Discussiones Mathematicae
Graph Theory 41 (2021) 5–38
doi:10.7151/dmgt.2369

A SURVEY ON THE CYCLIC COLORING AND ITS RELAXATIONS

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Abstract

A cyclic coloring of a plane graph is a vertex coloring such that any two vertices incident with the same face receive distinct colors. This type of coloring was introduced more than fifty years ago, and a lot of research in chromatic graph theory was sparked by it. This paper is a survey on the state of the art concerning the cyclic coloring and relaxations of this graph invariant.

Keywords: plane graph, edge coloring, vertex coloring.

2010 Mathematics Subject Classification: 05C15, 05C10.

1. Introduction and Notations

A *plane graph* is a particular drawing of a planar graph in the Euclidean plane such that no edges intersect except at their endvertices (for details concerning embeddings of graphs into surfaces see [33]). If all vertices of a plane graph are incident with the outer face, then the graph is called *outerplane graph*. Let G be a connected plane graph with vertex set $V(G)$, edge set $E(G)$, and face set

$F(G)$. The *boundary* of a face f is the boundary in the usual topological sense. It can be partitioned into vertices and edges contained in the closure of f and then organized into a closed walk in G traversing along a simple closed curve lying just inside the face f. This closed walk is unique up to the choice of the initial vertex and the direction, and is called the *boundary walk* of the face f (see [33], p. 101). Let f be a face having the boundary walk $v_0v_1 \cdots v_{k-1}v_0$ such that $v_i \in V(G)$ and v_i is adjacent to v_{i+1}, $i = 0, 1, \ldots, k-1$, subscripts taken modulo k. A *facial path* of f is a subpath $v_m v_{m+1} \cdots v_n$ (subscripts taken modulo k) of the boundary walk of f (i.e., a facial path is any path which is a consecutive part of a boundary walk of a face).

The *size* of a face f is the number of edges incident with f. The *degree* of a face f is the number of vertices incident with f. We use $\Sigma(G)$ and $\Delta^*(G)$ to denote the maximum face size and the maximum face degree of G, respectively. Let $V(f)$ and $E(f)$ denote the set of vertices and the set of edges incident with f, respectively.

Let $\Delta(G)$ denote the maximum vertex degree of a graph G, and let $\delta(G)$ be the minimum vertex degree of G.

For a cycle C (in a plane graph G) we denote the sets of vertices and edges of G lying inside C and outside C by $\mathrm{Int}(C)$ and $\mathrm{Ext}(C)$, respectively. We say C is a *separating cycle* of G if both $\mathrm{Int}(C)$ and $\mathrm{Ext}(C)$ are not empty.

The girth $g(G)$ of a graph G (that is not acyclic) is the length of a shortest cycle of G.

A graph G is *k-connected* (*k-edge-connected*) if G has at least $k+1$ vertices and $G - S$ is connected for any $S \subseteq V(G)$ ($S \subseteq E(G)$) with $|S| \leq k-1$. A *bridge* (a *cut-vertex*) of G is an edge (a vertex) whose removal from G yields a graph having more components than G does. A graph which contains no bridge is said to be *bridgeless*.

An edge (or a vertex) *coloring* of a graph G is an assignment of colors to edges (or vertices) of G, one color per edge (per vertex). An edge (or a vertex) coloring c of G is *proper* if for any two adjacent edges (or vertices) x_1 and x_2 of G, $c(x_1) \neq c(x_2)$ holds.

A *simple graph* is a graph without loops and parallel edges. In a *multigraph* parallel edges are allowed but loops are forbidden. In a *pseudograph* both loops and parallel edges are allowed.

2. CYCLIC COLORING

A *cyclic coloring* of a plane graph is a coloring of its vertices such that any two vertices incident with the same face receive distinct colors. The minimum number of colors needed for a cyclic coloring of a plane graph G, the *cyclic* *chromatic number*, is denoted by $\chi_c(G)$. Evidently, any cyclic coloring must use at least as many colors as the maximum number of vertices incident to a face of the involved graph, i.e., $\chi_c(G) \geq \Delta^*(G)$. The concept was introduced by Ore and Plummer [60] in 1969. The authors studied pseudographs. First they observed that $\chi_c(G) = k \leq 2$ if and only if $|V(G)| = k$. Then they showed that it suffices to consider simple 2-connected plane graphs because of the following considerations: Assume that e_1 and e_2 are parallel edges. If the cycle C formed by e_1 and e_2 is not separating, then one of the edges e_1 and e_2 can be omitted without changing $\chi_c(G)$. If C is separating, then $\chi_c(G) = \max\{\chi_c(G - \mathrm{Int}(C)), \chi_c(G - \mathrm{Ext}(C))\}$. Similar arguments apply when G contains a loop. Now assume that G has a cut-vertex v. Let G_1 and G_2 be two subgraphs obtained by separating G along v (i.e., $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{v\}$); note that G_1 and G_2 are not necessarily unique. G_1 has a face f_1 and G_2 has a face f_2 whose boundaries together form the boundary of f in G. In this case $\chi_c(G) = \max\{\chi_c(G_1), \chi_c(G_2), |V(f)|\}$.

It follows that in order to be able to find an upper bound for the cyclic chromatic number of a plane pseudograph it is sufficient to be able to find such a bound for a plane graph. Moreover, since above $|V(f)| \leq \Delta^*(G)$, it is useful to realize that when proving an upper bound for the cyclic chromatic number of a (connected) plane graph it suffices to work with 2-connected plane graphs. Therefore, in discussing cyclic coloring, we restrict our attention to simple graphs.

Ore and Plummer [60] proved that any plane graph has a cyclic coloring with at most $2\Delta^*(G)$ colors.

Theorem 2.1 [60]**.** *If $d_1 \geq d_2$ are the two largest face degrees in a plane graph G, then*

$$\chi_c(G) \leq d_1 + d_2 \leq 2\Delta^*(G).$$

There was no progress in this area during next 18 years, but many results were produced after 1987, when Plummer and Toft [61] proved that the upper bound $2\Delta^*(G)$ can be improved to $\Delta^*(G) + 9$ when G is 3-connected.

Theorem 2.2 [61]**.** *If G is a 3-connected plane graph, then*

$$\chi_c(G) \leq \Delta^*(G) + 9.$$

They also showed that if $\Delta^*(G)$ is sufficiently large or sufficiently small, then the bound from Theorem 2.2 can be improved.

Theorem 2.3 [61]**.** *If G is a 3-connected plane graph, then*

$$\chi_c(G) \leq \begin{cases} \Delta^*(G) + 8 & \text{for} \quad \Delta^*(G) \leq 10, \\ \Delta^*(G) + 7 & \text{for} \quad \Delta^*(G) \leq 9, \\ \Delta^*(G) + 6 & \text{for} \quad \Delta^*(G) \leq 8. \end{cases}$$

Theorem 2.4 [61]. *If G is a 3-connected plane graph, then*

$$\chi_c(G) \leq \begin{cases} \Delta^*(G)+8 & \textit{for} \quad \Delta^*(G) \geq 14, \\ \Delta^*(G)+7 & \textit{for} \quad \Delta^*(G) \geq 15, \\ \Delta^*(G)+6 & \textit{for} \quad \Delta^*(G) \geq 18, \\ \Delta^*(G)+5 & \textit{for} \quad \Delta^*(G) \geq 24, \\ \Delta^*(G)+4 & \textit{for} \quad \Delta^*(G) \geq 42. \end{cases}$$

Plummer and Toft [61] stated the following conjecture:

Conjecture 2.5 [61]. *If G is a 3-connected plane graph, then*

$$\chi_c(G) \leq \Delta^*(G)+2.$$

Moreover, they constructed an infinite family of 3-connected plane graphs for which this bound is attained, see Figure 1.

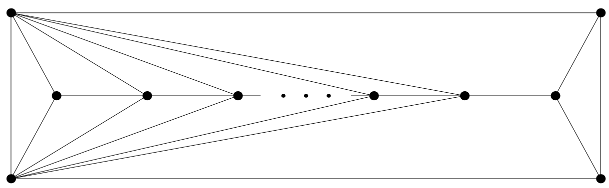

Figure 1. A 3-connected plane graph G with $\chi_c(G) = \Delta^*(G)+2$, [61].

The second infinite family of plane graphs depicted in Figure 2 shows that 3-connectivity in Conjecture 2.5 cannot be replaced by minimum degree three. In fact, given k there is even a 2-connected plane graph G with $\chi_c(G) > \Delta^*(G)+k$.

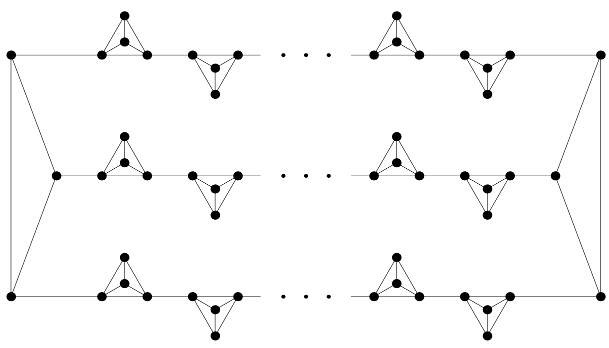

Figure 2. A 2-connected plane graph G with $\delta(G) = 3$ and $\chi_c(G) > \Delta^*(G)+k$, [61].

Another observation in [61] is that $\chi_c(G) = \frac{3}{2}\Delta^*(G)$ for the 2-connected plane graph G shown in Figure 3. Note that no plane graph G is known with $\chi_c(G) > \frac{3}{2}\Delta^*(G)$.

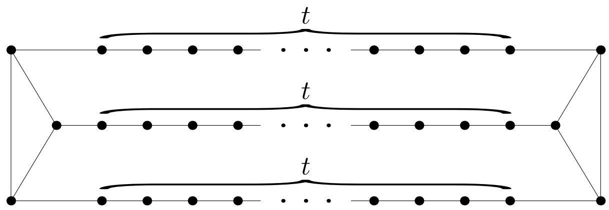

Figure 3. A 2-connected plane graph G with $\chi_c(G) = \frac{3}{2}\Delta^*(G)$, [61].

