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FORBIDDEN SUBGRAPHS FOR EXISTENCES OF (CONNECTED) 2-FACTORS OF A GRAPH

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Abstract

Clearly, having a 2-factor in a graph is a necessary condition for a graph to be hamiltonian, while having an even factor in graph is a necessary condition for a graph to have a 2-factor. In this paper, we completely characterize the forbidden subgraph and pairs of forbidden subgraphs that force a 2-connected graph admitting a 2-factor (a necessary condition) to be hamiltonian and a connected graph with an even factor (a necessary condition) to have a 2-factor, respectively. Our results show that these pairs of forbidden subgraphs become wider than those in Faudree, Gould and in Fujisawa, Saito, respectively, if we impose the two necessary conditions, respectively.

Keywords: forbidden subgraph, even factor, 2-factor, hamiltonian.

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1. Introduction

All graphs considered in this paper are finite, undirected and simple. For notation and terminology not defined here, see [3]. We denote by $V(G)$, $E(G)$, $\Delta(G)$ the vertex set, the edge set, the maximum degree of a graph G, respectively. We denote by $N_G(v)$ (or simply $N(v)$) and $d_G(v)$ (or simply $d(v)$) the neighborhood and the degree of a vertex v in G, respectively. For $S \subseteq V(G)$, we define $N_G(S) = \bigcup_{x\in S} N_G(x)$. Let $S \subseteq V(G)$ and $S' \subseteq E(G)$. The *induced subgraph* of G by S and S' is denoted by $G[S]$ and $G[S']$, respectively. We use $G \setminus S$ and $G \setminus S'$ to denote the subgraph $G[V(G) \setminus S]$ and $G[E(G) \setminus S']$, respectively. Let $X, Y \subseteq V(G)$ with $X \cap Y = \emptyset$, then we define $E(X,Y) = \{uv \in E(G) | u \in X, v \in Y\}$.

A *complete graph* on n vertices is denoted by K_n. A *complete bipartite graph* with m vertices in one set and n vertices in the other set is denoted by $K_{m,n}$. Let P_n and C_n denote the path and the cycle of order n, respectively. A *clique* is a complete subgraph of a graph. An *independent set* of a graph is a set of vertices no two of which are adjacent. The cardinality of a maximum independent set of G is denoted by $\alpha(G)$.

A spanning subgraph of a graph is called a *factor*. An *even factor* of G is a spanning subgraph of G in which every vertex has even positive degree. A 2-*factor* of a graph G is a spanning subgraph in which every vertex has degree 2. A *hamiltonian* graph has a 2-factor with exactly one component, i.e., a connected 2-factor.

Let $\mathcal{H}$ be a set of connected graphs. A graph G is said to be $\mathcal{H}$-*free* if G does not contain H as an induced subgraph for any H in $\mathcal{H}$, and we call each graph H of $\mathcal{H}$ a *forbidden subgraph.* If $\mathcal{H} = \{H\}$, then we simply say that G is H-*free.* We call $\mathcal{H}$ a *forbidden pair* if $|\mathcal{H}| = 2$. In order to state results clearly, we further introduce the following notation. For two sets $\mathcal{H}_1$ and $\mathcal{H}_2$ of connected graphs, we write $\mathcal{H}_1 \preceq \mathcal{H}_2$ if for every graph H' in $\mathcal{H}_2$, there exists a graph H'' in $\mathcal{H}_1$ such that H'' is an induced subgraph of H'. By the definition of the relation "$\preceq$", if $\mathcal{H}_1 \preceq \mathcal{H}_2$, then every $\mathcal{H}_1$-free graph is also $\mathcal{H}_2$-free.

The forbidden pairs that force the existence of a hamiltonian cycle or 2-factor of 2-connected graphs had been studied in [6] and [5], respectively. Further graphs used as forbidden induced subgraphs are shown in Figure 1.

Theorem 1 (Faudree and Gould, [6])**.** *Let R and S be connected graphs other than an induced subgraph of P_3. Then every* 2*-connected $\{R, S\}$-free graph of order at least* 10 *is hamiltonian if and only if $\{R, S\} \preceq \{K_{1,3}, P_6\}$, $\{K_{1,3}, Z_3\}$, $\{K_{1,3}, B_{1,2}\}$ or $\{K_{1,3}, N_{1,1,1}\}$.*

Theorem 2 (Faudree, Faudree and Ryjáček, [5])**.** *Let R and S be connected graphs other than an induced subgraph of P_3. Then every* 2*-connected $\{R, S\}$-free graph of order at least* 10 *has a* 2*-factor if and only if $\{R, S\} \preceq \{K_{1,3}, B_{1,4}\}$, $\{K_{1,3}, N_{3,1,1}\}$ or $\{K_{1,4}, P_4\}$.*

The following result reveals the existence of 2-factor in a connected graph.

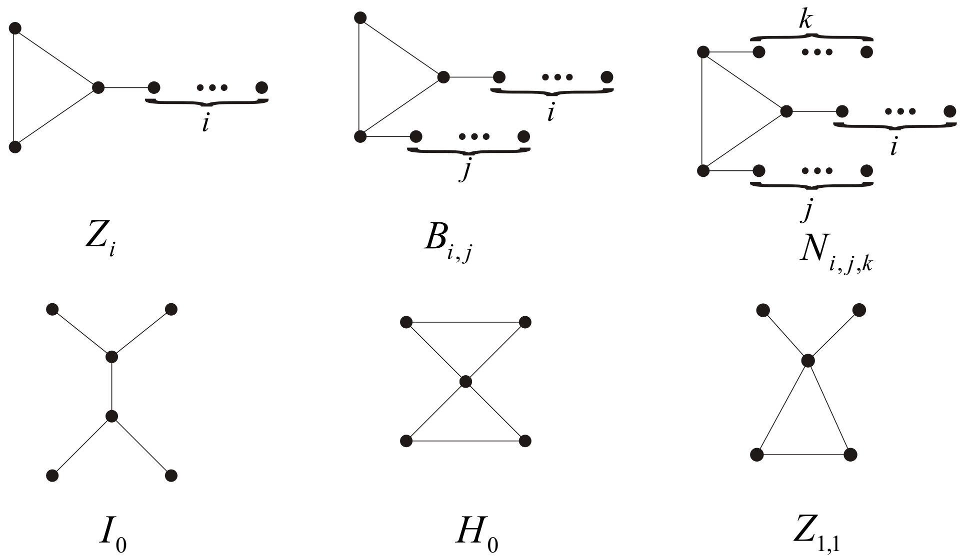

Figure 1. Some common induced subgraphs.

