# THE TURÁN NUMBER OF SPANNING STAR FORESTS ${ }^{1}$ 

Lin-Peng Zhang ${ }^{a, b}$, Ligong WANG ${ }^{a, b, 2}$<br>AND<br>Jiale Zhou ${ }^{a}$<br>${ }^{a}$ School of Mathematics and Statistics Northwestern Polytechnical University Xi’an, Shaanxi 710129, P.R. China<br>${ }^{b}$ Xi'an-Budapest Joint Research Center for Combinatorics<br>Northwestern Polytechnical University Xi'an, Shaanxi 710129, P.R. China<br>e-mail: lpzhangmath@163.com<br>lgwangmath@163.com<br>zj10508math@mail.nwpu.edu.cn


#### Abstract

Let $\mathcal{F}$ be a family of graphs. The Turán number of $\mathcal{F}$, denoted by $\operatorname{ex}(n, \mathcal{F})$, is the maximum number of edges in a graph with $n$ vertices which does not contain any subgraph isomorphic to some graph in $\mathcal{F}$. A star forest is a forest whose connected components are all stars and isolated vertices. Motivated by the results of Wang, Yang and Ning about the spanning Turán number of linear forests [J. Wang and W. Yang, The Turán number for spanning linear forests, Discrete Appl. Math. 254 (2019) 291-294; B. Ning and J. Wang, The formula for Turán number of spanning linear forests, Discrete Math. 343 (2020) \#111924]. In this paper, let $\mathcal{S}_{n, k}$ be the set of all star forests with $n$ vertices and $k$ edges. We prove that when $1 \leq k \leq n-1$, $e x\left(n, \mathcal{S}_{n, k}\right)=\left\lfloor\frac{k^{2}-1}{2}\right\rfloor$.


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## 1. Introduction

All graphs in this paper are finite, undirected and simple. Notations in this paper are standard as [1]. For a graph $G$, let $V(G)$ be the set of vertices, $E(G)$ be the set of edges and $e(G)=|E(G)|$ be the number of edges. For $v \in V(G)$, we define $N(v)$ to be the vertex set whose each vertex is adjacent to the vertex $v$. Furthermore, for a vertex set $S \subset V(G)$, we use $N(S)$ to denote the vertex set whose each vertex is in $V(G) \backslash S$ and adjacent to at least one vertex in $S$. Let $N[v]=N(v) \cup\{v\}$. We denote the degree of a vertex $v$ by $d(v)$ and the maximum degree of $G$ by $\Delta(G)$. For a vertex set $U \subset V(G)$, the subgraph of $G$ induced by $U$ is denoted by $G[U]$. For a subgraph $H$ of a graph $G$, the graph $G-H$ is the subgraph induced by the vertex set $V(G) \backslash V(H)$, i.e., $G[V(G) \backslash V(H)]$. A star forest is a forest whose connected components are all stars and isolated vertices. We denote the disjoint union of $k$ copies of the graph $F$ by $k \cdot F$. Let $H_{1}$ and $H_{2}$ be two disjoint graphs. The join of $H_{1}$ and $H_{2}$, denoted by $H_{1} \vee H_{2}$, is the graph whose vertex set is $V\left(H_{1} \vee H_{2}\right)=V\left(H_{1}\right) \cup V\left(H_{2}\right)$ and edge set is $E\left(H_{1} \vee H_{2}\right)=E\left(H_{1}\right) \cup E\left(H_{2}\right) \cup\left\{x y: x \in V\left(H_{1}\right), y \in V\left(H_{2}\right)\right\}$. We use $K_{n}$ and $E_{n}$ to denote the complete graph on $n$ vertices and the empty graph on $n$ vertices, respectively. We use $K_{s, t}$ to denote the complete bipartite graph with two sets of vertices, one with $s$ vertices and another with $t$ vertices. We denote the star graph on $n$ vertices by $S_{n-1}$, i.e., $S_{n-1}=K_{1, n-1}$. For any positive integer $n$, let $[n]=\{1,2,3, \ldots, n\}$.

