on n the following properties of trees T_n are essential.

Lemma 2.2. (i) *If $n \geq 6$ and $T_n \notin \{S_n, B_{n-3,3}\}$, then T_n contains vertices v_1 and v_2 of degree 1 with distance $d(v_1, v_2) \geq 3$ such that $T_n - \{v_1, v_2\}$ is a non-star tree of order $n-2$.*

(ii) *If $n \geq 5$ and $T_n \neq S_n$, then T_n contains a vertex v of degree 1 such that $T_n - \{v\}$ is a non-star tree of order $n-1$.*

Proof. Let $P = u_0u_1 \cdots u_\ell$ be a path of maximum length ℓ in T_n. Clearly, $d(u_0) = d(u_\ell) = 1$. Moreover, $T_n \neq S_n$ implies $\ell \geq 3$.

(i) Since in a tree any two vertices are connected by a unique path, $d(u_0, u_\ell) = \ell \geq 3$. Consider the tree $T^* = T_n - \{u_0, u_\ell\}$ of order $n-2$. Obviously, $T^* \neq S_{n-2}$ for $\ell \geq 5$. In case of $\ell = 3$, $T^* \neq S_{n-2}$ also holds, since otherwise one of the vertices u_1 and u_2 has to be the centre of S_{n-2}, and this yields $T_n = B_{n-3,3}$, a contradiction. It remains $\ell = 4$. Then we are done if $T^* \neq S_{n-2}$. In case of $T^* = S_{n-2}$, u_2 has to be the centre of S_{n-2} and among the $n-3 \geq 3$ vertices of degree 1 in T^* adjacent to u_2 we find a vertex w of degree 1 in T_n. But then u_0 and w are vertices of degree 1 with $d(u_0, w) \geq 3$ such that $T_n - \{u_0, w\}$ is a non-star tree of order $n-2$.

(ii) Consider the tree $T' = T_n - \{u_0\}$ of order $n-1$. Clearly, $T' \neq S_{n-1}$ for $\ell \geq 4$. It remains $\ell = 3$. Then we are done if $T' \neq S_{n-1}$. In case of $T' = S_{n-1}$, u_2 has to be the centre of S_{n-1} forcing $T_n = B_{n-3,3}$ where $n-3 \geq 2$. But then $T_n - \{u_3\}$ is a non-star tree of order $n-1$. ■

Besides Lemma 2.2 the values of $r(T_n, P_3)$ and $r(T_n, P_4)$ for $T_n \neq S_n$ will be used to prove Theorem 2.1. Chvátal and Harary [4] obtained a formula to derive $r(G, P_3)$ for any graph G depending on the edge independence number $\beta_1(\overline{G})$ of the complement $\overline{G}$ of G.

Theorem 2.3 (Chvátal and Harary [4]). *Let G be a graph of order n. Then*

$$r(G, P_3) = \begin{cases} n & \text{if } \overline{G} \text{ contains a 1-factor,} \\ 2n - 2\beta_1(\overline{G}) - 1 & \text{otherwise.} \end{cases}$$

For every tree $T_n \neq S_n$, $\beta_1(\overline{T_n}) = \lfloor n/2 \rfloor$. Applying Theorem 2.3 we obtain the following result.

Corollary 2.4. *Let $n \geq 4$ and $T_n \neq S_n$. Then $r(T_n, P_3) = n$.*

The next result on $r(T_n, P_4)$ was already mentioned without proof by Faudree, Rousseau and Schelp in [7].

Theorem 2.5. *Let $n \geq 4$ and $T_n \neq S_n$. Then $r(T_n, P_4) = n + 1$.*

Proof. Since $\chi(P_4) = 2$ and $s(P_4) = 2$ we obtain $r(T_n, P_4) \geq n + 1$ from (1). To prove that $r(T_n, P_4) \leq n + 1$ we use induction on n. It is easy to check that $r(T_n, P_4) \leq n + 1$ holds for $4 \leq n \leq 5$ if $T_n \neq S_n$, i.e., $T_n \in \{P_4, P_5, B_{2,3}\}$ (cf. also [4] and [5]). Now let $n \geq 6$. By the induction hypothesis, $r(T_k, P_4) \leq k + 1$ for every tree $T_k \neq S_k$ with $4 \leq k < n$. Suppose that a (T_n, P_4)-coloring of K_{n+1} with vertex set V exists for some tree $T_n \neq S_n$ of order n.

Case 1. $K_3 \subseteq [V]_g$. Let $U = \{u_1, u_2, u_3\}$ be the vertex set of a green K_3 and $W = V \setminus U$. Since $P_4 \not\subseteq [V]_g$, all edges between U and W have to be red. Thus $K_{n-2,3} \subseteq [V]_r$. Since $B_{n-3,3} \subseteq K_{n-2,3}$ and $T_n \not\subseteq [V]_r$ it follows that $T_n \neq B_{n-3,3}$. By Lemma 2.2(i), T_n contains two vertices v_1 and v_2 of degree 1 with $d(v_1, v_2) \geq 3$ such that the tree $T^* = T_n - \{v_1, v_2\}$ of order $n-2$ is not a star. The induction hypothesis yields $r(T^*, P_4) \leq n - 1$. Consider $V' = V \setminus \{u_1, u_2\}$. Since $|V'| = n - 1$ and $P_4 \not\subseteq [V']_g$, we obtain that $T^* \subseteq [V']_r$. Let a_1 and a_2 be the two vertices in T^* such that a_i is adjacent to v_i in T_n. Since $d(v_1, v_2) \geq 3$, $a_1 \neq a_2$. If $\{a_1, a_2\} \subseteq W$, then the edges a_1u_1 and a_2u_2 together with T^* would yield a red T_n, a contradiciton. If $a_1 = u_3$ or $a_2 = u_3$, say $a_1 = u_3$, then a vertex $w \in W$ exists where $w \notin V(T^*)$. But then the edges a_1w and a_2u_2 together with T^* again yield a red T_n.

Case 2. $K_3 \not\subseteq [V]_g$. Let v be a vertex in V with $d_g(v) = \Delta_g$. Corollary 2.4 and $T_n \not\subseteq [V]_r$ force $P_3 \subseteq [V]_g$, and this implies $\Delta_g \geq 2$. Let $W = V \setminus \{v\}$. As $K_3 \not\subseteq [V]_g$ and $P_4 \not\subseteq [V]_g$, in $[W]$ every $w \in N_g(v)$ is incident to red edges only. By Lemma 2.2(ii), T_n must contain a vertex u of degree 1 such that $T' = T_n - \{u\}$ is a tree of order $n-1$ different from S_{n-1}. Let $w \in V(T')$ be the neighbor of u in T_n. By the induction hypothesis, $r(T', P_4) \leq n$. Since $|W| = n$ and $P_4 \not\subseteq [W]_g$, a red T' occurs in $[W]$. If $w \in N_r(v)$, then T' together with vw yields a red T_n, a contradiction. It remains that $w \in N_g(v)$. We already know that in $[W]$ every $w \in N_g(v)$ is incident to red edges only. Since $|W| = n$, there is a vertex $w' \in W$ with $w' \notin V(T')$. But then T' together with ww' yields a red T_n and the proof is complete. ■

With these preparations we can now prove Theorem 2.1.