Theorem 3 (Fujisawa and Saito, [7])**.** *Let R and S be connected graphs of order at least three. Then there exists a positive integer n_0 such that every connected $\{R, S\}$-free graph of order at least n_0 and minimum degree at least two has a 2-factor if and only if $\{R, S\} \preceq \{K_{1,3}, Z_2\}$.*

Obviously, a hamiltonian graph should have a 2-factor and a 2-factor should be an even factor. However, the converse is not true in general. In other words, the condition that a graph has an even factor is a necessary for a graph to have a 2-factor and similarly the existence of a 2-factor is a necessary for a graph to be hamiltonian.

The problem of deciding whether or not a graph has a Hamilton cycle was one of first decision problems proved to be NP-complete by Karp [8, 9]. The problem remains NP-complete, even if the graphs are restricted to be $K_{1,3}$-free (see [2]). However, Anstee showed [1] that for any graph, there is an algorithm that either finds a k-factor or shows that it does not exist. This implies that 2-factors can be determined in polynomial time. Therefore, it is interesting to add a 2-factor condition when we consider whether a graph is hamiltonian.

Question 4. *Are there a wider set $\mathcal{H}$ of forbidden subgraphs when we impose a necessary condition on those 2-connected* (*connected*) *graphs to be hamiltonian* (*or to have a 2-factor, respectively*)?

A similar problem is considered in [10]. In this paper, we answer the question for $|\mathcal{H}| = 1, 2$ by proving the following results. Here, we use $K_4 - e$ to denote the graph by removing one edge from K_4.

Theorem 5. *It holds that*

(1) *Every* 2*-connected A-free graph admitting a* 2*-factor is hamiltonian if and only if* A *is* P_3*.*

(2) *Let* R*,* S *be two connected graphs other than an induced subgraph of* P_3*, and let* G *be a graph admitting a* 2*-factor. Then every* 2*-connected* $\{R, S\}$*-free graph* G *is hamiltonian if and only if* $\{R, S\} \preceq \{I_0, Z_1\}$*,* $\{P_4, K_4 - e\}$*,* $\{P_4, Z_{1,1}\}$ *or* $\{K_{1,3}, P_6\}$*,* $\{K_{1,3}, Z_3\}$*,* $\{K_{1,3}, B_{1,2}\}$*,* $\{K_{1,3}, N_{1,1,1}\}$*.*

Theorem 6. *It holds that*

(1) *Every* 2*-connected A-free graph admitting an even factor has a* 2*-factor if and only if* A *is* P_3*.*

(2) *Let* R*,* S *be two connected graphs other than an induced subgraph of* P_3*, and let* G *be a connected graph of order at least* 6 *admitting an even factor. Then every* $\{R, S\}$*-free graph* G *has a* 2*-factor if and only if* $\{R, S\} \preceq \{K_{1,4}, Z_1\}$*,* $\{K_{1,3}, H_0\}$ *or* $\{K_{1,3}, Z_2\}$*.*

Comparing both Theorems 1 and 5(2), and both Theorems 3 and 6(2), we know that pairs of forbidden subgraphs for a 2-connected graph to be hamiltonian (or to have 2-factor, respectively) become wider than those, if we impose a necessary condition that graphs in consideration have a 2-factor (or even factor, respectively).

2. Forbidden Subgraphs Guaranteeing a Graph with 2-Factor to be Hamiltonian: the Proof of Theorem 5

In this section, we completely characterize connected forbidden subgraphs and pairs of connected forbidden subgraphs that force a 2-connected graph admitting a 2-factor to be hamiltonian.

The following result was due to Egawa [4] who observed that the first one was proved implicitly by Faudree *et al.* [5].

Theorem 7 (Egawa, [4])**.** *Let* G *be a connected non-complete* P_4*-free graph and* S *be a smallest vertex-cut of* G*. Then each vertex in* S *is adjacent to all vertices in* $V(G) \setminus S$*.*

Lemma 8. *Every* 2*-connected* $\{K_4 - e, P_4\}$*-free graph is either a complete graph or a complete bipartite graph.*

Proof. Let G be a 2-connected $\{K_4 - e, P_4\}$-free graph and S be a smallest vertex-cut of G. Then $|S| \geq 2$. We suppose that G is a non-complete graph. Since G is P_4-free, by Theorem 7, each vertex in S is adjacent to all vertices in $V(G) \setminus S$. Suppose that there exists a pair of adjacent vertices $\{s_1, s_2\} \subseteq S$.