Let $\mathcal{F}$ be a family of graphs. A graph $G$ is called $\mathcal{F}$-free if $G$ does not contain any subgraph which is isomorphic with some $F \in \mathcal{F}$. The Turán number, denoted by $e x(n, \mathcal{F})$, is the maximum number of edges in an $\mathcal{F}$-free graph on $n$ vertices. When $\mathcal{F}$ contains only one graph $F$, we denote the Turán number of $F$ by ex $(n, F)$. Let $E X(n, F)$ denote an $\mathcal{F}$-free graph on $n$ vertices with ex $(n, F)$ edges. We call this graph an extremal graph for $F$. A traditional starting point of extremal graph theory (a significant branch of graph theory) is the Mantel's theorem (see e.g. [1]) that the maximum number of edges in a $K_{3}$-free graph on $n$ vertices is $\left\lfloor\frac{n^{2}}{4}\right\rfloor$. Turán $[15,16]$ generalized this result to determine the value $e x\left(n, K_{r+1}\right)$ and showed that the unique extremal graph for $K_{r+1}$ is the complete $r$-partite graph on $n$ vertices whose all parts are as equal in size as possible (the difference between any two parts is at most 1), denoted by $T_{r}(n)$. Generally, we call the graph $T_{r}(n)$ a Turán graph. In 1959, Erdős and Gallai [3] determined the value $e x\left(n,(k+1) \cdot K_{2}\right)$ and the extremal graph for it. In [14], Simonovits showed that the unique extremal graph for $k \cdot K_{r+1}$ is $K_{k-1} \vee T_{r}(n-p+1)$ when $n$ is sufficiently large. Later, by considering the graph that consists of $p$ disjoint copies of any connected graph $G$ on $n$ vertices, Gorgol [5] gave a lower bound for $e x(m, p \cdot G)$.

Theorem 1 [3]. When $n \geq 2 k+1$,

$$
e x\left(n,(k+1) \cdot K_{2}\right)=\max \left\{\binom{2 k+1}{2},\binom{n}{2}-\binom{n-k}{2}\right\} .
$$

Theorem 2 [5]. Let $G$ be any connected graph on $n$ vertices, $p$ be any positive integer and $m$ be an integer such that $m \geq p n$. Then $\operatorname{ex}(m, p \cdot G) \geq$ $\max \left\{e x(m-p n+1, G)+\binom{p n-1}{2}, e x(m-p+1, G)+(p-1) m-\binom{p}{2}\right\}$.

In [12], Lidický et al. investigated the Turán number of a star forest and determined the value ex $(n, F)$ when $n$ is sufficiently large, where $F=\bigcup_{i=1}^{k} S_{d_{i}}$ and $d_{1} \geq d_{2} \geq \cdots \geq d_{k}$. Yin and Rao [18] determined the value ex $\left(n, k \cdot S_{l}\right)$ when $n \geq \frac{1}{2} l^{2} k(k-1)+k-2+\max \left\{l k, l^{2}+2 l\right\}$, which improved the results of Lidický et al. Later, Lan et al. [10] determined the value ex $\left(n, k \cdot S_{l}\right)$ when $n \geq k\left(l^{2}+l+1\right)-\frac{l}{2}(l-3)$, which further improved these results. Recently, for a fixed $k$, Li et al. [11] determined the Turán number $e x\left(n, 2 \cdot S_{l}\right)$ for all positive integers $n$ and $l(\geq 4)$ and $e x\left(n, 3 \cdot S_{l}\right)$ for all positive integers $n$ and $l(\geq 3)$.

Recently, Ning and Wang [13] considered the forbidden family $\mathcal{L}_{n, k}$ of subgraphs (i.e., a family of all linear forests of order $n$ with $k$ edges) and determined the exact value $\operatorname{ex}\left(n, \mathcal{L}_{n, k}\right)$. The Hamiltonian completion number of a graph $G$ is the minimum number of edges to ensure that the graph $G$ is Hamiltonian by adding it. Their results determined also the maximum number of edges in a nonHamiltonian graph with fixed Hamiltonian completion number. Notice that the order of the forbidden linear forest and the order $n$ are dependent. Motivated by their results, we determine the Turán number $e x\left(n, \mathcal{S}_{n, k}\right)$ by considering a family of all star forests of order $n$ with $k$ edges, denoted by $\mathcal{S}_{n, k}$.
Theorem 3 [13]. Let $n, k$ be two positive integers and $1 \leq k \leq n-1$. Then