The general upper bound $2\Delta^*(G)$ (obtained by Ore and Plummer [60]) was first improved to $2\Delta^*(G) - 3$ (for $\Delta^*(G) \geq 8$) by Borodin [9] in 1992.

Theorem 2.6 [9]**.** *If G is a plane graph, then*

$$\chi_c(G) \leq \begin{cases} 2\Delta^*(G) - 3 & \textit{for} \quad \Delta^*(G) \geq 8, \\ 12 & \textit{for} \quad \Delta^*(G) \leq 7, \\ 11 & \textit{for} \quad \Delta^*(G) \leq 6, \\ 9 & \textit{for} \quad \Delta^*(G) \leq 5. \end{cases}$$

Four years later, in 1996, Borodin [10] and Horňák and Jendroľ [40] improved the upper bounds (obtained by Plummer and Toft [61]) for 3-connected plane graphs.

Theorem 2.7 [10]**.** *If G is a 3-connected plane graph, then*

$$\chi_c(G) \leq \begin{cases} 21 & \textit{for} \quad \Delta^*(G) \leq 16, \\ \Delta^*(G) + 5 & \textit{for} \quad \Delta^*(G) \geq 17, \\ \Delta^*(G) + 4 & \textit{for} \quad \Delta^*(G) \geq 19, \\ \Delta^*(G) + 3 & \textit{for} \quad \Delta^*(G) \geq 24. \end{cases}$$

Theorem 2.8 [40]**.** *If G is a 3-connected plane graph, then*

$$\chi_c(G) \leq \begin{cases} 19 & \textit{for} \quad \Delta^*(G) \leq 11, \\ 20 & \textit{for} \quad \Delta^*(G) \leq 12, \\ 21 & \textit{for} \quad \Delta^*(G) \leq 16, \\ \Delta^*(G) + 5 & \textit{for} \quad \Delta^*(G) \geq 17, \\ \Delta^*(G) + 4 & \textit{for} \quad \Delta^*(G) \geq 19, \\ \Delta^*(G) + 3 & \textit{for} \quad \Delta^*(G) \geq 24. \end{cases}$$

A significant progress has been made in years 1999–2002. Horňák and Jendroľ [38, 39] proved that Conjecture 2.5 holds for $\Delta^*(G) \geq 24$.

Theorem 2.9 [38, 39]**.** *If G is a 3-connected plane graph, then*

$$\chi_c(G) \leq \Delta^*(G) + 2 \quad \textit{for} \quad \Delta^*(G) \geq 24.$$

Borodin and Woodall [14] obtained even stronger result.

Theorem 2.10 [14]. *If G is a 3-connected plane graph, then*

$$\chi_c(G) \leq \Delta^*(G) + 1 \quad \textit{for} \quad \Delta^*(G) \geq 122.$$

This was improved by Enomoto, Horňák and Jendroľ [29].

Theorem 2.11 [29]. *If G is a 3-connected plane graph, then*

$$\chi_c(G) \leq \Delta^*(G) + 1 \quad \textit{for} \quad \Delta^*(G) \geq 60.$$

As the wheels show, we may not expect a bound for $\chi_c(G)$ better than $\Delta^*(G) + 1$ in the case of 3-connected plane graphs.

For general plane graphs, Borodin, Sanders and Zhao [15] showed the following.

Theorem 2.12 [15]. *If G is a plane graph, then*

$$\chi_c(G) \leq \left\lfloor \frac{9}{5}\Delta^*(G) \right\rfloor.$$

This bound was further improved by Sanders and Zhao [63].

Theorem 2.13 [63]. *If G is a plane graph, then*

$$\chi_c(G) \leq \left\lceil \frac{5}{3}\Delta^*(G) \right\rceil.$$

This is currently the best known general upper bound.

Borodin, Sanders and Zhao [15] also proved that $\chi_c(G) \leq 8$ for $\Delta^*(G) = 5$.

Four years later, Kriesell [50] showed that Conjecture 2.5 holds for locally connected 3-connected plane graphs. A graph is called *locally connected*, if the neighborhood of every vertex induces a connected subgraph.

Theorem 2.14 [50]. *If G is a locally connected 3-connected plane graph, then*

$$\chi_c(G) \leq \Delta^*(G) + 2.$$

He posed the following conjecture.

Conjecture 2.15 [50]. *If G is a locally connected 3-connected plane graph, then*

$$\chi_c(G) \leq \Delta^*(G) + 1.$$

In 2007, Borodin *et al.* [12] proved an upper bound for the cyclic chromatic number that depends on Δ^* and the following easily computable parameter of a graph. In a plane graph G, let k^*_G be the maximum number of vertices that two faces of G can have in common, i.e., $k^*_G = \max\{|V(f_1) \cap V(f_2)| : f_1, f_2 \in F(G), f_1 \neq f_2\}$. The following result was obtained.

Theorem 2.16 [12]. *If G is a plane graph, then*

$$\chi_c(G) \leq \max\left\{\Delta^*(G) + 3k^*_G + 2, \Delta^*(G) + 14, 3k^*_G + 6, 18\right\}.$$

Besides that, a challenging conjecture was proposed.

Conjecture 2.17 [12]. *If G is a plane graph and k^*_G is sufficiently large, then*

$$\chi_c(G) \leq \Delta^*(G) + k^*_G.$$

In 1996, it was known that every 3-connected plane graph admits a cyclic coloring with at most $\Delta^* + 8$ colors, see Theorem 2.3 and Theorem 2.8. This general upper bound (with no restriction on Δ^*) was improved in 2009 by Enomoto and Horňák [28].

Theorem 2.18 [28]. *If G is a 3-connected plane graph, then*

$$\chi_c(G) \leq \Delta^*(G) + 5.$$

This is the best general upper bound known so far for 3-connected plane graphs.

In the next year, Horňák and Zlámalová [41] showed that Conjecture 2.5 holds for $\Delta^* \geq 18$.

Theorem 2.19 [41]. *If G is a 3-connected plane graph, then*

$$\chi_c(G) \leq \Delta^*(G) + 2 \quad \textit{for} \quad \Delta^*(G) \geq 18.$$

They posed a stronger version of Conjecture 2.5.

Conjecture 2.20 [41]. *If G is a 3-connected plane graph with $\Delta^*(G) \neq 4$, then*

$$\chi_c(G) \leq \Delta^*(G) + 1.$$

Zlámalová [81] proved the validity of Conjecture 2.5 for some special classes of plane graphs.

Theorem 2.21 [81]. *If G is a 3-connected plane graph with $\delta(G) = 4$ and $\Delta^*(G) \geq 6$ or $\delta(G) = 5$, then*

$$\chi_c(G) \leq \Delta^*(G) + 2.$$

Azarija *et al.* [6] proved the upper bound $\Delta^* + 1$ for plane graphs having property that the faces of size at least four are in a sense far from each other.

Theorem 2.22 [6]**.** *If in a plane graph G all faces of size four or more are vertex disjoint, then*

$$\chi_c(G) \leq \Delta^*(G) + 1.$$

As we noted, no plane graph G is known with $\chi(G) > \frac{3}{2}\Delta^*(G)$. Already Borodin [11] (in 1984) has conjectured (implicitly) that no such graph exists.

Conjecture 2.23 [11]**.** *If G is a plane graph with $\Delta^*(G) \geq 3$, then*

$$\chi_c(G) \leq \left\lfloor \frac{3}{2}\Delta^*(G) \right\rfloor.$$

This conjecture is known as the Cyclic Coloring Conjecture; notice that the assumption $\Delta^*(G) \geq 3$ is only to avoid trivialities. In 2013, Amini, Esperet and van den Heuvel [1] showed that the Cyclic Coloring Conjecture is asymptotically true.

Theorem 2.24 [1]**.** *For every $\varepsilon > 0$, there exists Δ_ε such that every plane graph of maximum face degree $\Delta^* \geq \Delta_\varepsilon$ admits a cyclic coloring with at most $\left(\frac{3}{2}+\varepsilon\right)\Delta^*$ colors.*

The Cyclic Coloring Conjecture (Conjecture 2.23) was proven only for three values of Δ^*. In the case $\Delta^* = 3$ the result follows from the fact that every planar graph has a proper vertex coloring with at most four colors (Four Color Theorem, see [3–5, 62]).

Theorem 2.25 [3–5, 62]**.** *If G is a plane graph with $\Delta^*(G) = 3$, then*

$$\chi_c(G) \leq \left\lfloor \frac{3}{2}\Delta^*(G) \right\rfloor = \Delta^*(G) + 1 = 4.$$

In the case $\Delta^* = 4$ the conjecture follows from the fact that every 1-planar graph admits a proper vertex coloring with at most six colors, see [8, 11]. A graph is called 1-*planar* if it admits a drawing in the plane such that each edge is crossed at most once.

Theorem 2.26 [8, 11]**.** *If G is a plane graph with $\Delta^*(G) = 4$, then*

$$\chi_c(G) \leq \left\lfloor \frac{3}{2}\Delta^*(G) \right\rfloor = \Delta^*(G) + 2 = 6.$$

The case $\Delta^* = 6$ was proven by Hebdige and Kráľ [37] in 2016.

Theorem 2.27 [37]. *If G is a plane graph with $\Delta^*(G) = 6$, then*

$$\chi_c(G) \le \left\lfloor \frac{3}{2}\Delta^*(G) \right\rfloor = 9.$$

In addition to the aforementioned articles, there are two manuscripts dealing with cyclic coloring of plane graphs.

In [26], Dvořák *et al.* proved that Conjecture 2.5 holds for $\Delta^* = 16$ and $\Delta^* = 17$.

Theorem 2.28 [26]. *If G is a 3-connected plane graph with $\Delta^*(G) \in \{16, 17\}$, then*

$$\chi_c(G) \le \Delta^*(G) + 2.$$

So Conjecture 2.5 is open only for $\Delta^* \in \{5, 6, \dots, 15\}$ (see Theorems 2.19, 2.25, 2.26, 2.28).

