Discussiones Mathematicae
Graph Theory 43 (2023) 277–286
https://doi.org/10.7151/dmgt.2365

THE HILTON-SPENCER CYCLE THEOREMS VIA KATONA'S SHADOW INTERSECTION THEOREM

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Abstract

A family $\mathcal{A}$ of sets is said to be *intersecting* if every two sets in $\mathcal{A}$ intersect. An intersecting family is said to be *trivial* if its sets have a common element. A graph G is said to be *r-EKR* if at least one of the largest intersecting families of independent r-element sets of G is trivial. Let $\alpha(G)$ and $\omega(G)$ denote the independence number and the clique number of G, respectively. Hilton and Spencer recently showed that if G is the vertex-disjoint union of a cycle C raised to the power k and s cycles ${}_1C, \dots, {}_sC$ raised to the powers $k_1, \dots, k_s$, respectively, $1 \leq r \leq \alpha(G)$, and

$$\min\left(\omega\left({}_1C^{k_1}\right), \dots, \omega\left({}_sC^{k_s}\right)\right) \geq \omega\left(C^k\right),$$

then G is r-EKR. They had shown that the same holds if C is replaced by a path P and the condition on the clique numbers is relaxed to

$$\min\left(\omega\left({}_1C^{k_1}\right), \dots, \omega\left({}_sC^{k_s}\right)\right) \geq \omega\left(P^k\right).$$

We use the classical Shadow Intersection Theorem of Katona to obtain a significantly shorter proof of each result for the case where the inequality for the minimum clique number is strict.

Keywords: cycle, independent set, intersecting family, Erdős-Ko-Rado theorem, Hilton-Spencer theorem, Katona's shadow intersection theorem.

2010 Mathematics Subject Classification: 05D05, 05C35, 05C38, 05C69.

1. Introduction

Unless stated otherwise, we shall use small letters such as x to denote non-negative integers or elements of a set, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (sets whose members are sets themselves). The set of positive integers is denoted by $\mathbb{N}$. The set $\{i \in \mathbb{N} \colon m \leq i \leq n\}$ is denoted by $[m,n]$, $[1,n]$ is abbreviated to $[n]$, and $[0]$ is taken to be the empty set $\emptyset$. For a set X, the *power set of* X (that is, $\{A \colon A \subseteq X\}$) is denoted by 2^X. The family of r-element subsets of X is denoted by $\binom{X}{r}$. The family of r-element sets in a family $\mathcal{F}$ is denoted by $\mathcal{F}^{(r)}$. If $\mathcal{F} \subseteq 2^X$ and $x \in X$, then the family $\{A \in \mathcal{F} \colon x \in A\}$ is denoted by $\mathcal{F}(x)$ and called the *star of* $\mathcal{F}$ with *centre* x.

A family $\mathcal{A}$ is said to be *intersecting* if for every $A, B \in \mathcal{A}$, A and B intersect (that is, $A \cap B \neq \emptyset$). The stars of a family $\mathcal{F}$ are among the simplest intersecting subfamilies of $\mathcal{F}$. We say that $\mathcal{F}$ has the *star property* if at least one of the largest intersecting subfamilies of $\mathcal{F}$ is a star of $\mathcal{F}$.

