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THE TURÁN NUMBER OF SPANNING STAR FORESTS[1]

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Abstract

Let $\mathcal{F}$ be a family of graphs. The Turán number of $\mathcal{F}$, denoted by $ex(n, \mathcal{F})$, is the maximum number of edges in a graph with n vertices which does not contain any subgraph isomorphic to some graph in $\mathcal{F}$. A star forest is a forest whose connected components are all stars and isolated vertices. Motivated by the results of Wang, Yang and Ning about the spanning Turán number of linear forests [J. Wang and W. Yang, *The Turán number for spanning linear forests*, Discrete Appl. Math. 254 (2019) 291–294; B. Ning and J. Wang, *The formula for Turán number of spanning linear forests*, Discrete Math. 343 (2020) #111924]. In this paper, let $\mathcal{S}_{n,k}$ be the set of all star forests with n vertices and k edges. We prove that when $1 \leq k \leq n-1$, $ex(n, \mathcal{S}_{n,k}) = \left\lfloor \frac{k^2-1}{2} \right\rfloor$.

Keywords: spanning Turán problem, star forests, Loebl-Komlós-Sós type problems.

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1. Introduction

All graphs in this paper are finite, undirected and simple. Notations in this paper are standard as [1]. For a graph G, let $V(G)$ be the set of vertices, $E(G)$ be the set of edges and $e(G) = |E(G)|$ be the number of edges. For $v \in V(G)$, we define $N(v)$ to be the vertex set whose each vertex is adjacent to the vertex v. Furthermore, for a vertex set $S \subset V(G)$, we use $N(S)$ to denote the vertex set whose each vertex is in $V(G) \setminus S$ and adjacent to at least one vertex in S. Let $N[v] = N(v) \cup \{v\}$. We denote the degree of a vertex v by $d(v)$ and the maximum degree of G by $\Delta(G)$. For a vertex set $U \subset V(G)$, the subgraph of G induced by U is denoted by $G[U]$. For a subgraph H of a graph G, the graph $G - H$ is the subgraph induced by the vertex set $V(G)\backslash V(H)$, i.e., $G[V(G)\backslash V(H)]$. A star forest is a forest whose connected components are all stars and isolated vertices. We denote the disjoint union of k copies of the graph F by $k \cdot F$. Let H_1 and H_2 be two disjoint graphs. The *join* of H_1 and H_2, denoted by $H_1 \vee H_2$, is the graph whose vertex set is $V(H_1 \vee H_2) = V(H_1) \cup V(H_2)$ and edge set is $E(H_1 \vee H_2) = E(H_1) \cup E(H_2) \cup \{xy : x \in V(H_1), y \in V(H_2)\}$. We use K_n and E_n to denote the complete graph on n vertices and the empty graph on n vertices, respectively. We use $K_{s,t}$ to denote the complete bipartite graph with two sets of vertices, one with s vertices and another with t vertices. We denote the star graph on n vertices by S_{n-1}, i.e., $S_{n-1} = K_{1,n-1}$. For any positive integer n, let $[n] = \{1, 2, 3, \ldots, n\}$.

Let $\mathcal{F}$ be a family of graphs. A graph G is called $\mathcal{F}$-free if G does not contain any subgraph which is isomorphic with some $F \in \mathcal{F}$. The Turán number, denoted by $ex(n, \mathcal{F})$, is the maximum number of edges in an $\mathcal{F}$-free graph on n vertices. When $\mathcal{F}$ contains only one graph F, we denote the Turán number of F by $ex(n, F)$. Let $EX(n, F)$ denote an $\mathcal{F}$-free graph on n vertices with $ex(n, F)$ edges. We call this graph an extremal graph for F. A traditional starting point of *extremal graph theory* (a significant branch of graph theory) is the Mantel's theorem (see e.g. [1]) that the maximum number of edges in a K_3-free graph on n vertices is $\left\lfloor \frac{n^2}{4} \right\rfloor$. Turán [15, 16] generalized this result to determine the value $ex(n, K_{r+1})$ and showed that the unique extremal graph for K_{r+1} is the complete r-partite graph on n vertices whose all parts are as equal in size as possible (the difference between any two parts is at most 1), denoted by $T_r(n)$. Generally, we call the graph $T_r(n)$ a Turán graph. In 1959, Erdős and Gallai [3] determined the value $ex(n, (k+1) \cdot K_2)$ and the extremal graph for it. In [14], Simonovits showed that the unique extremal graph for $k \cdot K_{r+1}$ is $K_{k-1} \vee T_r(n-p+1)$ when n is sufficiently large. Later, by considering the graph that consists of p disjoint copies of any connected graph G on n vertices, Gorgol [5] gave a lower bound for $ex(m, p \cdot G)$.