In a plane graph G, a *subdivision* of an edge uv is the operation of replacing uv by a path of length two. Any graph derived from a graph G by a sequence of edge subdivisions is called a *subdivision* of G. A *regular subdivision* of G is a graph obtained from G by replacing each edge of G by a path of length k for some constant $k \ge 1$.

In [46], Jendroľ and Soták showed that the Cyclic Coloring Conjecture holds if and only if Conjecture 2.29 holds.

Conjecture 2.29 [46]. *If G is subdivision of a 3-connected simple plane graph, then*

$$\chi_c(G) \le \left\lfloor \frac{3}{2}\Delta^*(G) \right\rfloor.$$

They proved the following upper bound for subdivisions.

Theorem 2.30 [46]. *If G is a subdivision of a 3-connected plane graph R, then*

$$\chi_c(G) \le \left\lfloor \frac{3}{2} \max_{f \in F(G)} \left\{ \deg_G(f) - \deg_R(f') \right\} \right\rfloor + \chi_c(R),$$

where f' is the face of R corresponding to the face f of G. Moreover, the bound is tight.

For a plane graph G let $t(G)$ denote the number of vertices of a longest path in G induced by vertices of degree two.

Theorem 2.31 [46]. *If G is a subdivision of a 3-connected simple plane graph R, then*

$$\chi_c(G) \le \max_{f \in F(G)} \left\{ \deg_G(f) - \deg_R(f') \right\} + t(G) + \chi_c(R),$$

where f' is the face of R corresponding to the face f of G. Moreover, the bound is tight.

Applying Theorems 2.16, 2.30, and 2.31 one can easily show (see [46]) that Conjecture 2.29 (and so Conjecture 2.23) holds for subdivisions G of 3-connected simple plane graphs R with $\Delta^*(G) \geq \max\{6t(G)+16, 28\}$ or $\Delta^*(G) \geq 2\chi_{\mathrm{c}}(R) + 2t(G) - 6$, for subdivisions of 3-connected simple plane triangulations, for subdivisions of 3-connected simple plane quadrangulations, for subdivisions of 3-connected simple plane pentagulations with even maximum face degree, and for regular subdivisions of 3-connected simple plane graphs R with $\Delta^*(R) \geq 16$.

Jendroľ and Soták [46] posed the following generalized conjecture of Plummer and Toft [61], which if true is best possible.

Conjecture 2.32 [46]**.** *If G is a subdivision of a 3-connected simple plane graph, then*

$$\chi_{\mathrm{c}}(G) \leq \Delta^*(G) + t(G) + 2.$$

The 3-sided prism and its subdivision depicted in Figure 3 show tightness of Theorem 2.30, Theorem 2.31, and Conjecture 2.32.

The authors of Conjecture 2.32 proved that it holds for subdivisions of 3-connected simple plane triangulations, for subdivisions of 3-connected simple plane quadrangulations, and for regular subdivisions of 3-connected simple plane graphs R with $\Delta^*(R) \geq 16$.

It looks like cyclic coloring of plane graphs will remain an active area of research for a long time.

3. Facial Rainbow Coloring

The Cyclic Coloring Conjecture stimulated a lot of research, in particular, several restrictions and generalizations of the conjecture have been considered. A vertex coloring of a plane graph G is a *facial rainbow coloring* if any two distinct vertices of G connected by a facial path have distinct colors. The minimum number of colors needed for a facial rainbow coloring of G, the *facial rainbow number*, is denoted by $\mathrm{rb}(G)$.

This type of coloring was introduced in 2017 by Jendroľ and Kekeňáková [45]. Observe that if G is a 2-connected plane graph, then $\mathrm{rb}(G) = \chi_{\mathrm{c}}(G)$. In general, these two types of colorings differ. For example, for the star $G = K_{1,r}$, $r \geq 3$, we have $\mathrm{rb}(G) = 3$ if r is even, and $\mathrm{rb}(G) = 4$ if r is odd, while $\chi_{\mathrm{c}}(G) = r + 1$.

The following four theorems were proved by Jendroľ and Kekeňáková [44,45]. Let $L(G)$ denote the order (i.e., the number of vertices) of the longest facial path in a plane graph G. Trivially, $\mathrm{rb}(G) \geq L(G)$.

Theorem 3.1 [44]**.** *If G is a simple plane graph, then*

$$\mathrm{rb}(G) \leq \left\lceil \frac{5}{3} L(G) \right\rceil.$$

Theorem 3.2 [44]**.** *If T is a plane tree, then*

$$\mathrm{rb}(T) \le \left\lfloor \frac{3}{2} L(T) \right\rfloor .$$

Moreover, the bound is tight.

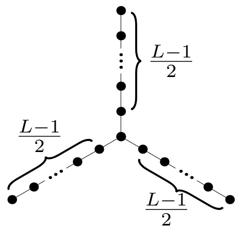

Figure 4. An example of a tree with $\mathrm{rb}(T) = \left\lfloor \frac{3}{2} L(T) \right\rfloor$ (for odd L).

For plane trees without vertices of degree two stronger results are available.

Theorem 3.3 [45]**.** *If T is a plane tree having no vertices of degree two, then*

$$\mathrm{rb}(T) \le \begin{cases} L(T)+1 & \text{for} \quad L(T) \ge 60, \\ L(T)+2 & \text{for} \quad L(T) \ge 16, \\ L(T)+5 & \text{for} \quad L(T) \ge 12. \end{cases}$$

Theorem 3.4 [44]**.** *For every $\varepsilon > 0$, there exists a constant L_ε such that every simple plane graph G with $L(G) \ge L_\varepsilon$ admits a facial rainbow coloring with $\left(\frac{3}{2} + \varepsilon\right) L(G)$ colors.*

The following conjecture is open.

Conjecture 3.5 [44]**.** *If G is a simple plane graph, then*

$$\mathrm{rb}(G) \le \left\lfloor \frac{3}{2} L(G) \right\rfloor .$$

4. ℓ-FACIAL COLORING

An *ℓ-facial coloring* of a plane graph G is a coloring of its vertices such that any two distinct vertices that lie on the same face and are at distance at most ℓ on that face (i.e., there exists a facial walk between them having at most ℓ edges) receive distinct colors. This type of coloring was introduced in 2005 by Kráľ, Madaras and Škrekovski [48, 49]. If $\Delta^*(G) \le 2\ell + 1$, then any cyclic coloring of G is an ℓ-facial coloring and, moreover, if G is 2-connected, then any ℓ-facial coloring of G is a cyclic coloring.

We denote the minimum number of colors needed for an ℓ-facial coloring of G by $\chi_\ell(G)$. It is easy to see that any upper bound for χ_ℓ in the class of simple connected plane graphs holds also for all plane graphs. Since $\chi_c(G) \geq \Delta^*(G)$, an arbitrary upper bound for $\chi_c(G)$ must somehow depend on $\Delta^*(G)$. On the other hand, this is not true concerning upper bounds for $\chi_\ell(G)$. Hence the concept of ℓ-facial colorings may be viewed as an extension of the concept of cyclic colorings conveniently tractable without restrictions imposed on $\Delta^*(G)$.

Kráľ, Madaras and Škrekovski [48, 49] obtained the following upper bounds.

Theorem 4.1 [48, 49]**.** *If G is a plane graph, then*

$$\chi_\ell(G) \leq \begin{cases} \left\lfloor \frac{18\ell}{5} \right\rfloor + 2 & \text{for} \quad \ell \geq 5, \\ 15 & \text{for} \quad \ell = 4, \\ 12 & \text{for} \quad \ell = 3, \\ 8 & \text{for} \quad \ell = 2. \end{cases}$$

The following conjecture was proposed.

Conjecture 4.2 [48]**.** *If G is a plane graph and $\ell \geq 1$, then*

$$\chi_\ell(G) \leq 3\ell + 1.$$

This conjecture is known as the $(3\ell+1)$-Conjecture (or Facial Coloring Conjecture). Note that the bound offered by the $(3\ell+1)$-Conjecture is tight: as shown by Figure 5, for every $\ell \geq 1$, there exists a plane graph that has no ℓ-facial coloring with 3ℓ colors.

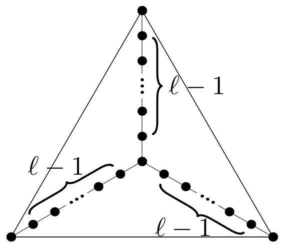

Figure 5. An example of a plane graph with $3\ell+1$ vertices and $\chi_\ell(G) = 3\ell+1$.

Observe that the $(3\ell+1)$-Conjecture implies the Cyclic Coloring Conjecture for all odd values of Δ^*. The $(3\ell+1)$-Conjecture is for $\ell = 1$ equivalent to the Four Color Theorem.

In 2006, Montassier and Raspaud [56] studied 2-facial coloring of certain families of plane graphs. They obtained the following results.

Theorem 4.3 [56]**.** *Every outerplane graph admits a* 2*-facial coloring using* 5 *colors.*

This result is best possible because the cycle on five vertices needs five colors.

Theorem 4.4 [56]**.** *Every* K_4*-minor free plane graph admits a* 2*-facial coloring using* 6 *colors.*

This result is also best possible: the graph formed by the cycle $v_1v_2v_3v_4v_5v_1$ and the path $v_1v_6v_3$ needs six colors.

Theorem 4.5 [56]**.** *Every plane graph* G *with girth* $g \geq 14$ $(10, 8,$ *respectively*$)$ *admits a* 2*-facial coloring using* 5 *colors* $(6, 7,$ *respectively*$)$*.*

In 2008, Havet, Sereni and Škrekovski [35] showed that the bound in Theorem 4.1 can be decreased by 1 for $\ell = 3$.

Theorem 4.6 [35]**.** *Every plane graph* G *admits a* 3*-facial coloring using* 11 *colors.*

Theorem 4.6 has a nice corollary.

Corollary 4.7 [35]**.** *If* G *is a plane graph with* $\Delta^*(G) = 7$*, then*

$$\chi_{\mathrm{c}}(G) \leq 11.$$

This bound is just one higher than that proposed by the Cyclic Coloring Conjecture.

Dvořák, Škrekovski and Tancer [27] posed the following so called 3ℓ-Conjecture.