Determining the size of a largest intersecting subfamily of a given family $\mathcal{F}$ is one of the most popular endeavours in extremal set theory. This started in [11], which features the classical result known as the Erdős-Ko-Rado (EKR) Theorem. The EKR Theorem states that if $r \leq n/2$ and $\mathcal{A}$ is an intersecting subfamily of $\binom{[n]}{r}$, then $|\mathcal{A}| \leq \binom{n-1}{r-1}$. Thus, $\binom{[n]}{r}$ has the star property for $r \leq n/2$ (clearly, for $n/2 < r \leq n$, $\binom{[n]}{r}$ itself is intersecting). There are various proofs of the EKR Theorem (see [9,16,24,25,27]), two of which are particularly short and beautiful: Katona's [25], which introduced the elegant cycle method, and Daykin's [9], using the fundamental Kruskal-Katona Theorem [26,28]. The EKR Theorem gave rise to some of the highlights in extremal set theory [1,14,27,30] and inspired many variants and generalizations; see [4,10,13,15,17,21,22].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We may represent an edge $\{v,w\}$ by vw. A subset I of $V(G)$ is an *independent set of* G if $vw \notin E(G)$ for every $v, w \in I$. Let $\mathcal{I}_G$ denote the family of independent sets of G. An independent set J of G is *maximal* if $J \not\subseteq I$ for each independent set I of G such that $I \neq J$. The size of a smallest maximal independent set of G is denoted by $\mu(G)$. The size of a largest independent set of G is denoted by $\alpha(G)$. A subset X of $V(G)$ is a *clique of* G if $vw \in E(G)$ for every $v, w \in X$ with $v \neq w$. The size of a largest clique of G is called the *clique number of* G and denoted by $\omega(G)$.

Holroyd and Talbot introduced the problem of determining whether ${\mathcal{I}_G}^{(r)}$ has the star property for a given graph G and an integer $r \geq 1$. Following their terminology, a graph G is said to be *r-EKR* if ${\mathcal{I}_G}^{(r)}$ has the star property. The Holroyd-Talbot (HT) Conjecture [22, Conjecture 7] claims that G is r-EKR if $\mu(G) \geq 2r$. This was verified by Borg [2] for $\mu(G)$ sufficiently large depending on r (see also [6, Lemma 4.4 and Theorem 1.4]). By the EKR Theorem, the conjecture is true if G has no edges. The HT Conjecture has been verified for several classes of graphs [2, 3, 7, 8, 12, 18–23, 29, 31]. As demonstrated in [8], for $r > \mu(G)/2$, whether G is r-EKR or not depends on G and r (both cases are possible). Naturally, graphs G of particular interest are those that are r-EKR for all $r \leq \alpha(G)$.

For $n \geq 1$, the graphs $\left([n], \binom{[n]}{2}\right)$ and $([n], \{\{i, i+1\} \colon i \in [n-1]\})$ are denoted by K_n and P_n, respectively. For $n \geq 3$, $([n], E(P_n) \cup \{\{n, 1\}\})$ is denoted by C_n. A copy of K_n is called a *complete graph*. A copy P of P_n is called an *n-path* or simply a *path*, and a vertex of P is called an *end-vertex* if it is not adjacent to more than one vertex. A copy of C_n is called an *n-cycle* or simply a *cycle* (normally, this terminology is used for $n \geq 3$, but we may include the case $n = 2$). If H is a subgraph of a graph G (that is, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$), then we say that G *contains* H. For $v, w \in V(G)$, the *distance* $d_G(v, w)$ is $\min\{k \colon v, w \in V(P)$ for some $(k+1)$-path P contained by $G\}$. The k^{th} *power of* G, denoted by G^k, is the graph with vertex set $V(G)$ and edge set $\{vw \colon v, w \in V(G),\ 1 \leq d_G(v, w) \leq k\}$; G^k is also referred to as G *raised to the power* k.

Note that ${P_n}^k = K_n$ for $n \leq k+1$, and ${C_n}^k = K_n$ for $n \leq 2k+1$. Also note that $\omega({P_n}^k) = k+1$ if $n \geq k+1$, $\omega({P_n}^k) = n$ if $n \leq k$, $\omega({C_n}^k) = k+1$ if $n \geq 2k+2$, $\omega({C_n}^k) = n$ if $n \leq 2k+1$, $\alpha({P_n}^k) = \lceil n/(k+1) \rceil$, $\alpha({C_n}^k) = \lfloor n/(k+1) \rfloor$ if $n \geq k+1$, and $\alpha({C_n}^k) = 1$ if $2 \leq n \leq k+1$.

The following remarkable analogue of the EKR theorem was obtained by Talbot [29].

Theorem 1 [29]**.** *For $1 \leq r \leq \alpha({C_n}^k)$, ${C_n}^k$ is r-EKR.*

Talbot introduced a compression technique to prove Theorem 1. In vague terms, his compression technique rotates anticlockwise the elements of the independent sets of the intersecting family which are distinct from a specified vertex (see Section 2).