***Proof of Theorem* 2.1.** By (1), $r(T_n, G) \geq 2n-1$ for any graph G with $\chi(G) = 3$. If $G \neq K_{1,1,4}$ and $G \not\subseteq K_{1,2,3}$, then $s(G) = 2$, and (1) yields $r(T_n, G) \geq 2n$. Moreover, $s(G) = 2$ and $G \neq K_{2,2,2}$ imply $G \subseteq K_{2,2,2} - e$. Thus, it suffices to prove $r(T_n, K_{1,2,3}) \le 2n-1$ and $r(T_n, K_{2,2,2}-e) \le 2n$ for every tree $T_n \ne S_n$ where $n \ge 4$. We use that the join $E_2 + P_4$ is isomorphic to $K_{2,2,2}-e$ and we write $\{v_1, v_2\} + P_4$ if $V(E_2) = \{v_1, v_2\}$. The proof consists of two parts: in (i) we derive the desired results for $T_n = B_{n-3,3}$, and in (ii) we consider the trees $T_n \notin \{S_n, B_{n-3,3}\}$.

(i) Let $T_n = B_{n-3,3}$ where the degenerated broom $B_{1,3} = P_4$ is included. Suppose we have a $(B_{n-3,3}, K_{1,2,3})$-coloring of K_{2n-1} or a $(B_{n-3,3}, K_{2,2,2}-e)$-coloring of K_{2n}. Let V denote the vertex sets of the complete graphs.

Claim 2.6. $S_{n-1} \subseteq [V]_r$.

Proof. From [11] we know that $r(S_{n-1}, G) \le 2n-1$ if $G = K_{1,2,3}$ or $G = K_{2,2,2}-e$ and $n \ge 5$. Because of $S_3 = P_3$, $r(P_3, G) = r(G, P_3)$ and Theorem 2.3 this upper bound also holds for $n = 4$. Thus, if $K_{1,2,3} \nsubseteq [V]_g$ or $K_{2,2,2}-e \nsubseteq [V]_g$, then $S_{n-1} \subseteq [V]_r$. □

Claim 2.7. $S_n \nsubseteq [V]_r$.

Proof. Assume that $S_n \subseteq [V]_r$ and let U be the vertex set of a red S_n with centre u_0. Since a red $B_{n-3,3}$ is forbidden, $[U \setminus \{u_0\}]$ has to be a green K_{n-1}. Moreover, all edges between $W = V \setminus U$ and $U \setminus \{u_0\}$ have to be green. This gives a green $K_6 - K_3$ in case of $|V| = 2n-1$, i.e., $|W| = n-1$, contradicting $K_{1,2,3} \nsubseteq [V]_g$. In case of $|V| = 2n$, i.e., $|W| = n$, Corollary 2.4 and $B_{n-3,3} \nsubseteq [V]_r$ imply that a green P_3 must occur in $[W]$. This yields a green $K_6 - e$, a contradiction to $K_{2,2,2}-e \nsubseteq [V]_g$. □

Now we use Claim 2.6 and consider a red S_{n-1} with vertex set U and centre u_0. By Claim 2.7 and $B_{n-3,3} \nsubseteq [V]_r$, all edges between U and $W = V \setminus U$ have to be green. In case of $|V| = 2n-1$ it follows that $|W| = n$, and Corollary 2.4 together with $B_{n-3,3} \nsubseteq [V]_r$ imply that a green $P_3 = w_1w_2w_3$ occurs in $[W]$. But then $\{w_2\} + \{w_1, w_3\} + \{u_0, u_1, u_2\}$ where $\{u_1, u_2\} \subseteq U \setminus \{u_0\}$ is a green $K_{1,2,3}$, a contradiction. In case of $|V| = 2n$ we obtain $|W| = n+1$, and Theorem 2.5 together with $B_{n-3,3} \nsubseteq [V]_r$ guarantee a green P_4 in $[W]$. But this forces $\{u_1, u_2\} + P_4$ to be a green $K_{2,2,2}-e$, a contradiction, and we are done for $T_n = B_{n-3,3}$.

(ii) It remains that $T_n \notin \{S_n, B_{n-3,3}\}$. We use induction on n to prove $r(T_n, K_{1,2,3}) \le 2n-1$ and $r(T_n, K_{2,2,2}-e) \le 2n$ for every tree $T_n \notin \{S_n, B_{n-3,3}\}$ with $n \ge 4$.

First we derive the desired results for $4 \le n \le 5$. There is only one tree $T_n \notin \{S_n, B_{n-3,3}\}$ with $4 \le n \le 5$, namely P_5. To prove $r(P_5, K_{1,2,3}) \le 9$ and $r(P_5, K_{2,2,2}-e) \le 10$ assume we have a $(P_5, K_{1,2,3})$-coloring of K_9 or a $(P_5, K_{2,2,2}-e)$-coloring of K_{10}. Let V denote the vertex sets of the complete graphs. Since $P_4 = B_{1,3}$, by the above result on brooms we already know that $r(P_4, K_{1,2,3}) \le 7$ and $r(P_4, K_{2,2,2}-e) \le 8$. Thus, a red $P_4 = u_1u_2u_3u_4$ must occur in $[V]$, and $P_5 \nsubseteq [V]_r$ forces all edges between $\{u_1, u_4\}$ and the vertices in $W = V \setminus \{u_1, u_2, u_3, u_4\}$ to be green. In K_9 we obtain $|W| = 5$, and $r(P_5, S_4) = 5$ (cf. [5]) guarantees a green S_4 in $[W]$ with centre w_0 and vertices w_1, w_2, w_3 of degree 1 yielding the green $K_{1,2,3} = \{w_0\} + \{u_1, u_4\} + \{w_1, w_2, w_3\}$, a contradiction. In K_{10} we have $|W| = 6$, and $r(P_5, P_4) = 6$ (see Theorem 2.5) forces a green P_4 in $[W]$. But then $\{u_1, u_4\} + P_4$ is a green $K_{2,2,2} - e$, a contradiction.

Now let $n \geq 6$. By the induction hypothesis, $r(T_k, K_{1,2,3}) \leq 2k - 1$ and $r(T_k, K_{2,2,2} - e) \leq 2k$ for every tree $T_k \notin \{S_k, B_{k-3,3}\}$ with $4 \leq k < n$. Suppose we have a $(T_n, K_{1,2,3})$-coloring of K_{2n-1} or a $(T_n, K_{2,2,2} - e)$-coloring of K_{2n} for some tree T_n where $T_n \notin \{S_n, B_{n-3,3}\}$. Again we use V to denote the vertex sets of the complete graphs. By Lemma 2.2(i), T_n contains two vertices v_1 and v_2 of degree 1 with distance $d(v_1, v_2) \geq 3$ such that the tree $T^* = T_n - \{v_1, v_2\}$ of order $n - 2$ is not a star. By the induction hypothesis and the above result on brooms, $r(T^*, K_{1,2,3}) \leq 2n - 5$ and $r(T^*, K_{2,2,2} - e) \leq 2n - 4$. Let a_1 and a_2 be the two vertices in T^* such that a_i is adjacent to v_i in T_n, where $1 \leq i \leq 2$. Since $r(T_n, K_4 - e) = 2n - 1$ (see [2]), one of the following two cases must occur.