Conjecture 4.8 [27]**.** *If* G *is a triangle-free plane graph and* $\ell \geq 1$*, then*

$$\chi_\ell(G) \leq 3\ell.$$

For $\ell = 1$ this statement is equivalent to Grötzsch's theorem [34], which states that every triangle-free planar graph admits a proper vertex coloring with at most three colors. The bound in this conjecture is tight, as shown by graphs depicted in Figure 6.

In 2010, Havet *et al.* [36] improved Theorem 4.1 for $\ell \geq 49$ and $\ell \in \{45, 47\}$.

Theorem 4.9 [36]**.** *If* G *is a plane graph and* $\ell \geq 1$*, then*

$$\chi_\ell(G) \leq \left\lfloor \frac{7\ell}{2} \right\rfloor + 6.$$

Two years later, Borodin and Ivanova [13] improved one case of Theorem 4.5.

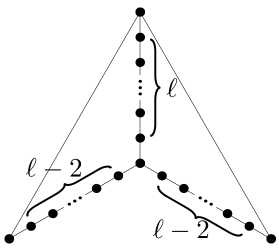

Figure 6. An example of a triangle-free plane graph with 3ℓ vertices and $\chi_\ell(G) = 3\ell$, $\ell \geq 2$.

Theorem 4.10 [13]**.** *Every plane graph G with girth $g \geq 12$ admits a 2-facial coloring using 5 colors.*

In 2018, Thomassen [68] proved that the square of any subcubic plane graph admits a proper vertex coloring with at most seven colors. This result implies that $(3\ell + 1)$-Conjecture with $\ell = 2$ holds for subcubic plane graphs.

Theorem 4.11 [68]**.** *Every subcubic plane graph G admits a 2-facial coloring using 7 colors.*

5. Odd Colorings

Let φ be a vertex coloring of a connected plane graph G. We say that a face f of G uses a color c (under the coloring φ) k times if this color appears k times in the sequence of colors of vertices of the boundary walk of f. Observe that if f is incident with a cut-vertex v, then v may occur more than once on the boundary walk of f, see Figure 7.

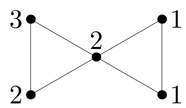

Figure 7. A vertex coloring of a plane graph.

The outer face of the graph depicted in Figure 7 uses the color 1 twice, the color 2 three times (note that the cut-vertex appears twice in the boundary walk of the outer face), and the color 3 only once.

If each face of G uses at least one color an odd number of times, then φ is a *weak odd coloring*. If for each face f and each color c, the face f uses the color c an odd number of times or does not use it at all, then φ is a *strong odd coloring*. Finally, a proper strong odd coloring is a *proper odd coloring*.

The problem is to determine the minimum number of colors $\chi_{\rm wo}(G)$ ($\chi_{\rm so}(G)$, $\chi_{\rm po}(G)$, respectively) used in a weak (strong, proper, respectively) odd coloring of a connected plane graph G.

The numbers $\chi_{\rm so}(G)$ and $\chi_{\rm po}(G)$ are correctly defined for 2-connected plane graphs, since any coloring of G using $|V(G)|$ colors is proper odd one. However, there are connected plane graphs that are not 2-connected and admit no strong (proper) odd coloring. One of such graphs is depicted in Figure 7.

Observe that any proper odd coloring of a 2-connected plane graph G with $\Delta^*(G) \leq 5$ is also a cyclic coloring. On the other hand any cyclic coloring of a 2-connected plane graph is a proper odd coloring.

Odd colorings were introduced in 2009 by Czap and Jendroľ [20]. The first result in this area was the following theorem.

Theorem 5.1 [20]**.** *If G is a connected loopless plane graph with minimum face size at least* 3*, then*

$$\chi_{\rm wo}(G) \leq 4.$$

Czap and Jendroľ [20] posed the following conjecture.

Conjecture 5.2 [20]**.** *If G is a connected loopless plane graph with minimum face size at least* 3*, then*

$$\chi_{\rm wo}(G) \leq 3.$$

The restriction to plane graphs with minimum face size at least 3 in this conjecture is essential. Consider a plane graph G with chromatic number four. Now add one parallel edge for each edge of G in order to obtain a plane graph H with minimum face size 2. Observe that every weak odd coloring of H is a proper coloring. Consequently, $\chi_{\rm wo}(H) = 4$.

The following result supports Conjecture 5.2.

Theorem 5.3 [20]**.** *If G is a* 2*-connected cubic plane graph, then*

$$\chi_{\rm wo}(G) \leq 3.$$

Moreover, the bound is tight.

In [20], 2-connected plane graphs with $\chi_{\rm so}(G) \geq 6$ were constructed and the following conjecture was proposed.

Conjecture 5.4 [20]**.** *There is a constant K such that for every* 2*-connected plane graph G*

$$\chi_{\rm so}(G) \leq K.$$

In 2011, Czap, Jendroľ and Voigt [24] showed that such a constant does exist.

Theorem 5.5 [24]**.** *If G is a 2-connected plane graph, then*

$$\chi_{\mathrm{so}}(G) \leq \chi_{\mathrm{po}}(G) \leq 118.$$

This upper bound was improved for 3-connected plane graphs having property that the faces of a certain size are in a sense far from each other by Czap, Jendroľ and Kardoš [22]. We write $v \in f$ if a vertex v is incident with a face f. Two distinct faces f and g *touch* each other, if there is a vertex v such that $v \in f$ and $v \in g$. Two distinct faces f and g *influence* each other, if they touch, or there is a face h such that h touches both f and g. We say that a face f of size i is *isolated* if there is no face g of size at least i touching f.

Theorem 5.6 [22]**.** *If G is a 3-connected plane graph in which the faces of size at least i pairwise do not influence each other, then*

$$\chi_{\mathrm{po}}(G) \leq \begin{cases} 6 & \text{for} \quad i = 4, \\ 8 & \text{for} \quad i = 5, \\ 10 & \text{for} \quad i = 6. \end{cases}$$

Theorem 5.7 [22]**.** *If G is a 3-connected plane graph such that any face of size at least i is isolated, then*

$$\chi_{\mathrm{po}}(G) \leq \begin{cases} 12 & \text{for} \quad i = 4, \\ 18 & \text{for} \quad i = 5, \\ 28 & \text{for} \quad i = 6. \end{cases}$$

In [19], Czap investigated proper odd colorings of 2-connected outerplane graphs.

Theorem 5.8 [19]**.** *If G is a 2-connected outerplane graph, then*

$$\chi_{\mathrm{po}}(G) \leq 12.$$

Theorem 5.9 [19]**.** *If G is a 2-connected bipartite outerplane graph, then*

$$\chi_{\mathrm{po}}(G) \leq 8.$$

Moreover, this bound is tight.

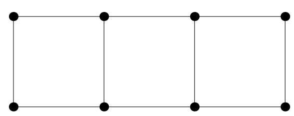

Figure 8. An example of an outerplane graph with $\chi_{\mathrm{po}}(G) = 8$.

In 2012, Wang, Finbow and Wang [72] improved the bound for outerplane graphs.

Theorem 5.10 [72]**.** *If G is a 2-connected outerplane graph, then $\chi_{\rm po}(G) \leq 10$. Moreover, $\chi_{\rm po}(G) \leq 9$ if and only if G is different from H_0 and H_1 depicted in Figure 9.*

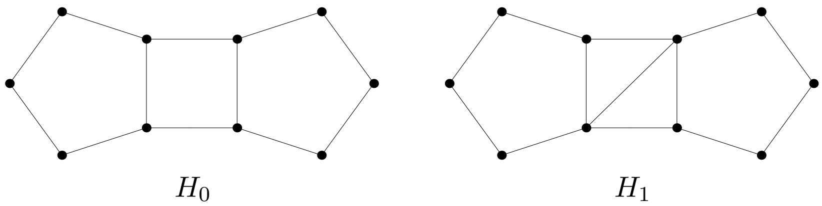

Figure 9. The graphs H_0 and H_1.

Theorem 5.11 [72]**.** *If G is a 2-connected outerplane graph, different from H_0 and H_1, and $|V(G)|$ is even, then*

$$\chi_{\rm po}(G) \leq 8.$$

Another improvement was obtained for Theorem 5.9. Let $\mathcal{G}$ denote the set of 2-connected outerplane graphs each of which has exactly three inner faces, and the degree of each end-face of G is divisible by four and the degree of the face which is not an end-face is four.

Theorem 5.12 [72]**.** *If G is a 2-connected bipartite outerplane graph, then $\chi_{\rm po}(G) = 8$ if and only if $G \in \mathcal{G}$.*

The best known general upper bound is due to Kaiser *et al.* [47].

Theorem 5.13 [47]**.** *If G is a 2-connected plane graph, then*

$$\chi_{\rm so}(G) \leq \chi_{\rm po}(G) \leq 97.$$

In 2016, Fabrici and Göring [30] proved a modified version of Conjecture 5.2.

Theorem 5.14 [30]**.** *Every simple plane graph has a vertex coloring with colors black, blue and red such that*

(1) *each face is incident with at most one red vertex, and*

(2) *each face that is not incident with a red vertex is incident with exactly one blue vertex.*

Motivated by Theorem 5.14, we can define strong and proper odd colorings for connected plane graphs (that are not necessarily 2-connected) in the following way. A strong (proper) odd coloring of a connected plane graph is a (proper) vertex coloring such that every face is incident with zero or an odd number of vertices of each color. Observe that this definition is equivalent to the original one for 2-connected plane graphs. Now $\chi_{\mathrm{so}}(G)$ and $\chi_{\mathrm{po}}(G)$ are correctly defined for arbitrary plane graph G.

The last result concerning odd colorings is from 2020. Štorgel [66] proved that there exists an infinite family of 2-connected plane graphs G with $\chi_{\mathrm{po}}(G) = 12$.

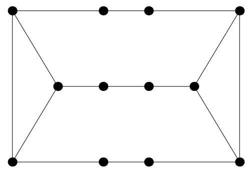

Figure 10. An example of a graph with $\chi_{\mathrm{po}}(G) = 12$, [66].