Theorem 1 [3]**.** *When $n \geq 2k+1$,*

$$ex(n,(k+1)\cdot K_2) = \max\left\{\binom{2k+1}{2}, \binom{n}{2} - \binom{n-k}{2}\right\}.$$

Theorem 2 [5]**.** *Let G be any connected graph on n vertices, p be any positive integer and m be an integer such that $m \geq pn$. Then $ex(m, p\cdot G) \geq \max\left\{ex(m-pn+1, G) + \binom{pn-1}{2}, ex(m-p+1, G) + (p-1)m - \binom{p}{2}\right\}$.*

In [12], Lidický *et al.* investigated the Turán number of a star forest and determined the value $ex(n,F)$ when n is sufficiently large, where $F = \bigcup_{i=1}^{k} S_{d_i}$ and $d_1 \geq d_2 \geq \cdots \geq d_k$. Yin and Rao [18] determined the value $ex(n, k\cdot S_l)$ when $n \geq \frac{1}{2}l^2k(k-1) + k - 2 + \max\{lk, l^2+2l\}$, which improved the results of Lidický *et al.* Later, Lan *et al.* [10] determined the value $ex(n, k\cdot S_l)$ when $n \geq k(l^2+l+1) - \frac{l}{2}(l-3)$, which further improved these results. Recently, for a fixed k, Li *et al.* [11] determined the Turán number $ex(n, 2\cdot S_l)$ for all positive integers n and $l(\geq 4)$ and $ex(n, 3\cdot S_l)$ for all positive integers n and $l(\geq 3)$.

Recently, Ning and Wang [13] considered the forbidden family $\mathcal{L}_{n,k}$ of subgraphs (i.e., a family of all linear forests of order n with k edges) and determined the exact value $ex(n, \mathcal{L}_{n,k})$. The Hamiltonian completion number of a graph G is the minimum number of edges to ensure that the graph G is Hamiltonian by adding it. Their results determined also the maximum number of edges in a non-Hamiltonian graph with fixed Hamiltonian completion number. Notice that the order of the forbidden linear forest and the order n are dependent. Motivated by their results, we determine the Turán number $ex(n, \mathcal{S}_{n,k})$ by considering a family of all star forests of order n with k edges, denoted by $\mathcal{S}_{n,k}$.

Theorem 3 [13]**.** *Let n, k be two positive integers and $1 \leq k \leq n-1$. Then*

$$ex(n, \mathcal{L}_{n,k}) = \max\left\{\binom{k}{2}, \binom{n}{2} - \binom{n - \lfloor\frac{k-1}{2}\rfloor}{2} + c\right\},$$

where $c = 0$ if k is odd, and $c = 1$ otherwise.

For convenience, we would like to give the definition of the almost d-regular graph. We define the almost d-regular graph as a graph that contains one vertex of degree $d-1$ and all the other vertices have degree d.

Theorem 4. *Let n, k be two positive integers and $1 \leq k \leq n-1$. Then*

$$ex(n, \mathcal{S}_{n,k}) = \left\lfloor \frac{k^2-1}{2} \right\rfloor.$$

Moreover, the unique extremal graph contains a connected component of size $k+1$ and $n-k-1$ isolated vertices. And the connected component is the almost $(k-1)$-regular graph or the $(k-1)$-regular graph on $k+1$ vertices.

Considering the median degree of a graph G, Loebl, Komlós and Sós [2] conjectured that every graph G of order n with at least $n/2$ vertices of degree at least k contains each tree T of order $k+1$ as a subgraph. This is a median degree version of the famous Erdős-Sós conjecture. For more results about Loebl-Komlós-Sós conjecture, we refer the reader to [6, 7, 8, 9].