Then $G[V \setminus S]$ is a clique, otherwise, assume that there exists a pair of non-adjacent vertices $\{u_1, u_2\} \subseteq V(G) \setminus S$, then $G[\{s_1, s_2, u_1, u_2\}] \cong K_4 - e$, a contradiction. Furthermore, $G[S]$ is a clique, otherwise, assume that there exists a pair of non-adjacent vertices $\{v_1, v_2\} \subseteq S$, then $G[\{v_1, v_2, w_1, w_2\}] \cong K_4 - e$, where $\{w_1, w_2\} \subseteq V(G) \setminus S$, a contradiction. Then G is a complete graph, contradicting our assumptions that G is a non-complete graph. This proves that S is an independent set. Then $V(G) \setminus S$ is an independent set, otherwise, assume that there exists a pair of adjacent vertices $\{u_1, u_2\} \subseteq V(G) \setminus S$, then $G[\{s_1, s_2, u_1, u_2\}] \cong K_4 - e$, where $\{s_1, s_2\} \subseteq S$, a contradiction. Therefore, G is a complete bipartite graph. ■

Theorem 9. *If G is a 2-connected $\{K_4 - e, P_4\}$-free graph admitting a 2-factor, then G is hamiltonian.*

Proof. By Lemma 8, G is a complete graph or a complete bipartite graph. If G is a complete graph, then G is hamiltonian. If G is a complete bipartite graph, then G is a balanced complete bipartite graph, i.e., $G \cong K_{m,m}$ (since G has a 2-factor). Then G is hamiltonian. ■

Let H and F be subgraphs of G. We define $H \triangle F$ by $H \triangle F = (V(H) \cup V(F), E(H) \triangle E(F))$, where $A \triangle B$ denotes the symmetric difference of the sets A and B. Note that if H and F are even subgraphs, then $H \triangle F$ is also an even graph, but $H \triangle F$ may have more components than H or F. Let $C(x_1 x_2 \cdots x_n x_1)$ denote the cycle $x_1 x_2 \cdots x_n x_1$.

Theorem 10. *Let G be a 2-connected graph admitting a 2-factor such that it satisfies one of the following.*

(1) *G is a $\{I_0, Z_1\}$-free graph, where I_0 is depicted in Figure* 1;

(2) *G is a $\{P_4, Z_{1,1}\}$-free graph, where $Z_{1,1}$ is depicted in Figure* 1.

Then G is hamiltonian.

Proof. Let G be a 2-connected graph admitting a 2-factor. Choose a 2-factor F of G with components $Q_1, \ldots, Q_t$ ($t \geq 1$) such that t is as small as possible. We shall show that $t = 1$. Otherwise, there exists an edge $e \in E(G) \setminus E(F)$ such that the two end-vertices of e are in different components of F. Take such an edge xy such that $x \in V(Q_i)$ and $y \in V(Q_j)$, $\{i, j\} \subseteq \{1, 2, \ldots, t\}$. Let $\{x_1, x_2\} \subseteq N_{Q_i}(x)$, $\{y_1, y_2\} \subseteq N_{Q_j}(y)$. For any $s, t \in \{1, 2\}$, we have that $x_s y_t \notin E(G)$, otherwise, $F \triangle C(xyy_t x_s x)$ is a 2-factor with fewer components than F, a contradiction. We claim that if $xy_s \in E(G)$, then $yx_t \notin E(G)$, otherwise, $F \triangle C(xx_t yy_s x)$ is a 2-factor with fewer components than F, a contradiction.

***Proof of* (1).** Let G be a $\{I_0, Z_1\}$-free graph. Then $x_1 y \notin E(G)$, otherwise, $G[\{x, x_1, y, y_2\}] \cong Z_1$, a contradiction. By symmetry, $\{yx_2, xy_1, xy_2\} \cap E(G) =$

$\emptyset$. Then $x_1x_2 \notin E(G)$, otherwise, $G[\{x, x_1, x_2, y\}] \cong Z_1$, a contradiction. By symmetry, $y_1y_2 \notin E(G)$. Then $G[\{x, x_1, x_2, y, y_1, y_2\}] \cong I_0$, a contradiction. Therefore, $t = 1$ and G is hamiltonian. $\square$

***Proof of* (2).** Let G be a $\{P_4, Z_{1,1}\}$-free graph. Since $G[\{x_1, x, y, y_1\}] \ncong P_4$, $\{xy_1, yx_1\} \cap E(G) \neq \emptyset$. Note that $|\{xy_1, yx_1\} \cap E(G)| \neq 2$. By symmetry, we suppose that $xy_1 \in E(G)$. Since $G[\{x_2, x, y, y_2\}] \ncong P_4$, $xy_2 \in E(G)$. Since $G[\{x, y, y_1, x_1, x_2\}] \ncong Z_{1,1}$, $x_1x_2 \in E(G)$. Since $G[\{x, x_1, x_2, y_1, y_2\}] \ncong Z_{1,1}$, $y_1y_2 \in E(G)$. Therefore, if $|V(Q_j)| = 3$, then $G[V(Q_j)] \cong K_3$ and x is adjacent to each vertex in $V(Q_j)$. If $|V(Q_j)| \geq 4$, let $Q_j = yy_1a_1a_2 \cdots a_{|V(Q_j)|-3}y_2y$. Since $G[\{x_1, x, y_1, a_1\}] \ncong P_4$, $xa_1 \in E(G)$. Since $G[\{x_1, x, a_1, a_2\}] \ncong P_4$, $xa_2 \in E(G)$. Then we claim that x is adjacent to each vertex in $V(Q_j)$, otherwise, $G[\{x_1, x, a_{i-1}, a_i\}] \cong P_4$, where $xa_i \notin E(G)$, a contradiction. Furthermore, $G[V(Q_j)]$ is a clique, otherwise, the subgraph induced by the two non-adjacent vertices in $V(Q_j)$ and $\{x, x_1, x_2\}$ is an induced $Z_{1,1}$, a contradiction.