6. Unique-Maximum Colorings

In a coloring of a graph we can use integers instead of colors. A *unique-maximum* k-coloring with respect to faces of a plane graph G is a coloring with "colors" $1, 2, \ldots, k$ such that, for each face f of G, the maximum color occurs exactly once on the vertices of f. The minimum k for which G has a unique-maximum (proper unique-maximum) k-coloring is denoted $\chi_{\mathrm{um}}(G)$ ($\chi_{\mathrm{pum}}(G)$, respectively).

Theorem 5.14 can be reformulated in the following way (red = 3, blue = 2, and black = 1).

Theorem 6.1 [30]. *If G is a simple plane graph, then*

$$\chi_{\mathrm{um}}(G) \leq 3.$$

Using the proof of Theorem 6.1 and the Four Color Theorem, Fabrici and Göring obtained the following upper bound for $\chi_{\mathrm{pum}}(G)$.

Theorem 6.2 [30]. *If G is a simple plane graph, then*

$$\chi_{\mathrm{pum}}(G) \leq 6.$$

They posed the following conjecture which is a strengthening of the Four Color Theorem.

Conjecture 6.3 [30]. *If G is a simple plane graph, then*

$$\chi_{\mathrm{pum}}(G) \leq 4.$$

Promptly, this coloring was considered by others. Wendland [74] decreased the upper bound to 5.

Theorem 6.4 [74]**.** *If G is a loopless plane graph without 2-faces, then*

$$\chi_{\text{pum}}(G) \leq 5.$$

Andova *et al.* [2] showed that Conjecture 6.3 holds for three classes of plane graphs.

Theorem 6.5 [2]**.** *If G is a simple plane subcubic graph, an outerplane graph, or a plane quadrangulation, then*

$$\chi_{\text{pum}}(G) \leq 4.$$

Moreover, the bound is tight.

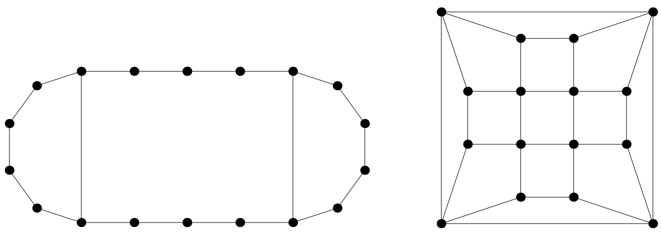

Figure 11. Plane graphs that show tightness of the upper bound, [2].

Conjecture 6.3 was disproved in 2018 by Lidický, Messerschmidt and Škrekovski [51].

Theorem 6.6 [51]**.** *There exists a plane graph G with*

$$\chi_{\text{pum}}(G) = 5.$$

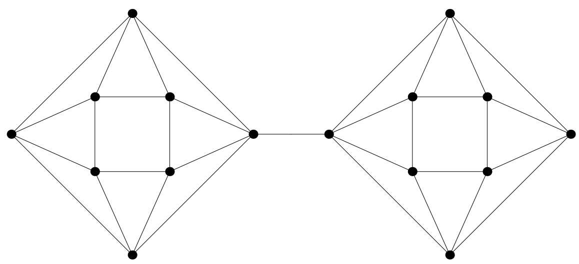

Figure 12. A counterexample to Conjecture 6.3, [51].

They introduced a variation of Conjecture 6.3 with maximum degree and connectivity conditions added.

Conjecture 6.7 [51]**.** *If G is a simple connected plane graph with maximum degree 4, then*

$$\chi_{\text{pum}}(G) \le 4.$$

Note that the counterexample to Conjecture 6.3 depicted in Figure 12 has maximum degree five, and Conjecture 6.3 is true for plane graphs with maximum degree three (see Theorem 6.5).

Recall that a *star* is a connected graph with at most one vertex with degree greater than 1 and a *star forest* is a graph consisting of vertex disjoint stars.

In 2019, Lidický, Messerschmidt and Škrekovski [52] extended Theorem 6.5 in the following way.

Theorem 6.8 [52]**.** *If G is a simple plane graph such that the vertices of degree at least four induce a star forest, then*

$$\chi_{\text{pum}}(G) \le 4.$$

Two new conjectures were proposed.

Conjecture 6.9 [52]**.** *If G is a simple plane graph such that the vertices of degree at least four induce an acyclic graph, then*

$$\chi_{\text{pum}}(G) \le 4.$$

Conjecture 6.10 [52]**.** *If G is a simple plane graph such that the vertices of degree at least four induce a graph of maximum degree 2, then*

$$\chi_{\text{pum}}(G) \le 4.$$

In the next part of the paper we deal with the edge versions of the mentioned colorings.

7. Cyclic Edge Coloring

Already in 1880 Tait [68] observed that the Four Color Problem (a colorability of vertices of a plane graph using four colors) is equivalent to the problem of a colorability of edges of a plane triangulation using three colors in such a way that edges of any face are colored with all three colors. The edge version of the cyclic coloring of a plane graph, the *cyclic edge coloring*, is an edge coloring such that any two edges incident with the same face receive distinct colors. The minimum number of colors needed in such a coloring is called the *cyclic chromatic index* and is denoted by $\chi'_{\text{c}}(G)$.

The cyclic edge coloring of a plane graph G can be seen as a proper edge coloring of the dual graph G^*. The dual G^* of G is an embedding to the plane of G obtained as follows: A face f of G corresponds to a vertex f^* of G^*, and an edge e of G corresponds to an edge e^* of G^*, in such a way that $f \mapsto f^*$ and $e \mapsto e^*$ are bijections; two vertices f^* and g^* are joined by an edge e^* in G^* if and only if their preimage faces f and g are separated by the preimage e of e^* in G (an edge of a plane graph separates the faces it is incident with). It is easy to see that the dual of a plane graph is itself a plane graph. We place each vertex f^* of G^* in the preimage face f of G, and then draw each edge e^* of G^* in such a way that the only edge of G crossed by e^* is the preimage e of e^*, see Figure 13 .

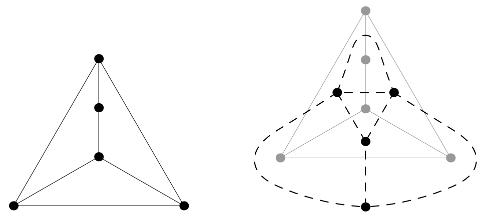

Figure 13. A plane graph and its dual.

Proper edge colorings were for the first time studied by Shannon [65] already in 1949. Denote by $\chi'(G)$ the *chromatic index* of a multigraph G, which is the minimum number of colors needed in a proper edge coloring of G. Shannon found out that $\chi'(G) \leq \left\lfloor \frac{3}{2}\Delta(G) \right\rfloor$ holds for any multigraph G. Note that the Shannon's bound is tight, and that its tightness is witnessed even by plane multigraphs. In 1964, Vizing [70] proved that for any multigraph G with maximum edge multiplicity $p(G)$ it holds $\chi'(G) \leq \Delta(G) + p(G)$. So for a simple graph G we have $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$. One year later, Vizing [69] proved that $\chi'(G) = \Delta(G)$ for any simple planar graph with $\Delta(G) \geq 8$ and observed that for any $\Delta \in \{2, 3, 4, 5\}$ there is a simple planar graph G with $\Delta(G) = \Delta$ and $\chi'(G) = \Delta + 1$. It took 36 years while, in 2001, Sanders and Zhao [64] proved that $\chi'(G) = \Delta(G)$ is true for any simple planar graph G with $\Delta(G) = 7$, too.

Using the fact that the edge connectivity of a plane graph G equals the girth of its dual G^* (see [32], p. 312), from the above results one can easily derive the following theorems.

Theorem 7.1. *If G is a 2-edge-connected plane graph with maximum face size $\Sigma(G)$, then*

$$\chi'_{\mathrm{c}}(G) \leq \left\lfloor \frac{3}{2}\Sigma(G) \right\rfloor .$$

Moreover, the bound is tight.

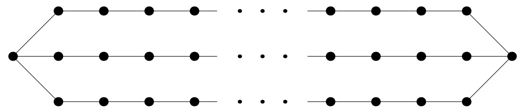

Figure 14. An example of a graph with $\chi'_{\mathrm{c}}(G) = \frac{3}{2}\Sigma(G)$.

Theorem 7.2. *If G is a 3-edge-connected plane graph with maximum face size $\Sigma(G)$, then*

(1) $\chi'_{\mathrm{c}}(G) = \Sigma(G)$ *for* $\Sigma(G) \geq 7$,

(2) $\chi'_{\mathrm{c}}(G) \leq \Sigma(G) + 1$ *for* $\Sigma(G)$ *satisfying* $2 \leq \Sigma(G) \leq 6$.

The bounds in Theorem 7.2 are tight if $2 \leq \Sigma(G) \leq 5$. The problem whether the upper bound is tight in the case $\Sigma(G) = 6$ is open.

8. Facial Rainbow Edge Coloring

In 2018, Jendroľ [43] introduced a *facial rainbow edge coloring* of a loopless plane graph G. It is an edge coloring of G in which two edges receive different colors if they lie on a common facial path of G. The minimum number of colors used in such a coloring is denoted by $\mathrm{erb}(G)$. Evidently, $\mathrm{erb}(G) \geq L'(G)$, where $L'(G)$ denotes the length of the longest facial path in G.

Jendroľ [43] proved the following four theorems.

Theorem 8.1 [43]**.** *If G is a loopless plane graph, then*

$$\mathrm{erb}(G) \leq \left\lfloor \frac{3}{2}(L'(G) + 1) \right\rfloor .$$

Moreover, the bound is tight.

Theorem 8.2 [43]**.** *If G is a plane tree, then*

$$\mathrm{erb}(G) \leq \left\lfloor \frac{3}{2} L'(G) \right\rfloor .$$

Moreover, the bound is tight.

Theorem 8.3 [43]**.** *If G is a plane tree without vertices of degree two, then*

(1) $\mathrm{erb}(G) = L'(G)$ *for* $L'(G) \geq 7$, *and*

(2) $\mathrm{erb}(G) \leq L'(G) + 1$ *for* $L'(G) \in \{2, 3, 4, 5, 6\}$.