Conjecture 5 (The Erdős-Sós Conjecture). *Every graph G order n with average degree greater than $k-1$ contains each tree T of order $k+1$ as a subgraph.*

Conjecture 6 (The Loebl-Komlós-Sós Conjecture, [2]). *Every graph G order n with median degree greater than $k-1$ contains each tree T of order $k+1$ as a subgraph.*

Füredi and Simonovits called it Loebl-Komlós-Sós type problem or the Median problem [4] as follows: for a given graph G of order n, which m and d ensure that if G has at least m vertices of degree $\geq d$, then G contains some subgraph H.

One could analogously define such problems for families $\mathcal{H}$ of graphs. We solved a problem of this type for $\mathcal{H} = \mathcal{S}_{n,k}$.

Theorem 7. *If a simple graph G on n $(n \geq k+2)$ vertices has at least $\frac{3k}{2}+2$ vertices of degree at least $\lceil \frac{k}{2} \rceil$, then G contains some graph in $\mathcal{S}_{n,k}$ as its subgraph.*

2. The Proof of Theorem 4

Proof. We prove this result mainly by induction on k. First, if $k=1$, then we know that $\mathcal{S}_{n,1}$ contains only one edge, i.e., $\mathcal{S}_{n,1} = K_2 \cup E_{n-2}$. Notice that if a graph G contains at least one edge, it must contain $\mathcal{S}_{n,1}$ as its subgraph. Then we have that $ex(n, \mathcal{S}_{n,1}) = 0 = \left\lfloor \frac{1^2-1}{2} \right\rfloor$ and the unique extremal graph is E_n. If $k=2$, we can see that $\mathcal{S}_{n,2} = \{S_2 \cup E_{n-3}, 2 \cdot K_2 \cup E_{n-4}\}$. Notice that the graph which does not contain any graph in $\mathcal{S}_{n,2}$ as its subgraph must have maximum degree at most 1. If there are two edges in distinct connected components of G, then we can find a copy of $2 \cdot K_2$ in G, a contradiction. Thus, we have that $ex(n, \mathcal{S}_{n,2}) \leq 1$. And, the graph $K_2 \cup E_{n-2}$ does not contain any graph in $\mathcal{S}_{n,2}$ as its subgraph and $e(K_2 \cup E_{n-2}) = 1$. Then $ex(n, \mathcal{S}_{n,2}) = 1 = \left\lfloor \frac{2^2-1}{2} \right\rfloor$ and the unique extremal graph is $K_2 \cup E_{n-2}$. We can see that $\mathcal{S}_{n,3} = \{S_3 \cup E_{n-4}, S_2 \cup K_2 \cup E_{n-5}, 3 \cdot K_2 \cup E_{n-6}\}$ when $k=3$. It is easy to know that the graph which does not contain any graph in $\mathcal{S}_{n,3}$ as its subgraph must have maximum degree at most 2. Also, we can see that there are at most two connected components in a extremal graph for $\mathcal{S}_{n,3}$ which is denoted by G. If there are exactly two connected components in G, then both of two connected components contain only an edge, thus $e(G) = 2$. If there is only one connected component in G, then we have that $\Delta(G) \leq 2$.

If $\Delta(G) = 1$, then $e(G) = 1$. If $\Delta(G) = 2$ and v is a vertex in G with degree 2, then let $N(v) = \{v_1, v_2\}$. If v_1v_2 is an edge of G, then $e(G) = 3$. If v_1v_2 is not an edge of G, then v_1 and v_2 can only be adjacent to the same vertex in $V(G) \setminus \{v, v_1, v_2\}$ and this fourth vertex is the last in the connected component. Otherwise, we can find a copy of $S_2 \cup K_2$ in G, a contradiction. Thus, we have that $ex(n, \mathcal{S}_{n,3}) = 4 = \left\lfloor \frac{3^2-1}{2} \right\rfloor$ and $K_{2,2} \cup E_{n-4}$ is the unique extremal graph.

Now, we assume that the result holds for all $k^{'} < k$. In the following, we will show that the result holds for k. From the above analyses, we can see that the extremal graph has maximum degree $k-1$ when $k \in \{1, 2, 3\}$. We claim that this conclusion holds for k. In order to prove that the claim is true, we first would like to construct an extremal graph for $\mathcal{S}_{n,k}$, denoted by G, and then to prove any graph G' with $\Delta(G') = k-1$ which contains no $\mathcal{S}_{n,k}$ as its subgraph can contain at most $e(G)$ edges. Then we would like to show that all other graphs with maximum degree less than $k-1$ have less edges than the constructed graph G.