$$
e x\left(n, \mathcal{L}_{n, k}\right)=\max \left\{\binom{k}{2},\binom{n}{2}-\binom{n-\left\lfloor\frac{k-1}{2}\right\rfloor}{ 2}+c\right\}
$$

where $c=0$ if $k$ is odd, and $c=1$ otherwise.
For convenience, we would like to give the definition of the almost $d$-regular graph. We define the almost $d$-regular graph as a graph that contains one vertex of degree $d-1$ and all the other vertices have degree $d$.
Theorem 4. Let $n, k$ be two positive integers and $1 \leq k \leq n-1$. Then

$$
e x\left(n, \mathcal{S}_{n, k}\right)=\left\lfloor\frac{k^{2}-1}{2}\right\rfloor .
$$

Moreover, the unique extremal graph contains a connected component of size $k+1$ and $n-k-1$ isolated vertices. And the connected component is the almost $(k-1)$ regular graph or the ( $k-1$ )-regular graph on $k+1$ vertices.

Considering the median degree of a graph $G$, Loebl, Komlós and Sós [2] conjectured that every graph $G$ of order $n$ with at least $n / 2$ vertices of degree at least $k$ contains each tree $T$ of order $k+1$ as a subgraph. This is a median degree version of the famous Erdős-Sós conjecture. For more results about Loebl-Komlós-Sós conjecture, we refer the reader to $[6,7,8,9]$.

Conjecture 5 (The Erdős-Sós Conjecture). Every graph G order $n$ with average degree greater than $k-1$ contains each tree $T$ of order $k+1$ as a subgraph.

Conjecture 6 (The Loebl-Komlós-Sós Conjecture, [2]). Every graph $G$ order $n$ with median degree greater than $k-1$ contains each tree $T$ of order $k+1$ as a subgraph.

Füredi and Simonovits called it Loebl-Komlós-Sós type problem or the Median problem [4] as follows: for a given graph $G$ of order $n$, which $m$ and $d$ ensure that if $G$ has at least $m$ vertices of degree $\geq d$, then $G$ contains some subgraph $H$.

One could analogously define such problems for families $\mathcal{H}$ of graphs. We solved a problem of this type for $\mathcal{H}=\mathcal{S}_{n, k}$.

Theorem 7. If a simple graph $G$ on $n(n \geq k+2)$ vertices has at least $\frac{3 k}{2}+2$ vertices of degree at least $\left\lceil\frac{k}{2}\right\rceil$, then $G$ contains some graph in $\mathcal{S}_{n, k}$ as its subgraph.

## 2. The Proof of Theorem 4

Proof. We prove this result mainly by induction on $k$. First, if $k=1$, then we know that $\mathcal{S}_{n, 1}$ contains only one edge, i.e., $\mathcal{S}_{n, 1}=K_{2} \cup E_{n-2}$. Notice that if a graph $G$ contains at least one edge, it must contain $\mathcal{S}_{n, 1}$ as its subgraph. Then we have that ex $\left(n, \mathcal{S}_{n, 1}\right)=0=\left\lfloor\frac{1^{2}-1}{2}\right\rfloor$ and the unique extremal graph is $E_{n}$. If $k=2$, we can see that $\mathcal{S}_{n, 2}=\left\{S_{2} \cup E_{n-3}, 2 \cdot K_{2} \cup E_{n-4}\right\}$. Notice that the graph which does not contain any graph in $\mathcal{S}_{n, 2}$ as its subgraph must have maximum degree at most 1. If there are two edges in distinct connected components of $G$, then we can find a copy of $2 \cdot K_{2}$ in $G$, a contradiction. Thus, we have that $e x\left(n, \mathcal{S}_{n, 2}\right) \leq 1$. And, the graph $K_{2} \cup E_{n-2}$ does not contain any graph in $\mathcal{S}_{n, 2}$ as its subgraph and $e\left(K_{2} \cup E_{n-2}\right)=1$. Then $e x\left(n, \mathcal{S}_{n, 2}\right)=1=\left\lfloor\frac{2^{2}-1}{2}\right\rfloor$ and the unique extremal graph is $K_{2} \cup E_{n-2}$. We can see that $\mathcal{S}_{n, 3}=\left\{S_{3} \cup E_{n-4}, S_{2} \cup K_{2} \cup E_{n-5}, 3 \cdot K_{2} \cup E_{n-6}\right\}$ when $k=3$. It is easy to know that the graph which does not contain any graph in $\mathcal{S}_{n, 3}$ as its subgraph must have maximum degree at most 2. Also, we can see that there are at most two connected components in a extremal graph for $\mathcal{S}_{n, 3}$ which is denoted by $G$. If there are exactly two connected components in $G$, then both of two connected components contain only an edge, thus $e(G)=2$. If there is only one connected component in $G$, then we have that $\Delta(G) \leq 2$.