Case 1. $K_4 \subseteq [V]_g$. Let $U = \{u_1, u_2, u_3, u_4\}$ be the vertex set of a green K_4 with minimal sum $d_r(u_1) + d_r(u_2) + d_r(u_3) + d_r(u_4)$ of r-degrees, and let $W = V \setminus U$. Since $|W| = 2n - 5$ in case of K_{2n-1} and $|W| = 2n - 4$ in case of K_{2n}, $T^* \subseteq [W]_r$. We distinguish two subcases depending on $q_r(a_i, U)$.

Case 1.1. $q_r(a_1, U) \geq 1$ and $q_r(a_2, U) \geq 1$. Then $T_n \subseteq [V]_r$, except for $q_r(a_1, U) = q_r(a_2, U) = 1$ where a_1 and a_2 have the same red neighbor in U, say u_1. But this gives the green $K_{1,2,3} = \{u_2\} + \{u_3, u_4\} + \{u_1, a_1, a_2\}$, a contradiction for $|V| = 2n - 1$. In the remaining case $|V| = 2n$ let $W' = W \setminus V(T^*)$. Note that $|W'| = n - 2$. If a_1 or a_2 has a red neighbor in W', then again a red T_n occurs. Otherwise all $n + 1$ vertices in $W' \cup \{u_2, u_3, u_4\}$ are common green neighbors of a_1 and a_2, and Theorem 2.5 guarantees a green P_4 in $[W' \cup \{u_2, u_3, u_4\}]$. But this forces $\{a_1, a_2\} + P_4$ to be a green $K_{2,2,2} - e$, a contradiction.

Case 1.2. $q_r(a_1, U) = 0$ or $q_r(a_2, U) = 0$, say $q_r(a_1, U) = 0$. Now let $U' = U \cup \{a_1\}$ and $W' = V \setminus U'$. Note that $[U']$ is a green K_5 and that $|W' \cap V(T^*)| = n - 3$. If $q_r(w, U') \leq 2$ for some $w \in W'$, then we find a green $K_{1,2,3}$ and a green $K_{2,2,2} - e$ in $[U' \cup \{w\}]$, a contradiction. Thus $q_r(w, U') \geq 3$ for every $w \in W'$ yielding $q_r(W', U') \geq 3|W'| \geq 3(2n - 6)$. This implies $q_r(u, W') = d_r(u) \geq n - 2$ for some $u \in U'$. In case of $d_r(a_1) \leq n - 3$ we may assume that $d_r(u_4) \geq n - 2$. But then the green $K_4 = [\{a_1, u_1, u_2, u_3\}]$ would have a smaller sum of r-degrees than the green $K_4 = [\{u_1, u_2, u_3, u_4\}]$. It remains $d_r(a_1) \geq n - 2$. This forces $q_r(a_1, W') \geq n - 2$ and we find a red neighbor w^* of a_1 in $W' \setminus V(T^*)$ since $|W' \cap V(T^*)| = n - 3$. Moreover, $q_r(w, U') \geq 3$ for every $w \in W'$ yields a red neighbor u^* of a_2 in U. But then T^* together with w^* and u^* produce a red T_n, a contradiction.

Case 2. $K_4 - e \subseteq [V]_g$ and $K_4 \not\subseteq [V]_g$. Let $U = \{u_1, u_2, u_3, u_4\}$ be the vertex set of a green $K_4 - e$ where u_1u_4 is red, and let $W = V \setminus U$. Since $K_4 \not\subseteq [V]_g$, $q_r(w, U) \geq 1$ for every $w \in W$. As in Case 1, $T^* \subseteq [W]_r$, and $T_n \not\subseteq [V]_r$ forces $q_r(a_1, U) = q_r(a_2, U) = 1$. Moreover, a_1 and a_2 must have the same red neighbor in U, and $K_4 \not\subseteq [V]_g$ implies that u_2 or u_3, say u_2, is the common red neighbor. But then we obtain the green $K_{1,2,3} = \{u_3\} + \{u_1, u_4\} + \{u_2, a_1, a_2\}$, a contradiction for $|V| = 2n - 1$. In the remaining case $|V| = 2n$ let $W' = W \setminus V(T^*)$. Note that $|W'| = n - 2$. If a_1 or a_2 has a red neighbor in W', then a red T_n occurs. Otherwise, the $n + 1$ vertices in $W' \cup \{u_1, u_3, u_4\}$ are common green neighbors of a_1 and a_2, and Theorem 2.5 guarantees a green P_4 in $[W' \cup \{u_1, u_3, u_4\}]$. But this gives a green $K_{2,2,2} - e$ and we are done. ∎

The two graphs G of order six with $\chi(G) = 3$ not considered in Theorem 2.1 are $G = K_{1,1,4}$ and $G = K_{2,2,2}$. The values of $r(T_n, K_{1,1,4})$ for $n \geq 9$ follow from a more general result due to Erdős, Faudree, Rousseau and Schelp [6] who investigated $r(T_n, B_m)$ for any tree T_n and the book-graph $B_m = K_{1,1,m}$.

Theorem 2.8 (Erdős, Faudree, Rousseau and Schelp [6])**.** *If $n \geq 3m - 3$, then*

$$r(T_n, B_m) = 2n - 1.$$

Applying Theorem 2.8 for $B_4 = K_{1,1,4}$ we obtain $r(T_n, K_{1,1,4}) = 2n - 1$ for any tree T_n with $n \geq 9$. A result due to Rousseau and Sheehan [18] implies $r(P_n, K_{1,1,4}) = 10$ for $4 \leq n \leq 5$ and $r(P_n, K_{1,1,4}) = 2n - 1$ for $n \geq 6$. Moreover, in [13] we determined the missing values of $r(T_n, K_{1,1,4})$ for $n \leq 8$. This proves that any non-star tree T_n with $n \geq 6$ is $K_{1,1,4}$-good.

Theorem 2.9. *Let $n \geq 4$ and $T_n \neq S_n$. Then*

$$r(T_n, K_{1,1,4}) = \begin{cases} 10 & \text{if } 4 \leq n \leq 5, \\ 2n - 1 & \text{if } n \geq 6. \end{cases}$$

For the remaining graph $G = K_{2,2,2}$ the situation is much more complicated. From (1) we obtain $r(T_n, K_{2,2,2}) \geq 2n$. On the other hand, for n sufficiently large we know that $r(S_n, K_{2,2,2}) > 2n - 2 + \lfloor \sqrt{n-1} - 6(n-1)^{11/40} \rfloor$ (see [12], note that $K_{2,2,2} = K_6 - 3K_2$) forcing $r(T_n, K_{2,2,2}) > 2n$ also for non-star trees T_n with maximum degree close to $n - 1$ if n is sufficiently large. Nevertheless, there are non-star trees with small maximum degree where the lower bound $2n$ is attained. For $T_n = P_n$ this follows from a more general result due to Gould and Jacobson [8] who proved that any path P_n with $n \geq 3$ is $(K_{2m} - mK_2)$-good.

Theorem 2.10 (Gould and Jacobson [8])**.** *If $n \geq 3$ and $m \geq 2$, then*

$$r(P_n, K_{2m} - mK_2) = (n-1)(m-1) + 2.$$

The following theorem shows that $r(T_n, K_{2,2,2}) = 2n$ also holds for a special class of trees T_n with $\Delta(T_n) = 3$.