The constructed extremal graph is as follows. G contains only one connected component containing an edge which is the almost $(k-1)$-regular graph or the $(k-1)$-regular graph on $k+1$ vertices. We denote the connected component of G by G_1. Notice that if G contains any graph in $\mathcal{S}_{n,k}$, then all edges of it must be contained in G_1. There is only one graph in $\mathcal{S}_{n,k}$ which contains only one connected component, that is $S_k \cup E_{n-k-1}$ and the only connected component is S_k. Since any graph in $\mathcal{S}_{n,k}$ other than $S_k \cup E_{n-k-1}$ has at least two connected components which contain at least $k+2$ vertices, G_1 cannot contain any of them as a subgraph. If G is not $\mathcal{S}_{n,k}$-free, then G_1 must contain S_k as its subgraph. It is easy to deduce that G_1 does not contain S_k as its subgraph since G_1 is the almost $(k-1)$-regular graph or the $(k-1)$-regular graph on $k+1$ vertices. Thus G is $\mathcal{S}_{n,k}$-free.

Next, we will prove that any graph G' with $\Delta(G') = k-1$ which does not contain any graph in $\mathcal{S}_{n,k}$ as its subgraph can contain at most $e(G)$ edges. Without loss of generality, we assume that $d(v) = k-1$ for $v \in V(G')$ and let $N(v) = \{v_1, v_2, \ldots, v_{k-1}\}$. Notice that there are no edges in $G'[V(G') \setminus N[v]]$, otherwise we can find a $S_{k-1} \cup K_2 \cup E_{n-k-2} \in \mathcal{S}_{n,k}$ in G', a contradiction. Thus, all edges of G' must be contained in the connected component which contains all vertices in $N(v)$. Any v_i can be adjacent to at most one vertex in $V(G') \setminus N[v]$, thus there are at most $k-1$ edges between $N(v)$ and $V(G') \setminus N[v]$. For any $v_i \in N(v)$, if there is an edge between the vertex v_i and the vertex set $V(G') \setminus N[v]$, then by the definition of Δ, there exist at most $k-3$ edges between the vertex v_i and the vertex set $N(v)$. Therefore, if there are $y \leq k-1$ edges between the two vertex sets $N(v)$ and $V(G') \setminus N[v]$, then at least $\left\lceil \frac{y}{2} \right\rceil$ edges are missing inside $N(v)$. Therefore, the total number of edges is

$$e(G') \le \binom{k}{2} - \left\lceil \frac{y}{2} \right\rceil + y \le \left\lfloor \frac{k^2-1}{2} \right\rfloor . \tag{1}$$

We can explain the first inequality of Equation (1) as follows. The first two terms are an upper bound on the number of edges inside $N[v]$, and the last term is the number of other edges by definition.

Notice that the extremal graph can contain only one vertex in $V(G') \setminus N[v]$ which is adjacent to every vertex in $N(v)$. Assume that there are two vertices which are adjacent to the vertex set $N(v)$, denoted by v_x, v_y. Then we have that $d_{N(v)}(v_x) + d_{N(v)}(v_y) = k-1$. Since $G'[N(v)]$ is an almost $(k-3)$-regular graph, we can find a copy of $S_2 \cup S_{k-2}$, a contradiction. Thus, we know that the constructed graph G is the unique extremal graph G' for $\Delta(G') = k-1$.

Our third step is to prove the following conclusion. All other graphs with maximum degree less than $k-1$ contain less edges than G.