If $\Delta(G)=1$, then $e(G)=1$. If $\Delta(G)=2$ and $v$ is a vertex in $G$ with degree 2 , then let $N(v)=\left\{v_{1}, v_{2}\right\}$. If $v_{1} v_{2}$ is an edge of $G$, then $e(G)=3$. If $v_{1} v_{2}$ is not an edge of $G$, then $v_{1}$ and $v_{2}$ can only be adjacent to the same vertex in $V(G) \backslash\left\{v, v_{1}, v_{2}\right\}$ and this fourth vertex is the last in the connected component. Otherwise, we can find a copy of $S_{2} \cup K_{2}$ in $G$, a contradiction. Thus, we have that $\operatorname{ex}\left(n, \mathcal{S}_{n, 3}\right)=4=\left\lfloor\frac{3^{2}-1}{2}\right\rfloor$ and $K_{2,2} \cup E_{n-4}$ is the unique extremal graph.

Now, we assume that the result holds for all $k^{\prime}<k$. In the following, we will show that the result holds for $k$. From the above analyses, we can see that the extremal graph has maximum degree $k-1$ when $k \in\{1,2,3\}$. We claim that this conclusion holds for $k$. In order to prove that the claim is true, we first would like to construct an extremal graph for $\mathcal{S}_{n, k}$, denoted by $G$, and then to prove any graph $G^{\prime}$ with $\Delta\left(G^{\prime}\right)=k-1$ which contains no $\mathcal{S}_{n, k}$ as its subgraph can contain at most $e(G)$ edges. Then we would like to show that all other graphs with maximum degree less than $k-1$ have less edges than the constructed graph $G$.

The constructed extremal graph is as follows. $G$ contains only one connected component containing an edge which is the almost $(k-1)$-regular graph or the $(k-1)$-regular graph on $k+1$ vertices. We denote the connected component of $G$ by $G_{1}$. Notice that if $G$ contains any graph in $\mathcal{S}_{n, k}$, then all edges of it must be contained in $G_{1}$. There is only one graph in $\mathcal{S}_{n, k}$ which contains only one connected component, that is $S_{k} \cup E_{n-k-1}$ and the only connected component is $S_{k}$. Since any graph in $\mathcal{S}_{n, k}$ other than $S_{k} \cup E_{n-k-1}$ has at least two connected components which contain at least $k+2$ vertices, $G_{1}$ cannot contain any of them as a subgraph. If $G$ is not $\mathcal{S}_{n, k}$-free, then $G_{1}$ must contain $S_{k}$ as its subgraph. It is easy to deduce that $G_{1}$ does not contain $S_{k}$ as its subgraph since $G_{1}$ is the almost $(k-1)$-regular graph or the $(k-1)$-regular graph on $k+1$ vertices. Thus $G$ is $\mathcal{S}_{n, k}$-free.