Theorem 8.4 [43]**.** *If G is a simple 3-connected plane graph, then*

(1) $\mathrm{erb}(G) = L'(G) + 1$ *for* $L'(G) \notin \{3, 4, 5\}$*, and*

(2) $L'(G) + 1 \leq \mathrm{erb}(G) \leq L'(G) + 2$ *for* $L'(G) \in \{3, 4, 5\}$.

Moreover, the lower bound is tight for all $L'(G)$*, the upper bound in* (2) *is tight for* $L'(G) = 3$.

Two conjectures were posed in the pioneering paper on facial rainbow edge colorings.

Conjecture 8.5 [43]**.**

(1) *There is a simple* 3*-connected plane graph* G *with* $L'(G) = 4$ *and* $\mathrm{erb}(G) = L'(G) + 2$.

(2) *There is no simple* 3*-connected plane graph* G *with* $L'(G) = 5$ *and* $\mathrm{erb}(G) = L'(G) + 2$.

If G is a simple 3-connected plane graph, then its dual G^* is also simple and 3-connected, see [55], p. 46. The restriction of a facial rainbow edge coloring of a 3-connected plane graph G to the edges bounding a face is injective, hence any such coloring of G induces a proper edge coloring of G^* and vice versa, i.e., $\mathrm{erb}(G) = \chi'(G^*)$. Therefore, Conjecture 8.5(2) is the 3-connected restriction of Vizing's Planar Graph Conjecture: Every simple planar graph G with maximum degree 6 is of class one (i.e., $\chi'(G) = \Delta(G)$). There are many papers, published in recent years, answering Vizing's conjecture in the affirmative, provided some additional conditions regarding the absence of cycles of given length are fulfilled. It is shown that every simple planar graph G with $\Delta(G) = 6$ is of class one if it is without 3-cycles, 4-cycles, or 5-cycles [80], 6-cycles [16], 7-cycles [42], chordal 4-cycles [16], chordal 5-cycles [71], chordal 6-cycles [57], 5-cycles with two chords [75], 6-cycles with two chords [76], 6-cycles with three chords [79], 7-cycles with three chords [77]. Vizing's Planar Graph Conjecture also holds for simple planar graphs in which no vertex is incident with four faces of size 3 [73], no 4-cycle is adjacent to a 5-cycle [58], 7-cycles are pairwise non-adjacent [78], there is $k \in \{3, 4, 5\}$ such that any k-cycle shares an edge with at most one other k-cycle [59]. Vizing's conjecture is still open in general.

Conjecture 8.5(1) was proven by Czap [18] in 2020.

9. ℓ-Facial Edge Coloring

An *ℓ-facial edge coloring* c of a plane graph G is an edge coloring such that for any pair of distinct edges e_1, e_2 of G that are at distance at most ℓ on the boundary of a face, $c(e_1) \neq c(e_2)$ holds (i.e., all the edges of any facial trail of length at most $\ell + 1$ receive pairwise distinct colors). The minimum number of colors for which G admits an ℓ-facial edge coloring is denoted by $\chi'_\ell(G)$.

Notice that all upper bounds established for $\chi_\ell(G)$ are valid for $\chi'_\ell(G)$ as well. Define the *simplified medial* $M(G)$ of a plane graph G as follows. Let $m : E(G) \to V(M(G))$ be a bijection, and let distinct vertices $m(e_1), m(e_2)$ of $M(G)$ be joined by an edge in $M(G)$ if and only if edges e_1 and e_2 are consecutive on the boundary of a face in G. It is easy to see that the simplified medial $M(G)$ of a plane graph G is a planar graph; moreover, there is a natural embedding of $M(G)$ in the plane of G, see Figure 15; for simplicity, we use the notation $M(G)$ for that natural plane embedding, too.

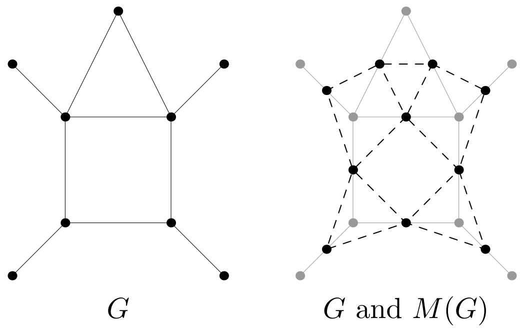

Figure 15. A plane graph and its simplified medial.

Observe that every ℓ-facial vertex coloring of $M(G)$ is an ℓ-facial edge coloring of G. Consequently, every plane graph admits a 1-facial edge coloring with at most four colors.

In 2015, Lužar *et al.* [53] proposed the following Facial Edge Coloring Conjecture.

Conjecture 9.1 [53]**.** *If G is a plane graph and $\ell \geq 1$, then*

$$\chi'_\ell(G) \leq 3\ell + 1.$$

Note that the bound offered by Conjecture 9.1 is tight (if the conjecture is true): as shown by Figure 16, for every $\ell \geq 2$, there exists a plane graph that has no ℓ-facial edge coloring with 3ℓ colors.

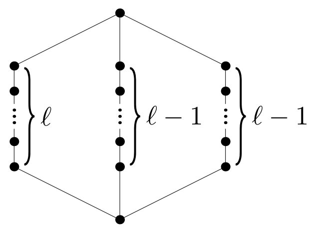

Figure 16. An example of a graph with $3\ell + 1$ edges and $\chi'_\ell(G) = 3\ell + 1$, [53].

The case with $\ell = 2$ was confirmed by Lužar *et al.* [53].

Theorem 9.2 [53]. *If G is a plane graph and $\ell = 2$, then*

$$\chi'_\ell(G) \leq 7.$$

The other cases are still open.

10. Odd Edge Coloring

Odd edge coloring of connected bridgeless plane graphs was introduced in 2011 by Czap, Jendroľ and Kardoš [21]. It is a 1-facial edge coloring such that for each face f and each color c, either no edge or an odd number of edges incident with f is colored with c. The minimum number of colors needed for an odd edge coloring of a connected bridgeless plane graph G is denoted by $\chi'_{\mathrm{o}}(G)$. In [21] it was shown that $\chi'_{\mathrm{o}}(G)$ is bounded from above by a constant.

Theorem 10.1 [21]. *If G is a connected bridgeless plane graph, then*

$$\chi'_{\mathrm{o}}(G) \leq 92.$$

The upper bound of Theorem 10.1 was significantly improved by the same authors with Soták [23].

Theorem 10.2 [23]. *If G is a connected bridgeless plane graph, then*

$$\chi'_{\mathrm{o}}(G) \leq 20.$$

For 3-edge-connected and 4-edge-connected plane graphs even stronger results were obtained.

Theorem 10.3 [23]. *If G is a 3-edge-connected plane graph, then*

$$\chi'_{\mathrm{o}}(G) \leq 12.$$

Theorem 10.4 [23]. *If G is a 4-edge-connected plane graph, then*

$$\chi'_{\mathrm{o}}(G) \leq 9.$$

Further, there is a graph G_1 such that $\chi'_{\mathrm{o}}(G_1) = 10$, and there is a 2-connected graph G_2 such that $\chi'_{\mathrm{o}}(G_2) = 9$, see Figure 17.

The bound of Theorem 10.2 can be improved for bridgeless outerplane graphs.

Theorem 10.5 [17]. *If G is a connected bridgeless outerplane graph, then*

$$\chi'_{\mathrm{o}}(G) \leq 15.$$

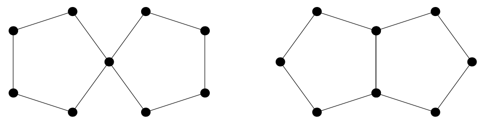

Figure 17. Two odd edge coloring extremal graphs, [23].

Theorem 10.6 [17]**.** *If G is a connected bridgeless plane cactus, then*

$$\chi'_{\mathrm{o}}(G) \leq 10.$$

Moreover, the bound is tight.

The best general upper bound of $\chi'_{\mathrm{o}}(G)$ known so far for a connected bridgeless plane graph G is 16, and was obtained by Lužar and Škrekovski [54] in 2013.

Theorem 10.7 [54]**.** *If G is a connected bridgeless plane graph, then*

$$\chi'_{\mathrm{o}}(G) \leq 16.$$

In 2015, Bálint and Czap [7] improved Theorem 10.5.

Theorem 10.8 [7]**.** *If G is a connected bridgeless outerplane graph different from G_1 depicted in Figure* 17*, then*

$$\chi'_{\mathrm{o}}(G) \leq 9.$$

Theorem 10.9 [7]**.** *There are infinitely many connected bridgeless outerplane graphs G with $\chi'_{\mathrm{o}}(G) = 9$.*

Such graphs can be obtained as follows: Let G be an outerplane graph created by identifying one vertex of a 5-cycle with one vertex of a $4k$-cycle, $k \geq 1$, such that the outer face has size $4k + 5$. Clearly, G is bridgeless. Observe that on the edges of the $4k$-cycle at least four different colors must appear in any odd edge coloring of G, and five different colors must appear on the 5-cycle. This graph has $\chi'_{\mathrm{o}}(G) = 9$ because no color used on the 5-cycle can be used on the $4k$-cycle if edges of the $4k$-cycle are colored with four colors.

In 2020, Štorgel [66] showed that there are connected bridgeless plane graphs G with $\chi'_{\mathrm{o}}(G) = 12$, so the general upper bound for $\chi'_{\mathrm{o}}(G)$ is between 12 and 16.

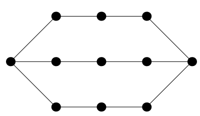

Figure 18. An example of a graph with $\chi'_{\mathrm{o}}(G) = 12$, [66].

In [25], Czap and Tuza dealt with the following question: For which integers k does there exist an odd edge coloring of a bridgeless plane graph G with exactly k colors?