First, we claim that the extremal graph for $\mathcal{S}_{n,k}$ contains only one connected component. Assume that there are two connected components in the extremal graph for $\mathcal{S}_{n,k}$ and there is one connected component such that the largest star forest in it has x edges. By the induction hypothesis, we have that the extremal graph contains at most $\left\lfloor \frac{(x+1)^2-1}{2} \right\rfloor + \left\lfloor \frac{(k-x)^2-1}{2} \right\rfloor \le \left\lfloor \frac{(k)^2-1}{2} \right\rfloor$ edges. Thus we have that the extremal graph for $\mathcal{S}_{n,k}$ contains only one connected component. Let H be an extremal graph for $\mathcal{S}_{n,k}$ that contains only one connected component H' containing an edge which has maximum degree $3 \le \Delta(H') = t < k-1$. Let $d(u) = t$ and $N(u) = \{u_1, u_2, \ldots, u_t\}$. In the following, we divide the edge set of the graph H into two parts: the edges of the graph $H[V(H') \setminus N[u]]$ and other edges. We denote the graph which is induced by the second set of edges by H_1.

Case 1. H_1 contains no star forest with more than t edges. Any u_i can be adjacent to at most one vertex in $V(H') \setminus N[u]$, otherwise we can find a copy of $S_2 \cup S_{t-1}$, a contradiction. Each vertex u_i can be adjacent to at most $t-1$ vertices in $N(u) - u_i$ for $i \in [1, t]$. For any $u_i \in N(u)$, if there is an edge between the vertex u_i and the vertex set $V(H_1) \setminus N[u]$, then by the definition of Δ, there exist at most $t-2$ edges between the vertex u_i and the vertex set $N(u)$. Therefore, if there are $z \le t$ edges between the two vertex sets $N(u)$ and $V(H_1) \setminus N[u]$, then at least $\left\lceil \frac{z}{2} \right\rceil$ edges are missing inside $N(u)$. Therefore, the total number of edges is

$$e(H_1) \le \binom{t+1}{2} - \left\lceil \frac{z}{2} \right\rceil + z \le \left\lfloor \frac{t(t+2)}{2} \right\rfloor . \tag{2}$$

We can explain the first inequality of Equation (2) as follows. The first two terms are an upper bound on the number of edges inside $N[u]$, and the last term is the number of other edges by definition.

Then by the induction hypothesis and Equation (2),

$$\begin{aligned} e(H) = e(H') &\le \left\lfloor \frac{(k-t)^2-1}{2} \right\rfloor + \left\lfloor \frac{t(t+2)}{2} \right\rfloor \\ &= \left\lfloor \frac{k^2-1+2t^2+2t-2kt}{2} \right\rfloor \le \left\lfloor \frac{k^2-1}{2} \right\rfloor = e(G). \end{aligned} \tag{3}$$

Case 2. H_1 contains a star forest with $y+t-1$ edges. But H_1 contains no star forest with more than $y+t-1$ edges. Similarly, any u_i can be adjacent to at most y vertices in $V(H') \setminus N[u]$, otherwise we can find a copy of $S_{y+1} \cup S_{t-1}$, a contradiction. Therefore, there are at most ty edges between the set $N(u)$ and the set $V(H') \setminus N[u]$. We assume that u_i is adjacent to y_i vertices in $V(H') \setminus N[u]$ for $i \in [1,t]$. Then

$$\begin{aligned} e(H_1) &\le \left\lfloor \frac{\sum_{i=1}^{t}(t-1-y_i)}{2} \right\rfloor + \sum_{i=1}^{t} y_i + t \\ &\le \left\lfloor \frac{t(t-1-y)}{2} \right\rfloor + t(y+1). \end{aligned} \tag{4}$$

Then by the induction hypothesis and Equation (4),

$$\begin{aligned} e(H) &\le \left\lfloor \frac{t(t-1-y)}{2} \right\rfloor + t(y+1) + \left\lfloor \frac{(k-y-t+1)^2-1}{2} \right\rfloor \\ &= \left\lfloor \frac{y^2+(4t-2k-2)y}{2} \right\rfloor + \left\lfloor \frac{k^2+2t^2-t+1-2kt+2k}{2} \right\rfloor. \end{aligned} \tag{5}$$

Further, we analyze two subcases as follows.