Since G is 2-connected, xy, xx_1 are in a cycle. Choose an induced cycle $C = xx_1w_1 \cdots w_{|V(C)|-3}yx$ of G such that $\{xy, xx_1\} \subseteq E(C)$. Since G is P_4-free, $|V(C)| \leq 4$. Recall that $yx_1 \notin E(G)$. Then $|V(C)| = 4$ and $C = xx_1w_1yx$. Since $G[V(Q_j)]$ is a clique and x is adjacent to each vertex in $V(Q_j)$, $w_1 \notin V(Q_j)$. Since F is a 2-factor of G, let $N_F(w_1) = \{v_1, v_2\}$. First we suppose that $w_1 \in V(Q_i)$. Since $G[\{w_1, x_1, x, y_1\}] \ncong P_4$, $y_1w_1 \in E(G)$. Recall that $E(\{v_1, v_2\}, \{y, y_1\}) = \emptyset$. Since $G[\{w_1, v_1, v_2, y, y_1\}] \ncong Z_{1,1}$, $v_1v_2 \in E(G)$. Therefore, $F \triangle (C(xy_2yw_1x_1x) \cup C(w_1v_1v_2w_1))$ is a 2-factor with fewer components than F, a contradiction. Next we suppose that $w_1 \in V(Q_k)$, where $k \in \{1, 2, \ldots, t\} \backslash \{i, j\}$. Since C is an induced cycle, $xw_1 \notin E(G)$. Since $G[\{x, x_1, w_1, v_1\}] \ncong P_4$, $\{v_1x, v_1x_1\} \cap E(G) \neq \emptyset$. If $v_1x \in E(G)$ or $v_1x_1 \in E(G)$, then $F \triangle C(xx_1w_1v_1x)$ or $F \triangle C(xx_1v_1w_1yy_1x)$ is a 2-factor with fewer components than F, a contradiction. This proves (2). The proof of this theorem is complete. $\square$

$\blacksquare$

Now, we present the proof of Theorem 5.

***Proof of Theorem* 5.** (1) If G is P_3-free, then G is a complete graph and hence G is hamiltonian. Conversely, graphs G_1, G_2, G_4 in Figure 2 are 2-connected admitting a 2-factor but non-hamiltonian.

Then A must be an induced subgraph of them. Without loss of generality, we assume that A is an induced subgraph of G_1. Then A is a tree with maximum degree at most 3 or contains a K_3. Note that G_2 is K_3-free. This implies that A contains no cycle. Thus, A is a tree. Since G_4 is $K_{1,3}$-free, A is a path. Note that the maximal induced path of G_1 is P_3. Therefore, A is P_3.

(2) By Theorems 1, 9 and 10, *the sufficiency* clearly holds. It remains to show *the necessity.* All graphs in Figure 2 are 2-connected with a 2-factor but non-hamiltonian. Then each graph contains at least one of R, S as an induced

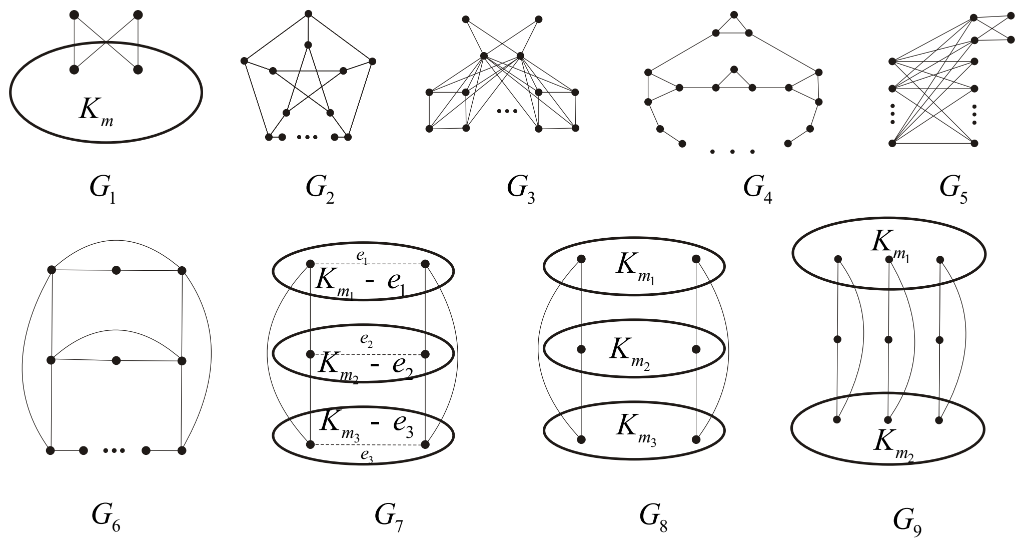

Figure 2. Graphs with 2-factor but non-hamiltonian.

subgraph. Without loss of generality, we assume that G_1 contains R as an induced subgraph. Then R is a tree with maximum degree at most 3 or contains a triangle.

Case 1. R is a tree. Then $\Delta(R) \leq 3$. We claim that $\Delta(R) = 3$, otherwise, R is a path and R is P_3, a contradiction. Since the maximal induced tree of G_1 is $K_{1,3}$, R is $K_{1,3}$. Since G_4, G_6, G_7, G_8, G_9 are $K_{1,3}$-free, S should be an induced subgraph of G_4, G_6, G_7, G_8, G_9. Then S is a path or contains a cycle. Note that the longest induced path of G_9 is P_6. Therefore, if S is a path, then S is an induced subgraph of P_6. Therefore, $\{R, S\} \preceq \{K_{1,3}, P_6\}$. Now we suppose that S contains a cycle. Note that the maximal common induced cycle of G_4, G_6, G_7, G_8, G_9 is K_3. Then S contains a K_3. Furthermore, S contains exactly one K_3. Since the maximal induced subgraph containing Z_i of G_9 is Z_3, we get $\{R, S\} \preceq \{K_{1,3}, Z_3\}$. Since the maximal induced subgraph containing $B_{i,j}$ of G_7 is $B_{1,2}$, we have $\{R, S\} \preceq \{K_{1,3}, B_{1,2}\}$. Finally, observe that the maximal induced subgraph containing $N_{i,j,k}$ of G_8 is $N_{1,1,1}$. Therefore, $\{R, S\} \preceq \{K_{1,3}, N_{1,1,1}\}$.

Case 2. R contains a triangle. First we suppose that R contains a K_4. Since G_2, G_3, G_4 are K_4-free, S is an induced subgraph of G_2, G_3, G_4. Since G_2 and G_4 have no common induced cycle, S is a tree. Since G_4 is $K_{1,3}$-free, S should be a path. Since the maximal induced path of G_3 is P_3, S is an induced subgraph of P_3, a contradiction.