Theorem 2.11. *Let T_n be a tree of order $n \geq 5$ with $\Delta(T_n) = 3$ containing a path P_{n-1}. Then*

$$r(T_n, K_{2,2,2}) = 2n.$$

To prove Theorem 2.11 we use a result due to Burr, Erdős, Faudree, Rousseau and Schelp [1] who obtained a formula to determine $r(T_n, C_4)$ depending on $r(S_{m+1}, C_4)$ where $m = \Delta(T_n)$.

Theorem 2.12 (Burr, Erdős, Faudree, Rousseau and Schelp [1])**.** *If T_n is a tree with $\Delta(T_n) = m$, then $r(T_n, C_4) = \max\{4, n+1, r(S_{m+1}, C_4)\}$.*

Thus, $r(T_n, C_4)$ is easily evaluated if $r(S_{m+1}, C_4)$ is known, but $r(S_{m+1}, C_4)$ has not yet been completely determined (see Parsons [15] and Wu, Sun, Zhang and Radziszowski [19]).

***Proof of Theorem* 2.11.** It suffices to prove that $r(T_n, K_{2,2,2}) \leq 2n$. Let T_n be a tree with $\Delta(T_n) = 3$ containing a path P_{n-1} and suppose we have a $(T_n, K_{2,2,2})$-coloring of K_{2n} with vertex set V.

Claim 2.13. $|N_g(v_1) \cap N_g(v_2)| \leq n$ *for any two vertices v_1 and v_2.*

Proof. Assume that there are vertices v_1 and v_2 with $|N_g(v_1) \cap N_g(v_2)| \geq n+1$. Since $r(S_4, C_4) = 6$ (cf. [4]), Theorem 2.12 states $r(T_n, C_4) = n+1$. Thus, $T_n \not\subseteq [V]_r$ forces a green $C_4 = (w_1w_2w_3w_4)$ in $[N_g(v_1) \cap N_g(v_2)]$. But this gives the green $K_{2,2,2} = \{v_1, v_2\} + \{w_1, w_3\} + \{w_2, w_4\}$, a contradiction. □

By Theorem 2.10 and $K_{2,2,2} = K_6 - 3K_2$, a red $P_{n-1} = u_1u_2 \cdots u_{n-1}$ must occur. First let n be odd or, in case of n even, let T_n not be isomorphic to the tree obtained from $u_1u_2 \cdots u_{n-1}$ by joining a vertex $w \in W = V \setminus \{u_1, \ldots, u_{n-1}\}$ to $u_{n/2}$. Then $T_n \not\subseteq [V]_r$ implies that there is some i with $1 \leq i \leq \lfloor (n-1)/2 \rfloor - 1$ such that u_{1+i} and u_{n-1-i} are joined green to all $n+1$ vertices in W, a contradiction to Claim 2.13. Consider now the remaining case for n even. Since $T_n \not\subseteq [V]_r$, all edges from $u_{n/2}$ to W have to be green, and then Claim 2.13 forces at least one red edge from every u_i with $i \neq n/2$ to W. Moreover, two independent red edges between $\{u_1, u_{n/2-1}\}$ and W would yield a red T_n. Thus we may assume that u_1 and $u_{n/2-1}$ have a common red neighbor $w^* \in W$ and that all edges between $\{u_1, u_{n/2-1}\}$ and $W \setminus \{w^*\}$ are green. Then $T_n \not\subseteq [V]_r$ forces u_1u_{n-1} to be green. Furthermore, by Claim 2.13, the edges $u_{n/2-1}u_{n-1}$ and $u_{n/2}u_{n-1}$ have to be red. Remind that a red edge $u_{n-1}w$ with $w \in W$ must occur. But then the red path $P_{n-1} = u_1 \cdots u_{n/2-1}u_{n-1}u_{n/2} \cdots u_{n-2}$ together with the red edge $u_{n-1}w$ yields the forbidden red T_n, a contradiction, and we are done. ■

3. Trees $T_n \in \{P_n, B_{2,n-2}\}$ and the Graphs G with $\chi(G) = 2$

It seems to be out of reach to determine the exact value of $r(T_n, G)$ for every tree T_n and all connected bipartite graphs G of order six, i.e., the star $S_6 = K_{1,5}$ and the connected spanning subgraphs of $K_{2,4}$ and $K_{3,3}$. Burr, Erdős, Faudree, Rousseau and Schelp [1] derived upper bounds for $r(T_n, K_{2,4})$ and $r(T_n, K_{3,3})$. They proved that for all sufficiently large n,

$$r(T_n, K_{2,4}) < n + 3n^{1/2}.$$

Moreover they showed that there exists a constant c such that for every tree T_n with maximum degree $\Delta(T_n) = m$,

$$r(T_n, K_{3,3}) \leq \max\left\{n + \left\lceil cn^{1/3} \right\rceil, r(S_{m+1}, K_{3,3})\right\}$$

and

$$r(S_{m+1}, K_{3,3}) < m + 3m^{2/3}$$

for m sufficiently large. Lower bounds can be obtained from $r(T_n, C_4)$ since $C_4 \subseteq K_{2,4}$ and $C_4 \subseteq K_{3,3}$. In [1] it was proved that for all sufficiently large n,

$$r(S_{m+1}, C_4) > m + \left\lfloor m^{1/2} - 6m^{11/40} \right\rfloor.$$

This together with Theorem 2.12 implies that $r(T_n, K_{2,4})$ and $r(T_n, K_{3,3})$ differ considerably from the lower bound (1) if n is sufficiently large and $\Delta(T_n) = m$ is close to $n-1$. Clearly, the same holds for $r(T_n, G)$ if G is any bipartite graph with $C_4 \subseteq G$. Here we restrict ourselves to study $r(T_n, G)$ for two trees with small maximum degree, namely $T_n \in \{P_n, B_{2,n-2}\}$. The choice of these two trees is essentielly due to our project to determine $r(T_n, G)$ for every connected graph of order six and all trees of order at most five — the only non-star trees on at most five vertices are the paths P_4 and P_5 and the broom $B_{2,3}$. Our results show that, except for some small n, the trees $T_n \in \{P_n, B_{2,n-2}\}$ are G-good for any connected bipartite graph G of order $p(G) = 6$, i.e., $r(T_n, G)$ attains the general lower bound from (1). Instead of $r(T_n, S_6)$ here we consider the more general case $r(T_n, S_m)$. We start by improving the lower bound (1) for $T_n \in \{P_n, B_{2,n-2}\}$ and any connected bipartite graph G in case of small n.

Lemma 3.1. *Let $G \subseteq K_{m_1,m_2}$ be a connected graph of order $m = m_1 + m_2$ where $1 \leq m_1 \leq m_2$. Then $r(P_n, G) \geq m - 1 + \lfloor n/2 \rfloor$ for $n \geq 2$ and $r(B_{2,n-2}, G) \geq m - 1 + \lfloor (n-1)/2 \rfloor$ for $n \geq 5$.*

Proof. From (1) it follows that $r(G, T_n) \geq m - 1 + s(T_n)$. Due to $r(F, G) = r(G, F)$, $s(P_n) = \lfloor n/2 \rfloor$ for $n \geq 2$ and $s(B_{2,n-2}) = \lfloor (n-1)/2 \rfloor$ for $n \geq 5$ we obtain the desired results. ∎

If G is a connected spanning subgraph of K_{m_1,m_2} with $1 \le m_1 \le m_2$, then $s(G) = m_1$, and the general lower bound (1) implies $r(T_n, G) \ge n+m_1-1$ for any tree T_n. Hence the general lower bound is improved by the lower bounds from Lemma 3.1 for $T_n = P_n$ if $n \le 2m_2 - 2$ and for $T_n = B_{2,n-2}$ if $n \le 2m_2 - 3$. The following lemma shows that in case of $T_n = B_{2,n-2}$ the general lower bound can also be improved for $n = 2m_2 - 2$ or $n = 2m_2$ and certain graphs $G \subseteq K_{m_1,m_2}$.