The *feasible set* $\mathcal{F} = \mathcal{F}(G)$ of a plane graph G consists of those integers k for which G admits an odd edge coloring with exactly k colors. Clearly, $\chi'_{\mathrm{o}}(G)$ and $|E(G)|$ are the smallest and the largest elements of $\mathcal{F}$, respectively. We say that the feasible set of G is

- *continuous* if it is an interval of integers, i.e., $\mathcal{F} = \{k : \chi'_{\mathrm{o}}(G) \leq k \leq |E(G)|\}$,
- *i-continuous* if $\{k : i \leq k \leq |E(G)|\} \subseteq \mathcal{F}$,
- *semi-continuous* if, for every $k \in \mathcal{F}$ with $k \leq |E(G)| - 2$, also $k+2 \in \mathcal{F}$ holds.

Czap and Tuza [25] obtained the following results.

Theorem 10.10 [25]**.** *There exist connected bridgeless plane graphs for which the feasible set is not continuous.*

For example, cycles are such graphs.

Theorem 10.11 [25]**.** *The feasible set is semi-continuous for any connected bridgeless plane graph.*

Theorem 10.12 [25]**.** *If G is a 3-edge-connected plane graph, then its feasible set is 12-continuous.*

They posed the following conjecture.

Conjecture 10.13 [25]**.** *If G is a 3-edge-connected plane graph, then its feasible set is continuous.*

Note that no similar results are known for odd vertex colorings.

11. Unique-Maximum Edge Colorings

In 2015, Fabrici, Jendroľ and Vrbjarová [31] introduced a *unique-maximum edge coloring* of a connected plane graph with respect to faces as an edge coloring with positive integers such that, for each face f, the maximum color occurs exactly once on the edges of the boundary walk of f. This definition is meaningful only for bridgeless plane graphs. Every edge of a bridgeless plane graph is incident with two different faces, i.e., it occurs at most once on the boundary walk of any face. Every edge of a tree T occurs twice on the boundary walk of the only face of T, thus no color of this face can be unique. The minimum k for which a connected bridgeless plane graph G has a unique-maximum edge coloring with colors $1, 2, \ldots, k$ is denoted by $\chi'_{\mathrm{um}}(G)$ and the minimum k for which G has a 1-facial unique-maximum edge coloring is denoted by $\chi'_{\mathrm{pum}}(G)$.

Theorem 11.1 [31]. *If G is a connected bridgeless plane graph, then*

$$\chi'_{\mathrm{um}}(G) \leq 3.$$

Moreover, the bound is tight.

No connected bridgeless plane graph with an odd number of faces has a unique-maximum edge coloring with colors 1 and 2, since in any such coloring every face has only one edge of color 2, and every edge is incident with two faces.

Theorem 11.2 [31]. *If G is a connected bridgeless plane graph, then*

$$\chi'_{\mathrm{pum}}(G) \leq 6.$$

Theorem 11.3 [31]. *Let G be a bridgeless plane graph and let G^* be the dual of G. If there exists a matching in G^* covering all vertices of G^* of degree at least 4, then*

$$\chi'_{\mathrm{pum}}(G) \leq 5.$$

Fabrici, Jendroľ and Vrbjarová [31] posed the following conjecture.

Conjecture 11.4 [31]. *If G is a connected bridgeless plane graph, then*

$$\chi'_{\mathrm{pum}}(G) \leq 4.$$

By a result of Wendland [74], obtained in 2016, the simplified medial of G admits a unique-maximum proper vertex coloring with colors $1, 2, \ldots, k$, $k \leq 5$. This immediately implies $\chi'_{\mathrm{pum}}(G) \leq 5$ for any connected bridgeless plane graph G.

Andova *et al.* [2] proved that Conjecture 11.4 is true for simple 2-connected plane graphs.

Theorem 11.5 [2]. *If G is a simple 2-connected plane graph, then*

$$\chi'_{\mathrm{pum}}(G) \leq 4.$$

This is the only result known so far supporting Conjecture 11.4.

Acknowledgement

This work was supported by the Slovak Research and Development Agency under the contract No. APVV-19-0153.

References

[1] O. Amini, L. Esperet and J. van den Heuvel, *A unified approach to distance-two colouring of graphs on surfaces*, Combinatorica **33** (2013) 253–296. doi:10.1007/s00493-013-2573-2

[2] V. Andova, B. Lidický, B. Lužar and R. Škrekovski, *On facial unique-maximum (edge-) coloring*, Discrete Appl. Math. **237** (2018) 26–32. doi:10.1016/j.dam.2017.11.024

[3] K. Appel and W. Haken, *Every planar map is four colorable*, Bull. Amer. Math. Soc. **82** (1976) 711–712. doi:10.1090/s0002-9904-1976-14122-5

[4] K. Appel and W. Haken, *Every planar map is four colorable, Part* I: *Discharging*, Illinois J. Math. **21** (1977) 429–490. doi:10.1215/ijm/1256049011

[5] K. Appel, W. Haken and J. Koch, *Every planar map is four colorable, Part* II: *Reducibility*, Illinois J. Math. **21** (1977) 491–567. doi:10.1215/ijm/1256049012

[6] J. Azarija, R. Erman, D. Král', M. Krnc and L. Stacho, *Cyclic colorings of plane graphs with independent faces*, European J. Combin. **33** (2012) 294–301. doi:10.1016/j.ejc.2011.09.011

[7] T. Bálint and J. Czap, *Facial parity* 9*-edge-coloring of outerplane graphs*, Graphs Combin. **31** (2015) 1177–1187. doi:10.1007/s00373-014-1472-7

[8] O.V. Borodin, *A new proof of the* 6 *color theorem*, J. Graph Theory **19** (1995) 507–521. doi:10.1002/jgt.3190190406

[9] O.V. Borodin, *Cyclic coloring of plane graphs*, Discrete Math. **100** (1992) 281–289. doi:10.1016/0012-365x(92)90647-x

[10] O.V. Borodin, *Cyclic degree and cyclic coloring of* 3*-polytopes*, J. Graph Theory **23** (1996) 225–231. doi:10.1002/(sici)1097-0118(199611)23:3⟨225::aid-jgt2⟩3.0.co;2-u

[11] O.V. Borodin, *Solution of Ringel's problems on vertex-face coloring of plane graphs and coloring of* 1*-planar graphs*, Met. Diskret. Anal. **41** (1984) 12–26.

[12] O.V. Borodin, H.J. Broersma, A. Glebov and J. van den Heuvel, *A new upper bound on the cyclic chromatic number*, J. Graph Theory **54** (2007) 58–72. doi:10.1002/jgt.20193

[13] O.V. Borodin and A.O. Ivanova, *List* 2*-facial* 5*-colorability of plane graphs with girth at least* 12, Discrete Math. **312** (2012) 306–314. doi:10.1016/j.disc.2011.09.018

[14] O.V. Borodin and D.R. Woodall, *Cyclic colorings of* 3*-polytopes with large maximum face size*, SIAM J. Discrete Math. **15** (2002) 143–154. doi:10.1137/s089548019630248x

[15] O.V. Borodin, D.P. Sanders and Y. Zhao, *On cyclic colorings and their generalizations*, Discrete Math. **203** (1999) 23–40. doi:10.1016/s0012-365x(99)00018-7

[16] Y. Bu and W. Wang, *Some sufficient conditions for a planar graph of maximum degree six to be class* 1, Discrete Math. **306** (2006) 1440–1445. doi:10.1016/j.disc.2006.03.032

[17] J. Czap, *Facial parity edge coloring of outerplane graphs*, Ars Math. Contemp. **5** (2012) 285–289. doi:10.26493/1855-3974.228.ee8

[18] J. Czap, *Facial rainbow edge-coloring of simple* 3*-connected plane graphs*, Opuscula Math. **40** (2020) 475–482. doi:10.7494/OpMath.2020.40.4.475

[19] J. Czap, *Parity vertex coloring of outerplane graphs*, Discrete Math. **311** (2011) 2570–2573. doi:10.1016/j.disc.2011.06.009

[20] J. Czap and S. Jendroľ, *Colouring vertices of plane graphs under restrictions given by faces*, Discuss. Math. Graph Theory **29** (2009) 521–543. doi:10.7151/dmgt.1462

[21] J. Czap, S. Jendroľ and F. Kardoš, *Facial parity edge colouring*, Ars Math. Contemp. **4** (2011) 255–269. doi:10.26493/1855-3974.129.be3

[22] J. Czap, S. Jendroľ and F. Kardoš, *On the strong parity chromatic number*, Discuss. Math. Graph Theory **31** (2011) 587–600. doi:10.7151/dmgt.1567

[23] J. Czap, S. Jendroľ, F. Kardoš and R. Soták, *Facial parity edge colouring of plane pseudographs*, Discrete Math. **312** (2012) 2735–2740. doi:10.1016/j.disc.2012.03.036

[24] J. Czap, S. Jendroľ and M. Voigt, *Parity vertex colouring of plane graphs*, Discrete Math. **311** (2011) 512–520. doi:10.1016/j.disc.2010.12.008

[25] J. Czap and Zs. Tuza, *Decompositions of plane graphs under parity constrains given by faces*, Discuss. Math. Graph Theory **33** (2013) 521–530. doi:10.7151/dmgt.1690

[26] Z. Dvořák, M. Hebdige, F. Hlásek, D. Kráľ and J.A. Noel: *Cyclic coloring of plane graphs with maximum face size* 16 *and* 17. arXiv:1603.06722.

[27] Z. Dvořák, R. Škrekovski and M. Tancer, *List-coloring squares of sparse subcubic graphs*, SIAM J. Discrete Math. **22** (2008) 139–159. doi:10.1137/050634049

[28] H. Enomoto and M. Horňák, *A general upper bound for the cyclic chromatic number of* 3*-connected plane graphs*, J. Graph Theory **62** (2009) 1–25. doi:10.1002/jgt.20383

[29] H. Enomoto, M. Horňák and S. Jendroľ, *Cyclic chromatic number of* 3*-connected plane graphs*, SIAM J. Discrete Math. **14** (2001) 121–137.
doi:10.1137/s0895480198346150

[30] I. Fabrici and F. Göring, *Unique-maximum coloring of plane graphs*, Discuss. Math. Graph Theory **36** (2016) 95–102.
doi:10.7151/dmgt.1846

[31] I. Fabrici, S. Jendroľ and M. Vrbjarová, *Unique-maximum edge-colouring of plane graphs with respect to faces*, Discrete Appl. Math. **185** (2015) 239–243.
doi:10.1016/j.dam.2014.12.002

[32] C. Godsil and G. Royle, Algebraic Graph Theory (Springer, 2001).