Next, we will prove that any graph $G^{\prime}$ with $\Delta\left(G^{\prime}\right)=k-1$ which does not contain any graph in $\mathcal{S}_{n, k}$ as its subgraph can contain at most $e(G)$ edges. Without loss of generality, we assume that $d(v)=k-1$ for $v \in V\left(G^{\prime}\right)$ and let $N(v)=\left\{v_{1}, v_{2}, \ldots, v_{k-1}\right\}$. Notice that there are no edges in $G^{\prime}\left[V\left(G^{\prime}\right) \backslash N[v]\right]$, otherwise we can find a $S_{k-1} \cup K_{2} \cup E_{n-k-2} \in \mathcal{S}_{n, k}$ in $G^{\prime}$, a contradiction. Thus, all edges of $G^{\prime}$ must be contained in the connected component which contains all vertices in $N(v)$. Any $v_{i}$ can be adjacent to at most one vertex in $V\left(G^{\prime}\right) \backslash N[v]$, thus there are at most $k-1$ edges between $N(v)$ and $V\left(G^{\prime}\right) \backslash N[v]$. For any $v_{i} \in N(v)$, if there is an edge between the vertex $v_{i}$ and the vertex set $V\left(G^{\prime}\right) \backslash N[v]$, then by the definition of $\Delta$, there exist at most $k-3$ edges between the vertex $v_{i}$ and the vertex set $N(v)$. Therefore, if there are $y \leq k-1$ edges between the two vertex sets $N(v)$ and $V\left(G^{\prime}\right) \backslash N[v]$, then at least $\left\lceil\frac{y}{2}\right\rceil$ edges are missing inside $N(v)$. Therefore, the total number of edges is

$$
\begin{equation*}
e\left(G^{\prime}\right) \leq\binom{ k}{2}-\left\lceil\frac{y}{2}\right\rceil+y \leq\left\lfloor\frac{k^{2}-1}{2}\right\rfloor . \tag{1}
\end{equation*}
$$

We can explain the first inequality of Equation (1) as follows. The first two terms are an upper bound on the number of edges inside $N[v]$, and the last term is the number of other edges by definition.

Notice that the extremal graph can contain only one vertex in $V\left(G^{\prime}\right) \backslash N[v]$ which is adjacent to every vertex in $N(v)$. Assume that there are two vertices which are adjacent to the vertex set $N(v)$, denoted by $v_{x}, v_{y}$. Then we have that $d_{N(v)}\left(v_{x}\right)+d_{N(v)}\left(v_{y}\right)=k-1$. Since $G^{\prime}[N(v)]$ is an almost $(k-3)$-regular graph, we can find a copy of $S_{2} \cup S_{k-2}$, a contradiction. Thus, we know that the constructed graph $G$ is the unique extremal graph $G^{\prime}$ for $\Delta\left(G^{\prime}\right)=k-1$.

Our third step is to prove the following conclusion. All other graphs with maximum degree less than $k-1$ contain less edges than $G$.

First, we claim that the extremal graph for $\mathcal{S}_{n, k}$ contains only one connected component. Assume that there are two connected components in the extremal graph for $\mathcal{S}_{n, k}$ and there is one connected component such that the largest star forest in it has $x$ edges. By the induction hypothesis, we have that the extremal graph contains at most $\left\lfloor\frac{(x+1)^{2}-1}{2}\right\rfloor+\left\lfloor\frac{(k-x)^{2}-1}{2}\right\rfloor \leq\left\lfloor\frac{(k)^{2}-1}{2}\right\rfloor$ edges. Thus we have that the extremal graph for $\mathcal{S}_{n, k}$ contains only one connected component. Let $H$ be an extremal graph for $\mathcal{S}_{n, k}$ that contains only one connected component $H^{\prime}$ containing an edge which has maximum degree $3 \leq \Delta\left(H^{\prime}\right)=t<k-1$. Let $d(u)=t$ and $N(u)=\left\{u_{1}, u_{2}, \ldots, u_{t}\right\}$. In the following, we divide the edge set of the graph $H$ into two parts: the edges of the graph $H\left[V\left(H^{\prime}\right) \backslash N[u]\right]$ and other edges. We denote the graph which is induced by the second set of edges by $H_{1}$.