Lemma 3.2. *Let $n \ge 6$ be even and let $m_1 \le m_2$. Then $r(B_{2,n-2}, G) \ge n + m_1$ if $m_1 \ge 1$, $n = 2m_2$ and $G = K_{m_1,m_2}$ or if $m_1 \ge 2$, $n = 2m_2 - 2$ and $G \in \{K_{m_1,m_2} - e, K_{m_1,m_2} - 2K_2\}$. Moreover, $r(B_{2,3}, K_{m_1,m_2}) \ge m_1 + m_2 + 2$.*

Proof. For $n = 2m_2$, the coloring of K_{n+m_1-1} with $[V]_g = 2K_{m_2} + \overline{K_{m_1-1}}$ contains no red $B_{2,n-2}$ and no green K_{m_1,m_2}. For $n = 2m_2 - 2$, the coloring of K_{n+m_1-1} with $[V]_g = 2K_{m_2-1} + \overline{K_{m_1-1}}$ contains no red $B_{2,n-2}$ and no green $K_{m_1,m_2} - 2K_2$. Moreover, the coloring of $K_{m_1+m_2+1}$ with $[V]_r = C_{m_1+m_2+1}$ contains no red $B_{2,3}$ and no green K_{m_1,m_2}. ■

Now we consider $r(T_n, S_m)$. Parsons [14] has already determined the exact value of $r(P_n, S_m)$ by explicit formulas and a recurrence.

Theorem 3.3 (Parsons [14]). *Let $n \ge 4$ and $m \ge 4$. Then*

$$r(P_n, S_m) = \begin{cases} 2m-3 & \text{if } m-1 \le n < 2m-3, \\ n & \text{if } n \ge 2m-3, \end{cases}$$

and $r(P_n, S_m) = \max\{r(P_{n-1}, S_m), r(P_n, S_{m-(n-1)}) + n - 1\}$ if $n < m - 1$.

Remark. For $n \ge 4$ and $m = 5$ only $r(P_4, S_6)$ is not explicitely given by Theorem 3.3. Applying the recurrence and Theorem 2.3 we derive $r(P_4, S_6) = 7$.

We use the result of Parsons to completely determine the exact values of $r(B_{2,n-2}, S_m)$ if $n \ge m - 1$.

Theorem 3.4. *Let $n \ge 5$ and $m \ge 4$. Then $r(B_{2,3}, S_4) = 6$ and*

$$r(B_{2,n-2}, S_m) = \begin{cases} 2m-3 & \text{if } m-1 \le n \le 2m-3 \text{ and } m \ge 5, \\ n+1 & \text{if } n = 2m-2, \\ n & \text{if } n \ge 2m-1. \end{cases}$$

To prove Theorem 3.4 the straightforward statements of the following lemma will be used.

Lemma 3.5. *Let $n \ge 5$ and let χ be a coloring of a complete graph with vertex set V and $P_n = u_1 \cdots u_n \subseteq [V]_r$, but $B_{2,n-2} \not\subseteq [V]_r$. Then u_1u_3, u_1u_{n-1}, u_2u_n and $u_{n-2}u_n$ have to be green. Furthermore, if $n \ge 7$ and u_1u_i is red for some i with $5 \le i \le n - 2$, then $u_{i-2}u_n$ has to be green.*

***Proof of Theorem* 3.4.** In [5] it was already shown that $r(B_{2,3}, S_4) = 6$. From (1) we obtain $r(B_{2,n-2}, S_m) \geq n$. Lemma 3.2 yields $r(B_{2,n-2}, S_m) \geq n+1$ if $n = 2m-2$. Moreover, the coloring of K_{2m-2} with $[V]_r = 2K_{m-2}$ shows $r(B_{2,n-2}, S_m) \geq 2m-3$ for $n \geq m-1$. Thus, to establish the results from Theorem 3.4 it suffices to prove that $r(B_{2,n-2}, S_m) \leq n$ for $n \geq 2m-1$ with $m \geq 4$ and for $n = 2m-3$ with $m \geq 5$ by using the monotonicity property $r(B_{2,n-2}, S_m) \leq r(B_{2,n'-2}, S_m)$ for $n < n'$. To obtain the desired upper bounds suppose that we have a $(B_{2,n-2}, S_m)$-coloring of K_n with vertex set V where $n \geq 2m-1$ with $m \geq 4$ or $n = 2m-3$ with $m \geq 5$. Then, by Theorem 3.3, a red $P_n = u_1u_2\cdots u_n$ occurs. Note that $S_m \not\subseteq [V]_g$ forces $\Delta_g \leq m-2$. By Lemma 3.5, u_1u_3, u_1u_{n-1},u_2u_n and $u_{n-2}u_n$ have to be green.

Case 1. $n \geq 2m-1$ where $m \geq 4$. Since $d_g(u_1) \leq m-2$, $q_g(u_1, \{u_5, \ldots, u_{n-2}\}) \leq m-4$. Thus, $q_r(u_1, \{u_5, \ldots, u_{n-2}\}) \geq n-6-(m-4) \geq 2m-1-m-2 = m-3$, and Lemma 3.5 implies $q_g(u_n, \{u_3, \ldots, u_{n-4}\}) \geq m-3$. But this yields $d_g(u_n) \geq m-1$, a contradiction.

Case 2. $n = 2m-3$ where $m \geq 5$. We distinguish two subcases depending on the color of u_1u_n.

Case 2.1. u_1u_n is green. Then $q_g(u_1, \{u_5, \ldots, u_{n-2}\}) \leq m-5$ because $d_g(u_1) \leq m-2$. This forces $q_r(u_1, \{u_5, \ldots, u_{n-2}\}) \geq n-6-(m-5) = 2m-3-m-1 = m-4$, and Lemma 3.5 yields $q_g(u_n, \{u_3, \ldots, u_{n-4}\}) \geq m-4$. Again we obtain $d_g(u_n) \geq m-1$, a contradiction.

Case 2.2. u_1u_n is red, i.e., $C_n = (u_1u_2\cdots u_n)$ is a red cycle. The remaining edges are the diagonals $u_iu_{i+\ell}$ of length ℓ with $\ell = 2, \ldots, m-2$ and $i = 1, \ldots, n$, where the indices should be read modulo n. To finish Case 2.2 we use the following properties of the diagonals of C_n.