[33] J.L. Gross and T.W. Tucker, Topological Graph Theory (Dover Publications, 2001).

[34] H. Grötzsch, *Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel*, Wiss. Z. Martin-Luther-Universität, Halle-Wittenberg, Math.-Nat. Reihe **8** (1959) 109–120.

[35] F. Havet, J.-S. Sereni and R. Škrekovski, 3*-facial coloring of plane graphs*, SIAM J. Discrete Math. **22** (2008) 231–247.
doi:10.1137/060664124

[36] F. Havet, D. Kráľ, J.-S. Sereni and R. Škrekovski, *Facial colorings using Hall's theorem*, European J. Combin. **31** (2010) 1001–1019.
doi:10.1016/j.ejc.2009.10.003

[37] M. Hebdige and D. Kráľ, *Third case of the cyclic coloring conjecture*, SIAM J. Discrete Math. **30** (2016) 525–548.
doi:10.1137/15m1019490

[38] M. Horňák and S. Jendroľ, *On a conjecture by Plummer and Toft*, J. Graph Theory **30** (1999) 177–189.
doi:10.1002/(sici)1097-0118(199903)30:3⟨177::aid-jgt3⟩3.0.co;2-k

[39] M. Horňák and S. Jendroľ, *On vertex types and cyclic colourings of* 3*-connected plane graphs*, Discrete Math. **212** (2000) 101–109.
doi:10.1016/s0012-365x(99)00212-5

[40] M. Horňák and S. Jendroľ, *Unavoidable sets of face types for planar maps*, Discuss. Math. Graph Theory **16** (1996) 123–141.
doi:10.7151/dmgt.1028

[41] M. Horňák and J. Zlámalová, *Another step towards proving a conjecture by Plummer and Toft*, Discrete Math. **310** (2010) 442–452.
doi:10.1016/j.disc.2009.03.016

[42] D. Huang and W. Wang, *Planar graphs of maximum degree six without* 7*-cycles are class one*, Electron. J. Combin. **19(3)** (2012) #P17.
doi:10.37236/2589

[43] S. Jendroľ, *Facial rainbow edge-coloring of plane graphs*, Graphs Combin. **34** (2018) 669–676.
doi:10.1007/s00373-018-1904-x

[44] S. Jendroľ and L. Kekeňáková, *Facial rainbow coloring of plane graphs*, Discuss. Math. Graph Theory **39** (2019) 889–897.
doi:10.7151/dmgt.2047

[45] S. Jendroľ and L. Kekeňáková, *Facial rainbow colorings of trees*, Australas. J. Combin. **69** (2017) 358–367.

[46] S. Jendroľ and R. Soták, *On the cyclic coloring conjecture*, Discrete Math. (2020) accepted.

[47] T. Kaiser, O. Rucký, M. Stehlík and R. Škrekovski, *Strong parity vertex coloring of plane graphs*, Discrete Math. Theor. Comput. Sci. **16** (2014) 143–158.

[48] D. Kráľ, T. Madaras and R. Škrekovski, *Cyclic, diagonal and facial colorings*, European J. Combin. **26** (2005) 473–490.
doi:10.1016/j.ejc.2004.01.016

[49] D. Kráľ, T. Madaras and R. Škrekovski, *Cyclic, diagonal and facial colorings—a missing case*, European J. Combin. **28** (2007) 1637–1639.
doi:10.1016/j.ejc.2006.07.005

[50] M. Kriesell, *Contractions, cycle double covers, and cyclic colorings in locally connected graphs*, J. Combin. Theory Ser. B **96** (2006) 881–900.
doi:10.1016/j.jctb.2006.02.009

[51] B. Lidický, K. Messerschmidt and R. Škrekovski, *A counterexample to a conjecture on facial unique-maximal colorings*, Discrete Appl. Math. **237** (2018) 123–125.
doi:10.1016/j.dam.2017.11.037

[52] B. Lidický, K. Messerschmidt and R. Škrekovski, *Facial unique-maximum colorings of plane graphs with restriction on big vertices*, Discrete Appl. Math. **242** (2019) 2612–2617.
doi:10.1016/j.disc.2019.05.029

[53] B. Lužar, M. Mockovčiaková, R. Soták, R. Škrekovski and P. Šugerek, *ℓ-facial edge colorings of graphs*, Discrete Appl. Math. **181** (2015) 193–200.
doi:10.1016/j.dam.2014.10.009

[54] B. Lužar and R. Škrekovski, *Improved bound on facial parity edge coloring*, Discrete Math. **313** (2013) 2218–2222.
doi:10.1016/j.disc.2013.05.022

[55] B. Mohar and C. Thomassen, Graphs on Surfaces (The Johns Hopkins University Press, 2001).

[56] M. Montassier and A. Raspaud, *A note on 2-facial coloring of plane graphs*, Inform. Process. Lett. **98** (2006) 235–241.
doi:10.1016/j.ipl.2006.02.013

[57] W.-P. Ni, *Edge colorings of planar graphs with $\Delta = 6$ without short cycles contain chords*, J. Nanjing Norm. Univ. Nat. Sci. Ed. **34** (2011) 19–24.

[58] W.-P. Ni, *Edge colorings of planar graphs without adjacent special cycles*, Ars Combin. **105** (2012) 247–256.

[59] W.-P. Ni and J. Wu, *Edge coloring of planar graphs which any two short cycles are adjacent at most once*, Theoret. Comput. Sci. **516** (2014) 133–138.
doi:10.1016/j.tcs.2013.11.011

[60] O. Ore and M.D. Plummer, *Cyclic coloration of plane graphs*, in: Recent Progress in Combinatorics (Proceedings of the Third Waterloo Conference on Combinatorics), (Academic Press, 1969) 287–293.

[61] M.D. Plummer and B. Toft, *Cyclic coloration of* 3*-polytopes*, J. Graph Theory **11** (1987) 507–515.
doi:10.1002/jgt.3190110407

[62] N. Robertson, D.P. Sanders, P.D. Seymour and R. Thomas, *The four-colour theorem*, J. Combin. Theory Ser. B **70** (1999) 2–44.
doi:10.1006/jctb.1997.1750

[63] D.P. Sanders and Y. Zhao, *A new bound on the cyclic chromatic number*, J. Combin. Theory Ser. B **83** (2001) 102–111.
doi:10.1006/jctb.2001.2046

[64] D.P. Sanders and Y. Zhao, *Planar graphs of maximum degree seven are class I*, J. Combin. Theory Ser. B **83** (2001) 201–212.
doi:10.1006/jctb.2001.2047

[65] C.E. Shannon, *A theorem on coloring the lines of a network*, J. Math. Phys. **28** (1949) 148–152.
doi:10.1002/sapm1949281148

[66] K. Štorgel, *Improved bounds for some facially constrained colorings*, Discuss. Math. Graph Theory, in press.
doi:10.7151/dmgt.2357

[67] P.G. Tait, *Remarks on the colouring of maps*, Proc. Roy. Soc. Edinburgh **10** (1880) 729.

[68] C. Thomassen, *The square of a planar cubic graph is* 7*-colorable*, J. Combin. Theory Ser. B **128** (2018) 192–218.
doi:10.1016/j.jctb.2017.08.010

[69] V.G. Vizing, *Critical graphs with a given chromatic class*, Met. Diskret. Anal. **5** (1965) 9–17.

[70] V.G. Vizing, *On an estimate of the chromatic class of a p-graph*, Diskret. Analiz **3** (1964) 25–30.

[71] W. Wang and Y. Chen, *A sufficient condition for a planar graph to be class* 1, Theoret. Comput. Sci. **385** (2007) 71–77.
doi:10.1016/j.tcs.2007.05.032

[72] W. Wang, S. Finbow and P. Wang, *An improved bound on parity vertex colourings of outerplane graphs*, Discrete Math. **312** (2012) 2782–2787.
doi:10.1016/j.disc.2012.04.009

[73] Y. Wang and L. Xu, *A sufficient condition for a plane graph with maximum degree* 6 *to be class* 1, Discrete Appl. Math. **161** (2013) 307–310.
doi:10.1016/j.dam.2012.08.011

[74] A. Wendland, *Coloring of plane graphs with unique maximal colors on faces*, J. Graph Theory **83** (2016) 359–371.
doi:10.1002/jgt.22002

[75] J. Wu and L. Xue, *Edge colorings of planar graphs without* 5*-cycles with two chords*, Theoret. Comput. Sci. **518** (2014) 124–127.
doi:10.1016/j.tcs.2013.07.027

[76] L. Xue and J. Wu, *Edge colorings of planar graphs without* 6*-cycles with two chords*, Open J. Discrete Math. **3** (2013) 83–85.
doi:10.4236/ojdm.2013.32016

[77] W. Zhang and J. Wu, *A note on the edge colorings of planar graphs without* 7*-cycles with three chords*, Ars Combin. **138** (2018) 393–402.

[78] W. Zhang and J. Wu, *Edge coloring of planar graphs without adjacent* 7*-cycles*, Theoret. Comput. Sci. **739** (2018) 59–64.
doi:10.1016/j.tcs.2018.05.006

[79] W. Zhang and J. Wu, *Edge colorings of planar graphs without* 6*-cycles with three chords*, Bull. Malays. Math. Sci. Soc. **41** (2018) 1077–1084.
doi:10.1007/s40840-016-0376-5

[80] G. Zhou, *A note on graphs of class* I, Discrete Math. **263** (2003) 339–345.
doi:10.1016/s0012-365x(02)00793-8

[81] J. Zlámalová, *A note on cyclic chromatic number*, Discuss. Math. Graph Theory **30** (2010) 115–122.
doi:10.7151/dmgt.1481

Received 26 June 2020
Revised 21 October 2020
Accepted 21 October 2020