Now we suppose that R contains a K_3 but no K_4. Since G_2, G_5 are K_3-free, S should be an induced subgraph of G_2, G_5. Since G_2, G_5 have no common induced cycle, S is a tree. Since $\Delta(G_2) = 3$, $\Delta(S) \leq 3$. If $\Delta(S) = 2$, then S is a path. Since the maximal induced path of G_5 is P_4, S is an induced subgraph of P_4.

Since G_1 and G_3 are P_4-free, R is an induced subgraph of G_1 and G_3. Recall that R contains a K_3. Next, observe that the maximal common induced subgraphs of G_1, G_3 containing a K_3 are $K_4 - e$ and $Z_{1,1}$. Therefore, $\{R, S\} \preceq \{P_4, K_4 - e\}$ or $\{P_4, Z_{1,1}\}$. If $\Delta(S) = 3$, then S contains a $K_{1,3}$. Since the maximal common induced subgraph containing a $K_{1,3}$ of G_2, G_5 is I_0, S is an induced subgraph of I_0. Since G_1, G_4 are I_0-free, R is an induced subgraph of G_1, G_4. Recall that R contains a K_3. Since the maximal common induced subgraph containing a K_3 of G_1, G_4 is Z_1, R is an induced subgraph of Z_1. Then $\{R, S\} \preceq \{I_0, Z_1\}$. This completes the proof of necessity. ■

3. Forbidden Subgraphs Guaranteeing a Graph with Even Factor to Have a 2-Factor: the Proof of Theorem 6

In this section, we completely characterize connected forbidden subgraph and pairs of connected forbidden subgraphs that force a graph admitting an even factor to have a 2-factor.

The *union* of two graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The union of m disjoint copies of the same graph G is denoted by mG. The *join* of two disjoint graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is obtained from their union by joining each vertex of G_1 to each vertex of G_2.

Theorem 11. *Let G be a connected graph with an even factor of order at least 6 such that it satisfies one of the following.*

(1) *G is $\{K_{1,4}, Z_1\}$-free;*

(2) *G is $\{K_{1,3}, H_0\}$-free;*

(3) *G is $\{K_{1,3}, Z_2\}$-free.*

Then G has a 2-factor.

Proof. Let G be a connected graph with an even factor of order at least 6. Choose an even factor $F = Q_1 \cup Q_2 \cup \cdots \cup Q_t$ ($t \geq 1$), of G such that

(i) $\Delta(F)$ is minimized;

(ii) $|\{x \in V(F) : d_F(x) = \Delta(F)\}|$ is minimized, subjected to (i).

(iii) t is minimized, subjected to (i) and (ii).

We shall prove that $\Delta(F) = 2$. Assume to the contrary that $\Delta(F) \geq 4$. Take a vertex $v \in V(F)$ such that $d_F(v) = \Delta(F) \geq 4$. Without loss of generality, let $v \in V(Q_i)$. Let $\{v_1, v_2, v_3, v_4\} \subseteq N_{Q_i}(v)$. Then we have the following claim.

Claim 12. (1) *If $v_iv_j \in E(G)$, then $v_iv_j \in E(F)$, for $\{i, j\} \subset \{1, 2, 3, 4\}$.*

(2) *If $v_iv_j \in E(G)$, then at least one of $\{v_i, v_j\}$ has degree 2 in F, for $\{i, j\} \subset \{1, 2, 3, 4\}$.*

Proof. (1) Assume to the contrary that there exist two vertices $\{v_i, v_j\} \subset \{v_1, v_2, v_3, v_4\}$ such that $v_iv_j \in E(G) \setminus E(F)$, then $F' = F \triangle C(vv_iv_jv)$ is an even factor of G with $d_{F'}(v) = \Delta(F) - 2$, contradicting (ii).

(2) Assume to the contrary that there exist two vertices $\{v_i, v_j\} \subset \{v_1, v_2, v_3, v_4\}$ such that $d_F(v_i) \geq 4$ and $d_F(v_j) \geq 4$, then $F' = F \triangle C(vv_iv_jv)$ is an even factor of G with $d_{F'}(v) = \Delta(F) - 2$, contradicting (ii). □

***Proof of* (1).** Let G be a $\{K_{1,4}, Z_1\}$-free graph.

Claim 13. *$G[\{v, v_1, v_2, v_3, v_4\}] \cong K_2 \vee 3K_1$.*

Proof. Since $G[\{v, v_1, v_2, v_3, v_4\}] \ncong K_{1,4}$, $E(G[\{v_1, v_2, v_3, v_4\}]) \neq \emptyset$. Without loss of generality, we suppose that $v_1v_2 \in E(G)$. By Claim 12(1), $v_1v_2 \in E(F)$. Since $G[\{v, v_1, v_2, v_3\}] \ncong Z_1$, $\{v_3v_1, v_3v_2\} \cap E(G) \neq \emptyset$. By symmetry, suppose that $v_3v_2 \in E(G)$. By Claim 12(1), $v_3v_2 \in E(F)$. Thus, $d_F(v_2) \geq 4$. Then $v_3v_4 \notin E(G)$, otherwise, by Claim 12(1), $v_3v_4 \in E(F)$ and hence $d_F(v_3) \geq 4$, contradicting Claim 12(2). By symmetry, $v_4v_1 \notin E(G)$. Since $G[\{v, v_2, v_3, v_4\}] \ncong Z_1$, $v_2v_4 \in E(G)$. By Claim 12(1), $v_2v_4 \in E(F)$. Since $d_F(v_2) \geq 4$ and $\{v_2v_1, v_2v_4, v_2v_3\} \subset E(G)$, by Claim 12(2), $d_F(v_1) = d_F(v_3) = d_F(v_4) = 2$. Then by Claim 12(1), $v_1v_3 \notin E(G)$. This implies that $G[\{v, v_1, v_2, v_3, v_4\}] \cong K_2 \vee 3K_1$. □