Case 1. $H_{1}$ contains no star forest with more than $t$ edges. Any $u_{i}$ can be adjacent to at most one vertex in $V\left(H^{\prime}\right) \backslash N[u]$, otherwise we can find a copy of $S_{2} \cup S_{t-1}$, a contradiction. Each vertex $u_{i}$ can be adjacent to at most $t-1$ vertices in $N(u)-u_{i}$ for $i \in[1, t]$. For any $u_{i} \in N(u)$, if there is an edge between the vertex $u_{i}$ and the vertex set $V\left(H_{1}\right) \backslash N[u]$, then by the definition of $\Delta$, there exist at most $t-2$ edges between the vertex $u_{i}$ and the vertex set $N(u)$. Therefore, if there are $z \leq t$ edges between the two vertex sets $N(u)$ and $V\left(H_{1}\right) \backslash N[u]$, then at least $\left\lceil\frac{z}{2}\right\rceil$ edges are missing inside $N(u)$. Therefore, the total number of edges is

$$
\begin{equation*}
e\left(H_{1}\right) \leq\binom{ t+1}{2}-\left\lceil\frac{z}{2}\right\rceil+z \leq\left\lfloor\frac{t(t+2)}{2}\right\rfloor . \tag{2}
\end{equation*}
$$

We can explain the first inequality of Equation (2) as follows. The first two terms are an upper bound on the number of edges inside $N[u]$, and the last term is the number of other edges by definition.

Then by the induction hypothesis and Equation (2),

$$
\begin{align*}
e(H)=e\left(H^{\prime}\right) & \leq\left\lfloor\frac{(k-t)^{2}-1}{2}\right\rfloor+\left\lfloor\frac{t(t+2)}{2}\right\rfloor \\
& =\left\lfloor\frac{k^{2}-1+2 t^{2}+2 t-2 k t}{2}\right\rfloor \leq\left\lfloor\frac{k^{2}-1}{2}\right\rfloor=e(G) . \tag{3}
\end{align*}
$$

Case 2. $H_{1}$ contains a star forest with $y+t-1$ edges. But $H_{1}$ contains no star forest with more than $y+t-1$ edges. Similarly, any $u_{i}$ can be adjacent to at most $y$ vertices in $V\left(H^{\prime}\right) \backslash N[u]$, otherwise we can find a copy of $S_{y+1} \cup S_{t-1}$, a contradiction. Therefore, there are at most ty edges between the set $N(u)$ and the set $V\left(H^{\prime}\right) \backslash N[u]$. We assume that $u_{i}$ is adjacent to $y_{i}$ vertices in $V\left(H^{\prime}\right) \backslash N[u]$ for $i \in[1, t]$. Then

$$
\begin{align*}
e\left(H_{1}\right) & \leq\left\lfloor\frac{\sum_{i=1}^{t}\left(t-1-y_{i}\right)}{2}\right\rfloor+\sum_{i=1}^{t} y_{i}+t  \tag{4}\\
& \leq\left\lfloor\frac{t(t-1-y)}{2}\right\rfloor+t(y+1) .
\end{align*}
$$

Then by the induction hypothesis and Equation (4),

$$
\begin{align*}
e(H) & \leq\left\lfloor\frac{t(t-1-y)}{2}\right\rfloor+t(y+1)+\left\lfloor\frac{(k-y-t+1)^{2}-1}{2}\right\rfloor  \tag{5}\\
& =\left\lfloor\frac{y^{2}+(4 t-2 k-2) y}{2}\right\rfloor+\left\lfloor\frac{k^{2}+2 t^{2}-t+1-2 k t+2 k}{2}\right\rfloor .
\end{align*}
$$

Further, we analyze two subcases as follows.
Subcase 2.1. $t \leq\left\lfloor\frac{k-1}{2}\right\rfloor+1$. Considering the term $y$ and $2 \leq y \leq t-1$, we can see that the right part of the above inequality about $y$ is a parabola with an upward opening, which obtains a value of 0 at points $y=0$ and $y=$ $2 k+2-4 t$. If $t-1 \geq 2 k+2-4 t$ (i.e., $t \geq(2 k+3) / 5$ ), then $e(H)$ can attains its maximum value at $y=t-1$. Otherwise, $e(H)$ can attain its maximum value at $y=2$. When $t \geq(2 k+3) / 5$, we can calculate the maximum number of edges as $\left\lfloor\frac{k^{2}+7 t^{2}-9 t+4-4 k t+4 k}{2}\right\rfloor$. And the maximum number of edges is $\left\lfloor\frac{k^{2}+2 t^{2}+7 t+1-2 k t-2 k}{2}\right\rfloor$ when $t<(2 k+3) / 5$. The two values are both less than $e(G)=\left\lfloor\frac{k^{2}-1}{2}\right\rfloor$. Thus, we prove the above conclusion when $t \leq\left\lfloor\frac{k-1}{2}\right\rfloor+1$.