Subcase 2.1. $t \le \lfloor \frac{k-1}{2} \rfloor + 1$. Considering the term y and $2 \le y \le t-1$, we can see that the right part of the above inequality about y is a parabola with an upward opening, which obtains a value of 0 at points $y = 0$ and $y = 2k+2-4t$. If $t-1 \ge 2k+2-4t$ (i.e., $t \ge (2k+3)/5$), then $e(H)$ can attains its maximum value at $y = t-1$. Otherwise, $e(H)$ can attain its maximum value at $y = 2$. When $t \ge (2k+3)/5$, we can calculate the maximum number of edges as $\left\lfloor \frac{k^2+7t^2-9t+4-4kt+4k}{2} \right\rfloor$. And the maximum number of edges is $\left\lfloor \frac{k^2+2t^2+7t+1-2kt-2k}{2} \right\rfloor$ when $t < (2k+3)/5$. The two values are both less than $e(G) = \left\lfloor \frac{k^2-1}{2} \right\rfloor$. Thus, we prove the above conclusion when $t \le \lfloor \frac{k-1}{2} \rfloor + 1$.

Subcase 2.2. $\lfloor \frac{k-1}{2} \rfloor + 1 < t \le k-1$. It is easy to know that $2 \le y \le k-t$. By a simple calculation for Equation (5), we know that the maximum value of the right part of Equation (5) can be obtained when $y = k-t$. Substituting

$y = k - t$ into Equation (5), we obtain that the maximum value is $\left\lfloor \frac{kt+t}{2} \right\rfloor$. We can obtain that it is smaller than $e(G)$ by a simple calculation. Thus we prove that the conclusion holds for all k.

Thus, our claim is true that the extremal graph has maximum degree $k-1$. This completes our proof. ■

3. The Proof of Theorem 7

Proof. To prove this conclusion, we just need to prove that there are at most $\frac{3k}{2}+1$ vertices of degree at least $\lceil \frac{k}{2} \rceil$ if G contains no graph in $\mathcal{S}_{n,k}$ as its subgraph. We define d as the smallest number that is at least $\lceil \frac{k}{2} \rceil$ and there is a vertex of degree exactly d. Without loss of generality, let $v \in V(G)$, $d(v) = d$ and $N(v) = \{v_1, v_2, \dots, v_d\}$. Notice that every vertex not in $N[v]$ nor in $N(N(v))$ has degree at most $k-d-1$, otherwise we can find a copy of $S_d \cup S_{k-d} \cup E_{n-k-2} \in \mathcal{S}_{n,k}$, a contradiction. Thus we know that every vertex which has degree at least $\lceil \frac{k}{2} \rceil$ can be contained only in $N[v]$ or $N(N(v))$. Every vertex in $N(v)$ has at most $k-d$ neighbors in $V(G) \setminus N[v]$. Thus we know that there are at most $d(k-d)$ edges between the vertex set $N(v)$ and $N(N(v))$.

Claim 8. *There are at most $\frac{3k}{2} - d$ vertices in $N(N(v))$ which have degree at least d.*

In the following proof, we call a vertex which has degree at least $\lceil \frac{k}{2} \rceil$ as the large degree vertex.

Proof. Suppose that there are at least $\frac{3k}{2}-d+1$ large degree vertices in $N(N(v))$. By Theorem 4, there are at most $\left\lfloor \frac{(k-d)^2-1}{2} \right\rfloor$ edges in $G[V(G) \setminus N[v]]$. It follows that the large degree vertices in $N(N(v))$ are incident to at least $\left(\frac{3k}{2} - d + 1\right)d$ edges, but there are at most $(k-d)^2 - 1$ incidences inside $G[V(G) \setminus N[v]]$, thus at least $\left(\frac{3k}{2} - d + 1\right)d - ((k-d)^2 - 1) = \frac{7kd}{2} - 2d^2 - k^2 + d + 1$ incidences are by edges between $N(v)$ and $N(N(v))$. Thus, the number of edges between $N(v)$ and $N(N(v))$ must be no less than $\frac{7kd}{2} - 2d^2 - k^2 + d + 1$, which is greater than $d(k-d)$. This contradicts with the fact that there are at most $d(k-d)$ edges between the vertex set $N(v)$ and $N(N(v))$. Thus, we have that there are at most $\frac{3k}{2} - d$ vertices in G which have degree at least d. □

Combining Claim 8 and the fact that there are at most $d+1$ vertices in $N[v]$ which have degree at least d, we conclude that there are at most $\frac{3k}{2}+1$ vertices of degree at least $\lceil \frac{k}{2} \rceil$ if G contains no graph in $\mathcal{S}_{n,k}$ as its subgraph. This completes our proof. ■

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