Claim 3.6. *If a diagonal $u_iu_{i+\ell}$ of length ℓ with $2 \leq \ell \leq m-2$ and $1 \leq i \leq n$ is red, then also $u_{i+1}u_{i+\ell+1}$ has to be red.*

Proof. If $u_iu_{i+\ell}$ is red and $u_{i+1}u_{i+\ell+1}$ is green, then the end-vertices of the red $P_n = u_{i+\ell+1}u_{i+\ell+2}\cdots u_nu_1\cdots u_iu_{i+\ell}u_{i+\ell-1}\cdots u_{i+1}$ are joined green, a situation already considered in Case 2.1. □

Claim 3.7. *For $2 \leq \ell \leq m-2$, all diagonals of length ℓ must have the same color.*

Proof. This is an immediate consequence of Claim 3.6. □

Claim 3.8. *If the diagonals of length ℓ with $2 \leq \ell \leq m-3$ are red, then the diagonals of length $\ell+1$ have to be green.*

Proof. Assume that the diagonals of length $\ell+1$ are also red. Using the diagonal $u_1u_{\ell+1}$ of length ℓ, the diagonal $u_2u_{\ell+3}$ of lenght $\ell+1$ and edges from the red C_n we obtain the red $B_{2,n-2}$ with bristles $u_1u_{\ell+1}$, $u_{\ell+1}u_{\ell+2}$ and handle $u_{\ell+1}u_\ell\cdots u_2u_{\ell+3}\cdots u_n$, a contradiction. □

Claim 3.9. *The diagonals of length ℓ with $2\le\ell\le3$ and, for $m\ge6$, also the diagonals of lenght $\ell=4$ have to be green.*

Proof. Assume that for some ℓ with $2\le\ell\le4$ the diagonals of length ℓ are red. If $\ell=2$, then u_1u_3 together with edges of the red C_n give the red $B_{2,n-2}$ with bristles u_1u_3, u_2u_3 and handle $u_3u_4\cdots u_n$, a contradiction. If $\ell=3$, then the red $B_{2,n-2}$ with bristles u_1u_2, u_1u_n and handle $u_1u_4u_3u_6\cdots u_{n-4}u_{n-1}u_{n-2}$ would occur. If $\ell=4$ and $m\ge6$, then the diagonals u_1u_5 and u_3u_7 together with edges from the red C_n would yield the red $B_{2,n-2}$ with bristles u_2u_3, u_3u_4 and handle $u_3u_7u_6u_5u_1u_nu_{n-1}\cdots u_8$. □

Now we finish Case 2.2 by deriving a contradiction to $\Delta_g\le m-2$. Note that for $2\le\ell\le m-2$ every u_i is incident to two diagonals of lenght ℓ. Thus, Claim 3.9 yields the desired contradiction for $5\le m\le7$. In the remaining case $m\ge8$ we additionally have to consider the diagonals of length $\ell\ge5$. There are $m-6$ different diagonal lengths ℓ with $5\le\ell\le m-2$ and Claim 3.8 implies that at least $\lfloor(m-6)/2\rfloor$ of them belong to green diagonals. Hence $d_g(u_i)\ge6+2\lfloor(m-6)/2\rfloor\ge m-1$, a contradiction, and we are done. ■

In the following two theorems $r(P_n,G)$ and $r(B_{2,n-2},G)$ are determined for any connected spanning subgraph G of $K_{2,4}$.

Theorem 3.10. *Let $n\ge4$ and let G be a connected graph of order six where $G\subseteq K_{2,4}$. Then*

$$r(P_n,G)=\begin{cases}7 & \text{if } 4\le n\le5,\\ 8 & \text{if } n=6,\\ n+1 & \text{otherwise.}\end{cases}$$

Proof. From (1) we obtain $r(P_n,G)\ge n+1$. Moreover, Lemma 3.1 implies $r(P_n,G)\ge7$ for $4\le n\le5$ and $r(P_6,G)\ge8$. To establish equality it suffices to show $r(P_5,K_{2,4})\le7$ and $r(P_n,K_{2,4})\le n+1$ for $n\ge7$. Consider any coloring of K_7 not containing a red P_5 and any coloring of K_{n+1}, $n\ge7$, not containing a red P_n. We have to prove that a green $K_{2,4}$ occurs. Let $P_k=u_1\cdots u_k$ be a red path of maximum length, $U=\{u_1,\dots,u_k\}$ and $W=V\setminus U$ where V denotes the vertex sets of the complete graphs. If $k=1$, then only green edges occur and we find a green $K_{2,4}$. Now let $k\ge2$. The maximality of k forces that u_1 and u_k are joined green to all vertices in W. This yields a green $K_{2,4}$ if $|W|\ge4$. It remains $|W|=3$ in case of K_7 and $2\le|W|\le3$ in case of K_{n+1}, $n\ge7$.

Case 1. $|W| = 3$. Then $k = n - 1 = 4$ in case of K_7 and $k = n - 2 \geq 5$ in case of K_{n+1}, $n \geq 7$. Let $W = \{w_1, w_2, w_3\}$. Only green edges between W and $\{u_2, u_{k-1}\}$ imply a green $K_{2,4}$. Otherwise we may assume that u_2w_1 is red. Since $P_{k+1} \not\subseteq [V]_r$, w_1 has to be joined green to w_2, w_3 and u_3. Furthermore, u_1u_3 and u_1u_k have to be green, and we obtain the green $K_{2,4} = \{u_1, w_1\} + \{w_2, w_3, u_3, u_k\}$.

Case 2. $|W| = 2$ in case of K_{n+1}, $n \geq 7$. This implies $k = n - 1$. Let $W = \{w_1, w_2\}$. If $K_{2,4} \not\subseteq [V]_g$, then at most one vertex from $\{u_2, \ldots, u_{n-2}\}$ is joined green to w_1 and to w_2. Therefore we may assume that every vertex in $\{u_2, \ldots, u_{\lfloor (n-1)/2 \rfloor}\}$ is joined red to w_1 or to w_2. Note that $\lfloor (n-1)/2 \rfloor \geq 3$. Since $P_{k+1} \not\subseteq [V]_r$, a common red neighbor of u_2 and u_3 in W is forbidden. Thus, we may assume that u_2w_1 and u_3w_2 are red. Then $P_{k+1} \not\subseteq [V]_r$ forces w_1w_2, w_1u_3, w_1u_4, u_1u_3, u_1u_4 and u_1u_{n-1} to be green, and this yields the green $K_{2,4} = \{u_1, w_1\} + \{w_2, u_3, u_4, u_{n-1}\}$. ∎

Theorem 3.11. *Let $n \geq 5$ and let G be a connected graph of order six where $G \subseteq K_{2,4}$. Then, if $G \neq K_{2,4}$,*

$$r(B_{2,n-2}, G) = \begin{cases} 7 & \text{if } n = 5, \\ 8 & \text{if } n = 6 \text{ and } K_{2,4} - 2K_2 \subseteq G, \\ n+1 & \text{otherwise,} \end{cases}$$

and

$$r(B_{2,n-2}, K_{2,4}) = \begin{cases} 8 & \text{if } n \leq 7, \\ 10 & \text{if } n = 8, \\ n+1 & \text{otherwise.} \end{cases}$$

Proof. From (1) we obtain $r(B_{2,n-2}, G) \geq n+1$, and Lemma 3.1 yields $r(B_{2,3}, G) \geq 7$. Lemma 3.2 gives $r(B_{2,3}, K_{2,4}) \geq 8$, $r(B_{2,4}, K_{2,4} - 2K_2) \geq 8$ and $r(B_{2,6}, K_{2,4}) \geq 10$. To establish equality it suffices to show that $r(B_{2,3}, K_{2,4} - e) \leq 7$, $r(B_{2,4}, G^*) \leq 7$ for G^* obtained from $K_{2,4}$ by deleting two edges incident to the same vertex of degree 4 and $r(B_{2,n-2}, G) \leq n+1$ for $G = K_{2,4}$ if $n = 7$ or $n \geq 9$ and for $G = K_{2,4} - e$ if $n = 8$.