Suppose that $d_F(v) \geq 6$ and there exists a vertex $v' \in V(Q_i)$ such that $v'v \in E(F)$. Since $G[\{v, v', v_1, v_3, v_4\}] \ncong K_{1,4}$, $\{v'v_1, v'v_3, v'v_4\} \cap E(G) \neq \emptyset$. By symmetry, assume that $v'v_3 \in E(G)$. Since $d_F(v_3) = 2$, $v'v_3 \notin E(F)$. Hence $F' = F \triangle C(vv_3v'v)$ is an even factor of G with $d_{F'}(v) = \Delta(F) - 2$, contradicting (ii). This implies that $d_F(v) = 4$. By symmetry, $d_F(v_2) = 4$. By Claim 12(2), $d_F(v_1) = d_F(v_3) = d_F(v_4) = 2$. Then $G[\{v, v_1, v_2, v_3, v_4\}]$ is a component of F. By the arbitrariness of v, we have that $\Delta(F) = 4$ and the vertex with maximum degree is in $K_2 \vee 3K_1$. Since $n \geq 6$, $E(\{v, v_1, v_2, v_3, v_4\}, V(G) \setminus \{v, v_1, v_2, v_3, v_4\}) \neq \emptyset$. Then there exists a vertex $w \in V(Q_j)$ such that $E(\{w\}, \{v, v_1, v_2, v_3, v_4\}) \neq \emptyset$.

First suppose that $v_3w \in E(G)$. Since $G[\{v, v_2, v_3, w\}] \ncong Z_1$, $\{wv, wv_2\} \cap E(G) \neq \emptyset$. By symmetry, assume that $wv_2 \in E(G)$. Then $d_F(w) = \Delta(F) = 4$, otherwise, $F' = F \triangle C(vv_3wv_2v)$ is an even factor of G satisfying that $d_{F'}(v) = \Delta(F) - 2$, $d_{F'}(w) = \Delta(F)$ but v_3 and w are in the same component of F', contradicting (iii). Let $\{w_1, w_2, w_3, w_4\} \subseteq N_{Q_j}(w)$. By the arbitrariness of v and by Claim 13, $G[\{w, w_1, w_2, w_3, w_4\}] \cong K_2 \vee 3K_1$. Without loss of generality, let $d_F(w_2) = 4$. Since $G[\{w, w_1, w_3, w_4, v_2\}] \ncong K_{1,4}$, $\{v_2w_1, v_2w_3, v_2w_4\} \cap E(G) \neq \emptyset$. By symmetry, assume that $v_2w_1 \in E(G)$. Then $F' = F \triangle C(ww_1v_2vv_3w)$ is an even factor of G with $d_{F'}(v) = \Delta(F) - 2$, contradicting (ii). This implies that $v_3w \notin E(G)$. By symmetry, $\{wv_1, wv_4\} \cap E(G) = \emptyset$.

Then $\{wv_2, wv\} \cap E(G) \neq \emptyset$. By symmetry, assume that $wv_2 \in E(G)$. Since $G[\{v, v_2, v_3, w\}] \ncong Z_1$ and $wv_3 \notin E(G)$, $wv \in E(G)$. Since $G[\{v, v_1, v_3, v_4, w\}] \ncong K_{1,4}$, $\{wv_1, wv_3, wv_4\} \cap E(G) \neq \emptyset$. Without loss of generality, let $wv_1 \in E(G)$. Then $d_F(w) = \Delta(F) = 4$, otherwise, $F' = F \triangle C(wv_2vv_1w)$ is an even factor of G with fewer components than F, contradicting (iii). Let $\{w_1, w_2, w_3, w_4\} \subseteq N_{Q_j}(w)$. By the arbitrariness of v and by Claim 13, $G[\{w, w_1, w_2, w_3, w_4\}] \cong K_2 \vee 3K_1$. Let $d_F(w_2) = 4$. Since $G[\{w, w_1, w_3, w_4, v_2\}] \ncong K_{1,4}$, $\{v_2w_1, v_2w_3, v_2w_4\} \cap E(G) \neq \emptyset$. By symmetry, assume that $v_2w_1 \in E(G)$. Then $F' = F \triangle C(ww_1v_2vw)$ is an even factor of G with fewer components than F, contradicting (iii). This proves (1). $\square$

In the following, let G be a $K_{1,3}$-free graph. Before present the proofs of (2), (3), we show the following claim.

Claim 14. *$G[\{v, v_1, v_2, v_3, v_4\}] \cong H_0$.*

Proof. Since $G[\{v, v_1, v_2, v_3\}] \ncong K_{1,3}$, $\{v_1v_2, v_1v_3, v_2v_3\} \cap E(G) \neq \emptyset$. Without loss of generality, we suppose that $v_1v_2 \in E(G)$. By Claim 12(1), $v_1v_2 \in E(F)$ and at least one of $\{v_1, v_2\}$ has degree 2 in F. Without loss of generality, let $d_F(v_1) = 2$. By Claim 12(1), $\{v_1v_4, v_1v_3\} \cap E(G) = \emptyset$. Since $G[\{v, v_1, v_3, v_4\}] \ncong K_{1,3}$, $v_3v_4 \in E(G)$. By Claim 12, $v_3v_4 \in E(F)$ and at least one of $\{v_3, v_4\}$ has degree 2 in F. Then $v_2v_3 \notin E(G)$, otherwise, $F' = F \triangle E(vv_2v_3v)$ is an even factor of G with $d_{F'}(v) = d_F(v) - 2$, contradicting (ii). By symmetry, $v_2v_4 \notin E(G)$. Then $G[\{v, v_1, v_2, v_3, v_4\}] \cong H_0$ and $E(G[\{v, v_1, v_2, v_3, v_4\}]) \subseteq E(F)$. $\square$