If $G, G_1, \ldots, G_k$ are graphs such that the vertex sets of $G_1, \ldots, G_k$ are pairwise disjoint and $G = \left(\bigcup_{i=1}^k V(G_i), \bigcup_{i=1}^k E(G_i)\right)$, then G is said to be the *disjoint union of* $G_1, \ldots, G_k$, and $G_1, \ldots, G_k$ are said to be *vertex-disjoint*.

Inspired by the work of Talbot, Hilton and Spencer [19] went on to prove the following result, which is stated with notation used in [19, 20].

Theorem 2 [19]**.** *If G is the disjoint union of a path P raised to the power k and s cycles ${}_1C, \ldots, {}_sC$ raised to the powers $k_1, \ldots, k_s$, respectively, $1 \leq r \leq \alpha(G)$, and*

$$\min\left(\omega\left({}_1C^{k_1}\right), \ldots, \omega\left({}_sC^{k_s}\right)\right) \geq \omega(P^k), \tag{1}$$

then G is r-EKR. Moreover, for any end-vertex x of P, ${\mathcal{I}_G}^{(r)}(x)$ is a largest intersecting subfamily of ${\mathcal{I}_G}^{(r)}$.

However, it was desired to obtain a generalization of Theorem 1, and this was eventually achieved by Hilton and Spencer [20] with the following theorem.

Theorem 3 [20]**.** *If G is the disjoint union of $s+1$ cycles $C, {}_1C, \dots, {}_sC$ raised to the powers $k, k_1, \dots, k_s$, respectively, $1 \le r \le \alpha(G)$, and*

$$\min\left(\omega\left({}_1C^{k_1}\right), \dots, \omega\left({}_sC^{k_s}\right)\right) \ge \omega(C^k), \tag{2}$$

then G is r-EKR. Moreover, for any $x \in V(C)$, ${\mathcal{I}_G}^{(r)}(x)$ is a largest intersecting subfamily of ${\mathcal{I}_G}^{(r)}$.

Hilton and Spencer [20] conjectured that every disjoint union of powers of cycles is r-EKR.

The proof of Theorem 3 is also inspired by Talbot's proof of Theorem 1. In particular, an essential ingredient in the proof of Theorem 3 is the use of Theorem 2 for the special case where P^k is a complete graph as the base case of an induction argument.

In this paper, we give a significantly shorter and simpler proof of Theorem 2 and of Theorem 3, except for the cases of equality in conditions (1) and (2), respectively. In other words, we prove the following two results.

Theorem 4. *Theorem* 2 *is true if the inequality in* (1) *is strict.*

Theorem 5. *Theorem* 3 *is true if the inequality in* (2) *is strict.*

Our argument is based on the Shadow Intersection Theorem of Katona [27], hence demonstrating yet another application of this classical and useful result in extremal set theory.

2. The New Proof

Let $P, {}_1C, \dots, {}_sC$ be as in Theorem 2. Let $p = |V(P)|$ and $c_i = |V({}_iC)|$. For $1 \le i \le s$, we label the vertices of ${}_iC$ by $v_{1,i}, v_{2,i}, \dots, v_{c_i,i}$, where $E({}_iC) = \{v_{j,i}v_{j+1,i} \colon j \in [c_i - 1]\} \cup \{v_{c_i,i}v_{1,i}\}$. We may assume that $P = P_p$, that is, $V(P) = [p]$ and $E(P) = \{\{i, i+1\} \colon i \in [p-1]\}$. Let H be the union of ${}_1C^{k_1}, \dots, {}_sC^{k_s}$, and let $f : V(H) \to V(H)$ be the bijection given by

$$f(v_{c_i,i}) = v_{1,i} \quad \text{and} \quad f(v_{j,i}) = v_{j+1,i} \quad \text{for} \quad 1 \le i \le s \quad \text{and} \quad 1 \le j \le c_i - 1.$$