To verify that $r(B_{2,3}, K_{2,4} - e) \leq 7$ and $r(B_{2,4}, G^*) \leq 7$ consider any coloring of K_7 with vertex set V. If a green $K_{2,4}$ occurs, then we are done. Otherwise, by Theorem 3.10, a red $P_5 = u_1 \cdots u_5$ must occur. Let $U = \{u_1, \ldots, u_5\}$ and let $W = V \setminus U = \{w_1, w_2\}$. Assume first that $B_{2,3} \not\subseteq [V]_r$. Then all edges between W and $\{u_2, u_3, u_4\}$ have to be green. Moreover, at least one edge from u_1 to W must be green yielding a green $K_{2,4} - e$. Suppose now that $B_{2,4} \not\subseteq [V]_r$. Then all edges between W and $\{u_2, u_4\}$ have to be green. If w_1 or w_2 is joined green to both u_1 and u_5, then a green G^* occurs. Neither u_1 nor u_5 can be joined red to w_1 and to w_2 since $B_{2,4} \not\subseteq [V]_r$. Thus we may assume that u_1w_1 and u_5w_2 are green and that u_1w_2 and u_5w_1 are red. But then $B_{2,4} \not\subseteq [V]_r$ forces u_3w_1 to be green, and we obtain a green G^*.

To prove that $r(B_{2,n-2}, G) \leq n+1$ for $G = K_{2,4}$ if $n = 7$ or $n \geq 9$ and for $G = K_{2,4} - e$ if $n = 8$ consider any coloring of K_{n+1}, $n \geq 7$, not containing a red $B_{2,n-2}$. Let $V = V(K_{n+1})$. We have to show that a green $K_{2,4} - e$ occurs in case of $n = 8$ and a green $K_{2,4}$ otherwise.

Case 1. There is a red cycle $C_k = (u_1 \cdots u_k)$ of length $k = n$ or $k = n+1$. Let $U = \{u_1, \ldots, u_k\}$. We consider two subcases depending on k.

Case 1.1. $k = n$. Then $B_{2,n-2} \not\subseteq [V]_r$ implies that all edges between U and the vertex $w \in V \setminus U$ are green. By Theorem 3.4, $r(B_{2,n-2}, S_5) = n$ if $n = 7$ or $n \geq 9$, and $r(B_{2,n-2}, S_4) = n$ if $n = 8$. This yields a green S_5 in $[U]$ for $n = 7$ and for $n \geq 9$ and a green S_4 in $[U]$ for $n = 8$. Together with the green edges incident to w we obtain a green $K_{2,4}$ and a green $K_{2,4} - e$, respectively.

Case 1.2. $k = n+1$. Then $B_{2,n-2} \not\subseteq [V]_r$ forces all diagonals of length $\ell \leq 3$ to be green. If, in addition, all diagonals of length $\ell = 4$ are green, then $\{u_1, u_2\} + \{u_4, u_5, u_{n-1}, u_n\}$ is a green $K_{2,4}$. The remaining case is that at least one diagonal of length 4, say u_1u_5, is red. Any red diagonal of length $\ell \geq 4$ incident to u_3 yields a red $B_{2,n-2}$ with bristles u_2u_3 and u_3u_4, a contradiction. Otherwise all diagonals of length $\ell \geq 2$ incident to u_3 are green. Thus, u_1, u_6 and u_7 are common green neighbors of u_3 and u_4. If u_4u_{n+1} is also green, then $\{u_3, u_4\} + \{u_1, u_6, u_7, u_{n+1}\}$ is a green $K_{2,4}$. On the other hand, if u_4u_{n+1} is red, then all diagonals incident to u_2 have to be green since $B_{2,n-2} \not\subseteq [V]_r$. But then u_2 and u_3 have at least four common green neighbors and again a green $K_{2,4}$ occurs.

Case 2. Every red cycle has length at most $n-1$. If $K_{2,4} \subseteq [V]_g$, then we are done. Otherwise, by Theorem 3.10, a red $P_n = u_1 \cdots u_n$ occurs. Let $U = \{u_1, \ldots, u_n\}$ and let w be the vertex in $V \setminus U$. Since $B_{2,n-2} \not\subseteq [V]_r$, the edges wu_2, wu_3, wu_{n-2} and wu_{n-1} have to be green. By Lemma 3.5, the edges u_1u_3, u_1u_{n-1}, u_2u_n and $u_{n-2}u_n$ are green. Moreover, $C_n \not\subseteq [V]_r$ forces u_1u_n to be green, and $C_{n+1} \not\subseteq [V]_r$ implies that at least one of the edges wu_1 and wu_n, say wu_n, is green. To avoid a green $K_{2,4} = \{u_1, w\} + \{u_2, u_{n-2}, u_{n-1}, u_n\}$, u_1u_{n-2} has to be red. Then, by Lemma 3.5, $u_{n-4}u_n$ must be green. Furthermore, $C_n \not\subseteq [V]_r$ implies that $u_{n-3}u_{n-1}$ is green, and $B_{2,n-2} \not\subseteq [V]_r$ forces $u_{n-3}u_n$ to be green. If $n = 7$, then wu_1, wu_4 and u_3u_6 have to be red as otherwise $\{w, u_7\} + \{u_1, u_2, u_3, u_4, u_5\}$ contains a green $K_{2,4}$ or $\{u_6, u_7\} + \{u_1, u_3, u_4, w\}$ is a green $K_{2,4}$. But this yields a red C_n, a contradiction. If $n \geq 8$, then wu_{n-4} has to be green if no red $B_{2,n-2}$ with bristles $u_{n-4}u_{n-3}$ and $u_{n-4}w$ shall occur. Hence $\{w, u_n\} + \{u_1, u_2, u_3, u_{n-4}, u_{n-2}\}$ contains a green $K_{2,4}$ or u_3u_n and wu_1 are red. But then we obtain a red $B_{2,n-2}$ with bristles u_1u_2 and u_1w, a contradiction. ■

Finally we determine $r(P_n, G)$ and $r(B_{2,n-2}, G)$ for all connected spanning subgraphs G of $K_{3,3}$.

Theorem 3.12. *Let $n \geq 4$ and let G be a connected graph of order six where $G \subseteq K_{3,3}$. Then*

$$r(P_n, G) = \begin{cases} 7 & \textit{if } n = 4, \\ n+2 & \textit{otherwise.} \end{cases}$$