***Proof of* (2).** Let G be a $\{K_{1,3}, H_0\}$-free graph. By Claim 14, $G[\{v, v_1, v_2, v_3, v_4\}] \cong H_0$, contradicting that G is H_0-free. Then (2) clearly holds. $\square$

***Proof of* (3).** Let G be a $\{K_{1,3}, Z_2\}$-free graph. By Claim 14 and Claim 12 (2), we suppose that $d_F(v_2) = d_F(v_3) = 2$. Suppose that $d_F(v) = \Delta(F) \geq 6$ and $\{v_1, v_2, v_3, v_4, v'\} \subseteq N_F(v)$. Since $G[\{v, v_2, v_3, v'\}] \ncong K_{1,3}$, $\{v'v_2, v'v_3\} \cap E(G) \neq \emptyset$. By symmetry, assume that $v'v_2 \in E(G)$. Recall that $d_F(v_2) = 2$, then $v'v_2 \notin E(F)$. Hence $F' = F \triangle C(vv_2v'v)$ is an even factor of G with $d_{F'}(v) = d_F(v) - 2$, contradicting (ii). This implies that $d_F(v) = \Delta(F) = 4$.

We claim that $d_F(v_1) = 2$. Suppose, otherwise, that $d_F(v_1) = 4$ and $N_F(v_1) = \{v_5, v_6, v_2, v\}$. If $v_2v_6 \in E(G)$, then $v_2v_6 \notin E(F)$ (since $d_F(v_2) = 2$). Thus, $F' = F \triangle C(v_1v_2v_6v)$ is an even factor of G with $d_{F'}(v_1) = 2$, contradicting (ii). This implies that $v_2v_6 \notin E(G)$. By symmetry, $v_2v_5 \notin E(G)$. Since $G[\{v_1, v_2, v_5, v_6\}] \ncong K_{1,3}$, $v_5v_6 \in E(G)$. Furthermore, $v_5v_6 \in E(F)$, otherwise, $F' = F \triangle C(v_1v_5v_6v)$ is an even factor of G with $d_{F'}(v_1) = 2$, contradicting (ii). Recall that $d_F(v) = 4$ and $d_F(v_3) = 2$. If $vv_5 \in E(G)$ (or $v_3v_5 \in E(G)$), then $vv_5 \notin E(F)$ (or $v_3v_5 \notin E(F)$). Thus, $F' = F \triangle C(vv_1v_5v)$ (or $F \triangle C(vv_1v_5v_3v)$) is an even factor of G with $d_{F'}(v_1) = 2$, contradicting (ii). Thus, $vv_5, v_3v_5 \notin E(G)$. If $v_4v_5 \in E(G)$, then $F' = F \triangle C(vv_1v_5v_4v)$ is an even factor of G with $d_{F'}(v_1) =$

2, contradicting (ii). Thus, $v_4v_5 \notin E(G)$. Then $G[\{v, v_3, v_4, v_1, v_5\}] \cong Z_2$, a contradiction. This proves that $d_F(v_1) = 2$. By symmetry, $d_F(v_4) = 2$. Then $G[\{v, v_1, v_2, v_3, v_4\}] \cong H_0$ is a component of F.

Since $n \geq 6$, $V(G) \setminus V(H_0) \neq \emptyset$. First, we suppose that there exists a component Q_i of F such that $E(v, V(Q_i)) \neq \emptyset$ and $vu \in E(G)$, where $u \in V(Q_i)$. Let $\{u_1, u_2\} \subseteq N_{Q_i}(u)$. Then $E(\{u_1, u_2\}, \{v_1, v_2, v_3, v_4\}) = \emptyset$. Otherwise, by symmetry, suppose that $v_1u_1 \in E(G)$, then $F' = F\triangle C(u_1v_1vuu_1)$ is an even factor of G such that H_0 and Q_i are in the same component of F', but the other components are the same with F, contradicting (iii). We have that $vu_1 \notin E(G)$, otherwise, $G[\{v, v_1, v_4, u_1\}] \cong K_{1,3}$, a contradiction. By symmetry, $vu_2 \notin E(G)$. Since $G[\{u, u_1, u_2, v\}] \ncong K_{1,3}$, $u_1u_2 \in E(G)$. Since $G[\{u, u_1, u_2, v, v_1\}] \ncong Z_2$, $uv_1 \in E(G)$. Since $G[\{u, u_1, u_2, v, v_4\}] \ncong Z_2$, $uv_4 \in E(G)$. Then $G[\{u, v_1, v_4, u_1\}] \cong K_{1,3}$, a contradiction. This implies that $E(v, V(G) \setminus V(H_0)) = \emptyset$.

Then $E(\{v_1, v_2, v_3, v_4\}, V(Q_i)) \neq \emptyset$. Without loss of generality, we suppose that $v_1w \in E(G)$, where $w \in V(Q_i)$. Let $\{w_1, w_2\} \subseteq N_{Q_i}(w)$. Since $G[\{v, v_3, v_4, v_1, w\}] \ncong Z_2$ and $vw \notin E(G)$, $\{wv_3, wv_4\} \cap E(G) \neq \emptyset$. Without loss of generality, we suppose that $wv_4 \in E(G)$. Then $w_1v_4 \notin E(G)$ and $v_1w_1 \notin E(G)$, otherwise, $F' = F\triangle C(wv_1vv_4w_1w)$ and $F\triangle C(w_1v_1vv_4ww_1)$ is an even factor of G with $d_{F'}(v) = 2$, contradicting (ii). Thus, $G[\{w, v_1, v_4, w_1\}] \cong K_{1,3}$, a contradiction. This prove that $\Delta(F) = 2$. □

■

Now we prove Theorem 6.