Let $f^1 = f$, and for any integer $t \ge 2$, let $f^t = f \circ f^{t-1}$ and $f^{-t} = f^{-1} \circ f^{-(t-1)}$. Note that for $t \ge 1$, one can think of f^t as t clockwise rotations, and of f^{-t} as t anticlockwise rotations. For $I \in \mathcal{I}_H$, we denote the set $\{f^t(x) \colon x \in I\}$ by $f^t(I)$, and for $\mathcal{A} \subseteq \mathcal{I}_H$, we denote the family $\{f^t(A) \colon A \in \mathcal{A}\}$ by $f^t(\mathcal{A})$. The notation $f^{-t}(I)$ and $f^{-t}(\mathcal{A})$ is defined similarly.

The new argument presented in this paper lies entirely in the proof of the following important case, which both Theorem 4 and Theorem 5 pivot on.

Lemma 6. *Theorem* 2 *is true if* P^k *is a complete graph and the inequality in* (1) *is strict.*

As shown in this section, Theorem 4 follows from Lemma 6 by applying the compression method in [21], and Theorem 5 follows from Lemma 6 by applying the same compression method of Talbot in [29].

We now start working towards the proof of Lemma 6.

Let $\mathcal{A}$ be a family of r-element sets. The *shadow of* $\mathcal{A}$, denoted by $\partial\mathcal{A}$, is the family $\bigcup_{A\in\mathcal{A}}\binom{A}{r-1}$. A special case of Katona's Shadow Intersection Theorem [27] is that

$$|\mathcal{A}| \leq |\partial\mathcal{A}| \quad \text{if } \mathcal{A} \text{ is intersecting.} \tag{3}$$

***Proof of Lemma* 6.** Suppose that P^k is a complete graph. Then, $\omega(P^k) = p$. Suppose $\min\big(\omega({}_1C^{k_1}), \ldots, \omega({}_sC^{k_s})\big) > p$. Note that this implies that for every $i \in [s]$, $c_i \geq p+1$ and, for every $v_{h,i}, v_{j,i} \in V({}_iC^{k_i})$,

$$\text{if } v_{j,i} \in \big\{f^{-q}(v_{h,i})\colon q \in [p]\big\} \cup \big\{f^{q}(v_{h,i})\colon q \in [p]\big\}, \text{ then } v_{h,i}v_{j,i} \in E\big({}_iC^{k_i}\big). \tag{4}$$

It is worth pointing out that the strict inequality is only used for (4), from which we obtain Claim 7.

Let $\mathcal{A}$ be an intersecting subfamily of $\mathcal{I}_G{}^{(r)}$. Recall that $V(P^k) = [p]$. Let $\mathcal{A}_0 = \{A \in \mathcal{A}\colon A \cap [p] = \emptyset\}$ and $\mathcal{A}_i = \{A \in \mathcal{A}\colon A \cap [p] = \{i\}\}$ for $1 \leq i \leq p$. Since P^k is a complete graph, the families $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_p$ partition $\mathcal{A}$. Let $\mathcal{A}'_0 = \mathcal{A}_0$ and $\mathcal{A}'_i = \{A\backslash\{i\}\colon A \in \mathcal{A}_i\}$ for $1 \leq i \leq p$. Since $\mathcal{A}$ is intersecting,

$$\text{for any } i,j \in \{0\}\cup[p] \text{ with } i \neq j, \text{ each set in } \mathcal{A}'_i \text{ intersects each set in } \mathcal{A}'_j. \tag{5}$$

Claim 7. *The families* $\partial\mathcal{A}_0, f^1(\mathcal{A}'_1), f^2(\mathcal{A}'_2), \ldots, f^p(\mathcal{A}'_p)$ *are pairwise disjoint.*

Proof. Suppose $B \in f^i(\mathcal{A}'_i) \cap f^j(\mathcal{A}'_j)$ for some $i, j \in [p]$ with $i < j$. Then, $B = f^i(A_i) = f^j(A_j)$ for some $A_i \in \mathcal{A}'_i$ and $A_j \in \mathcal{A}'_j$. Thus, $A_i = f^{j-i}(A_j)$. Since $1 \leq j-i < p$, (4) gives us $A_i \cap A_j = \emptyset$, but this contradicts (5). Therefore, $f^1(\mathcal{A}'_1), f^2(\mathcal{A}'_2), \ldots, f^p(\mathcal{A}'_p)$ are pairwise disjoint.