Proof. By (1), $r(P_n, G) \geq n+2$. Moreover, Lemma 3.1 yields $r(P_4, G) \geq 7$. To establish equality it suffices to prove that $r(P_n, K_{3,3}) \leq n+2$ for $n \geq 5$. Consider any coloring of K_{n+2}, $n \geq 5$, not containing a red P_n. We have to show that a green $K_{3,3}$ occurs. Let $P_k = u_1 \cdots u_k$ be a red path of maximum length, $U = \{u_1, \ldots, u_k\}$ and $W = V \setminus U = \{w_1, w_2, \ldots, w_{n+2-k}\}$. In case of $k \leq 2$ either at most one red edge occurs or any two red edges are independent. This yields a green $K_6 - 2K_2$ containing a green $K_{3,3}$. Now let $k \geq 3$. All edges between $\{u_1, u_k\}$ and W have to be green. Since $k \leq n-1$, $|W| = n+2-k \geq 3$. If $q_g(u_i, W) \geq 3$ for some i with $2 \leq i \leq k-1$, then a green $K_{3,3}$ occurs. Otherwise $q_r(u_i, W) \geq 1$ for every i with $2 \leq i \leq k-1$, and we may assume that u_2w_1 is red. Since $P_{k+1} \not\subseteq [V]_r$, all edges incident to w_1 in $[W]$ have to be green. This produces a green $K_{3,3}$ if $|W| \geq 4$. The remaining case is $|W| = 3$ which implies $k = n-1 \geq 4$. Again we apply $P_{k+1} \not\subseteq [V]_r$. Thus, u_1u_k, u_1u_3 and u_3w_1 must be green. Moreover we may assume that u_3w_2 is red, and this forces u_2w_2 and u_2u_k to be green. If u_2w_3 is green, then we obtain the green $K_{3,3} = \{u_1, u_2, w_1\} + \{u_k, w_2, w_3\}$. If u_2w_3 is red, then w_2w_3 and u_3w_3 have to be green yielding the green $K_{3,3} = \{u_1, w_1, w_3\} + \{u_3, u_k, w_2\}$, and the proof is complete. ∎

Theorem 3.13. *Let $n \geq 5$ and let G be a connected graph of order six where $G \subseteq K_{3,3}$. Then*

$$r(B_{2,n-2}, G) = \begin{cases} n+3 & \textit{if } G = K_{3,3} \textit{ and } 5 \leq n \leq 6, \\ n+2 & \textit{otherwise.} \end{cases}$$

Proof. By (1), $r(B_{2,n-2}, G) \geq n+2$. Moreover, Lemma 3.2 yields $r(B_{2,3}, K_{3,3}) \geq 8$ and $r(B_{2,4}, K_{3,3}) \geq 9$. To establish equality we prove $r(B_{2,n-2}, K_{3,3}) \leq n+3$ as well as $r(B_{2,n-2}, K_{3,3}-e) \leq n+2$ for $n \geq 5$ and $r(B_{2,n-2}, K_{3,3}) \leq n+2$ for $n \geq 7$. Consider any coloring of K_m with $n+2 \leq m \leq n+3$ and $n \geq 5$ not containing a red $B_{2,n-2}$. Let $V = V(K_m)$. If a green $K_{3,3}$ occurs, then we are done. Otherwise Theorem 3.12 guarantees a red $P_n = u_1 \cdots u_n$. Let $U = \{u_1, \ldots, u_n\}$. $B_{2,n-2} \not\subseteq [V]_r$ forces only green edges between $\{u_2, u_3, u_{n-2}, u_{n-1}\}$ and $W = V \setminus U$. Hence $K_{3,3} \subseteq [V]_g$ in case of $m = n+3$, i.e., $|W| = 3$, a contradiction. It remains $m = n+2$. Let $W = \{w_1, w_2\}$. By Lemma 3.5, u_1u_3, u_1u_{n-1}, u_2u_n and $u_{n-2}u_n$

have to be green. Thus we find a green $K_{3,3} - e$ in $\{u_1, w_1, w_2\} + \{u_2, u_3, u_{n-1}\}$ proving that $r(B_{2,n-2}, K_{3,3} - e) \leq n + 2$ if $n \geq 5$. Now let $n \geq 7$. To avoid that $\{u_1, w_1, w_2\} + \{u_3, u_{n-2}, u_{n-1}\}$ or $\{w_1, w_2, u_n\} + \{u_2, u_3, u_{n-2}\}$ is a green $K_{3,3}$, u_1u_{n-2} and u_3u_n have to be red, and then $B_{2,n-2} \not\subseteq [V]_r$ implies that u_1w_1 and u_1w_2 are green. This forces u_1u_n to be red as otherwise $\{u_1, u_2, u_{n-2}\} + \{w_1, w_2, u_n\}$ is a green $K_{3,3}$. Consequently, since $B_{2,n-2} \not\subseteq [V]_r$, all edges between U and W have to be green. By Theorem 3.4, $r(B_{2,n-2}, S_4) = n$ for $n \geq 7$. But this implies a green S_4 in $[U]$ yielding a green $K_{3,3}$ together with w_1 and w_2, a contradiction, and we are done. ■

4. Concluding Remarks

Summarizing Theorems 2.1, 2.9, the results from [11] concerning non-bipartite graphs G and the results from [13] for $r(S_n, K_{1,1,4})$, we see that $r(T_n, G)$ has been determined for any tree T_n and all connected graphs $G \neq K_{2,2,2}$ of order six with $\chi(G) \geq 3$, except for $T_n = S_n$ in case of some small n and some G where $\chi(G) = 3$. The exact values of $r(S_n, G)$ are still missing in the following cases (the numbering of G corresponds to the numbering of the 112 connected graphs of order six used in [11]): $G = G_{100} = K_{1,2,3}$ with $n \in \{7, 9, 11\}$, $G = G_{94} = E_2 + (E_1 \cup P_3)$ with $n = 7$, $G = G_{92} = K_{3,3} + e$ with $6 \leq n \leq 12$, $G = G_{78} = E_2 + (E_2 \cup K_2)$ with $6 \leq n \leq 8$, $G = G_{60}$ and $G = G_{79}$ (the two graphs obtained from $K_{1,1,3}$ by joining an additional vertex to one or two of the three vertices of degree 2) with $n = 6$. In all these cases we know that the value of $r(S_n, G)$ differs by at most 2 from the lower bound given in (1). By a detailed case analysis, perhaps assisted by computer algorithms, it should be possible to determine the missing exact values.

To achieve significant progress in evaluating $r(T_n, K_{2,2,2})$ seems to be difficult, especially for trees T_n with maximum degree $\Delta(T_n)$ close to $n-1$, where we know that, for n sufficiently large, $r(T_n, K_{2,2,2})$ differs considerably from the lower bound $2n$ obtained from (1). In contrast, for some trees with small maximum degree as P_n and a special class of trees with $\Delta(T_n) = 3$, $r(T_n, K_{2,2,2})$ attains the bound $2n$ (see Theorems 2.10 and 2.11). It seems to be promising to study $r(T_n, K_{2,2,2})$ for further trees with small maximum degree, in particular it would be desirable to obtain a characterization of all $K_{2,2,2}$-good trees T_n.

As already explained, it seems to be extremely difficult to evaluate $r(T_n, G)$ for trees T_n with maximum degree $\Delta(T_n)$ close to $n-1$ and all connected bipartite graphs G of order six, i.e., all connected spanning subgraphs of K_{m_1,m_2} with $1 \leq m_1 \leq m_2$ and $m_1 + m_2 = 6$. If $\Delta(T_n)$ is small, then the situation is entirely different. For $T_n \in \{P_n, B_{2,n-2}\}$ we have shown that, except for small n, T_n is G-good for any connected bipartite graph G of order six, and there might be other trees T_n with small maximum degree where the general lower bound (1) is attained. Especially, by Theorems 3.3, 3.10 and 3.12, P_n is G-good if and only if $n \geq 2m_2 - 1$. This improves in a very special case a general result due to Pokrovskiy and Sudakov [16] who recently have shown that P_n is G-good for any graph G on $p(G)$ vertices if $n \geq 4p(G)$.

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