Subcase 2.2. $\left\lfloor\frac{k-1}{2}\right\rfloor+1<t \leq k-1$. It is easy to know that $2 \leq y \leq k-t$. By a simple calculation for Equation (5), we know that the maximum value of the right part of Equation (5) can be obtained when $y=k-t$. Substituting
$y=k-t$ into Equation (5), we obtain that the maximum value is $\left\lfloor\frac{k t+t}{2}\right\rfloor$. We can obtain that it is smaller than $e(G)$ by a simple calculation. Thus we prove that the conclusion holds for all $k$.

Thus, our claim is true that the extremal graph has maximum degree $k-1$. This completes our proof.

## 3. The Proof of Theorem 7

Proof. To prove this conclusion, we just need to prove that there are at most $\frac{3 k}{2}+1$ vertices of degree at least $\left\lceil\frac{k}{2}\right\rceil$ if $G$ contains no graph in $\mathcal{S}_{n, k}$ as its subgraph. We define $d$ as the smallest number that is at least $\left\lceil\frac{k}{2}\right\rceil$ and there is a vertex of degree exactly $d$. Without loss of generality, let $v \in V(G), d(v)=d$ and $N(v)=\left\{v_{1}, v_{2}, \ldots, v_{d}\right\}$. Notice that every vertex not in $N[v]$ nor in $N(N(v))$ has degree at most $k-d-1$, otherwise we can find a copy of $S_{d} \cup S_{k-d} \cup E_{n-k-2} \in \mathcal{S}_{n, k}$, a contradiction. Thus we know that every vertex which has degree at least $\left\lceil\frac{k}{2}\right\rceil$ can be contained only in $N[v]$ or $N(N(v))$. Every vertex in $N(v)$ has at most $k-d$ neighbors in $V(G) \backslash N[v]$. Thus we know that there are at most $d(k-d)$ edges between the vertex set $N(v)$ and $N(N(v))$.

Claim 8. There are at most $\frac{3 k}{2}-d$ vertices in $N(N(v))$ which have degree at least d.

In the following proof, we call a vertex which has degree at least $\left\lceil\frac{k}{2}\right\rceil$ as the large degree vertex.

Proof. Suppose that there are at least $\frac{3 k}{2}-d+1$ large degree vertices in $N(N(v))$. By Theorem 4, there are at most $\left\lfloor\frac{(k-d)^{2}-1}{2}\right\rfloor$ edges in $G[V(G) \backslash N[v]]$. It follows that the large degree vertices in $N(N(v))$ are incident to at least $\left(\frac{3 k}{2}-d+1\right) d$ edges, but there are at most $(k-d)^{2}-1$ incidences inside $G[V(G) \backslash N[v]]$, thus at least $\left(\frac{3 k}{2}-d+1\right) d-\left((k-d)^{2}-1\right)=\frac{7 k d}{2}-2 d^{2}-k^{2}+d+1$ incidences are by edges between $N(v)$ and $N(N(v))$. Thus, the number of edges between $N(v)$ and $N(N(v))$ must be no less than $\frac{7 k d}{2}-2 d^{2}-k^{2}+d+1$, which is greater than $d(k-d)$. This contradicts with the fact that there are at most $d(k-d)$ edges between the vertex set $N(v)$ and $N(N(v))$. Thus, we have that there are at most $\frac{3 k}{2}-d$ vertices in $G$ which have degree at least $d$.

Combining Claim 8 and the fact that there are at most $d+1$ vertices in $N[v]$ which have degree at least $d$, we conclude that there are at most $\frac{3 k}{2}+1$ vertices of degree at least $\left\lceil\frac{k}{2}\right\rceil$ if $G$ contains no graph in $\mathcal{S}_{n, k}$ as its subgraph. This completes our proof.

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    ${ }^{2}$ Corresponding author.