Proof of Theorem 6. (1) If G is P_3-free, then G is a complete graph and hence G has a 2-factor. Conversely, G_1, G_2, G_5 in Figure 3 are connected and contain an even factor but no 2-factor. Then A must be an induced subgraph of them. Without loss of generality, we assume that A is an induced subgraph of G_1. Then A is $K_{1,s}$ or $K_{2,s}$ $(s \geq 2)$. Since G_2 is $K_{2,s}$-free and G_5 is $K_{1,3}$-free, A is a path. Since the maximal induced path of G_1 is P_3, A is P_3.

(2) By Theorem 11, *the sufficiency* clearly holds. It remains to show *the necessity*. All graphs in Figure 3 are connected and have an even factor but no 2-factor. Then each graph contains at least one of R, S as an induced subgraph. Without loss of generality, we assume that G_1 contains R as an induced subgraph. Then R is $K_{1,t}$ $(t \geq 3)$ or $K_{2,s}$ $(s \geq 2)$.

Case 1. R is $K_{1,t}$ $(t \geq 5)$ or $K_{2,s}$ $(s \geq 2)$. Since G_2, G_4, G_5 are $\{K_{1,5}, K_{2,s}\}$-free, they must contain S as an induced subgraph. Since G_2, G_4 have no common induced cycle and G_5 is $K_{1,3}$-free, S should be a path. Since the maximal induced path of G_2 is P_3, S is an induced subgraph of P_3, a contradiction.

Case 2. R is $K_{1,4}$. Since G_3, G_5, G_9 are $K_{1,4}$-free, they must contain S as an induced subgraph. Since G_5 is $K_{1,3}$-free, S should be a path or contain a cycle. Note that the maximal induced path of G_9 is P_3. If S is a path, then S is an induced subgraph of P_3, a contradiction. Then S contains a cycle. Note that the maximal common induced cycle of G_3 and G_9 is K_3. Furthermore, S contains exactly one K_3. Since the maximal common induced subgraph containing a K_3 of G_3, G_5, G_9 is Z_1, S is an induced subgraph of Z_1. Therefore, $\{R, S\} \preceq \{K_{1,4}, Z_1\}$.

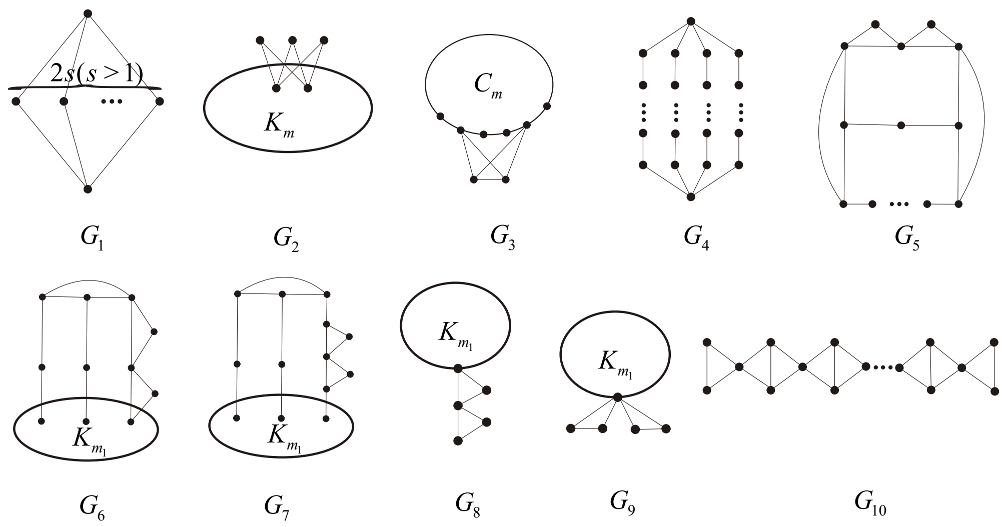

Figure 3. Graphs with even factor but no 2-factor.

Case 3. R is $K_{1,3}$. Since G_5, G_6, G_7, G_8, G_{10} are $K_{1,3}$-free, they must contain S as an induced subgraph. Then S should be a path or contain a cycle. Note that the maximal induced path of G_8 is P_4. Thus, if S is a path, then S is an induced subgraph of P_4. Then $\{R, S\} \preceq \{K_{1,3}, P_4\}$. Now we suppose that S contains a cycle. Since G_5 is K_4-free, S contains no K_4. Note that the maximal common induced cycle of G_5, G_6, G_7, G_8, G_{10} is K_3. Furthermore, S contains at most two triangles. Note that G_{10} is $B_{i,j}$-free. Thus, if S contains exactly one triangle, then S is Z_i. Since the maximal induced subgraph containing Z_i of them is Z_2, S should be an induced subgraph of Z_2. Therefore, $\{R, S\} \preceq \{K_{1,3}, Z_2\}$. Since the maximal common induced subgraph containing exactly two triangles of them is H_0, S should be an induced subgraph of H_0. Then $\{R, S\} \preceq \{K_{1,3}, H_0\}$. Note that $P_4 \preceq Z_2$. Therefore, $\{R, S\} \preceq \{K_{1,4}, Z_1\}$, $\{K_{1,3}, Z_2\}$, $\{K_{1,3}, H_0\}$. This completes the necessity. ■

4. Concluding Remarks

In this paper, we consider what happen for pairs of forbidden subgraphs for a graph to be hamiltonian or to have 2-factor if we impose a necessary conditions (Theorems 5 and 6). In fact, they hold also for graphs with any sufficiently large order, from their proof.

It remains to consider the problem how to determine all pairs of forbidden subgraphs for guaranteeing a 2-connected graph with an even factor to have a 2-factor. We have tried this problem, however, it would be very complicated (there are many pairs of forbidden subgraphs). More generally, it would be interesting to consider the following question:

Question 15. *Whether does forbidden pairs become wider for graphs with a high connectivity if we impose a necessary condition? i.e.,*

- *How to determine all forbidden pairs for a k-connected graph with 2-factor to be hamiltonian?*
- *How to determine all forbidden pairs for a k-connected graph with even factor to have a 2-factor?*

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