Suppose $B \in \partial\mathcal{A}_0 \cap f^i(\mathcal{A}'_i)$ for some $i \in [p]$. Then, $C\backslash\{x\} = B = f^i(A_i)$ for some $C \in \mathcal{A}_0$, $x \in C$, and $A_i \in \mathcal{A}'_i$. Since $1 \leq i \leq p$, (4) gives us $C \cap A_i = \emptyset$, but this contradicts (5). The claim follows. □

Let $\mathcal{A}^*_0 = \{A \cup \{1\}\colon A \in \partial\mathcal{A}_0\}$ and $\mathcal{A}^*_i = \{A \cup \{1\}\colon A \in f^i(\mathcal{A}'_i)\}$ for $1 \leq i \leq p$. For $0 \leq i \leq p$, $\mathcal{A}^*_i \subseteq \mathcal{I}_G{}^{(r)}(1)$. By Claim 7, $\sum_{i=0}^{p}|\mathcal{A}^*_i| = \big|\bigcup_{i=0}^{p}\mathcal{A}^*_i\big| \leq \big|\mathcal{I}_G{}^{(r)}(1)\big|$. By (3), $|\mathcal{A}_0| \leq |\partial(\mathcal{A}_0)| = |\mathcal{A}^*_0|$. We have

$$|\mathcal{A}| = \sum_{i=0}^{p}|\mathcal{A}_i| = |\mathcal{A}_0| + \sum_{i=1}^{p}|\mathcal{A}^*_i| \leq \sum_{i=0}^{p}|\mathcal{A}^*_i| \leq \big|\mathcal{I}_G{}^{(r)}(1)\big|,$$

and the lemma is proved. ■

The full Theorem 4 is now obtained by the line of argument laid out in [21], hence making use of established facts regarding compressions on independent sets.

For any edge uv of a graph G, let $\delta_{u,v} \colon \mathcal{I}_G \to \mathcal{I}_G$ be defined by

$$\delta_{u,v}(A) = \begin{cases} (A\backslash\{v\}) \cup \{u\} & \text{if } v \in A, u \notin A, \text{ and } (A\backslash\{v\}) \cup \{u\} \in \mathcal{I}_G; \\ A & \text{otherwise,} \end{cases}$$

and let $\Delta_{u,v} \colon 2^{\mathcal{I}_G} \to 2^{\mathcal{I}_G}$ be the *compression operation* (also called a *shifting operation*) defined by

$$\Delta_{u,v}(\mathcal{A}) = \{\delta_{u,v}(A) \colon A \in \mathcal{A}\} \cup \{A \in \mathcal{A} \colon \delta_{u,v}(A) \in \mathcal{A}\}.$$

It is well known, and easy to see, that

$$|\Delta_{u,v}(\mathcal{A})| = |\mathcal{A}|$$

(see [11, 15]). For any $x \in V(G)$, let $N_G(x)$ denote the set $\{y \in V(G) \colon xy \in E(G)\}$. The following is given by [8, Lemma 2.1] (which is actually stated for ${\mathcal{I}_G}^{(r)}$ but proved for $\mathcal{I}_G$) and essentially originated in [21]. We omit the proof.

Lemma 8 [8, 21]**.** *If G is a graph, $uv \in E(G)$, $\mathcal{A}$ is an intersecting subfamily of $\mathcal{I}_G$, $\mathcal{B} = \Delta_{u,v}(\mathcal{A})$, $\mathcal{B}_0 = \{B \in \mathcal{B} \colon v \notin B\}$, $\mathcal{B}_1 = \{B \in \mathcal{B} \colon v \in B\}$, and $\mathcal{B}_1' = \{B\backslash\{v\} \colon B \in \mathcal{B}_1\}$, then*

(i) *$\mathcal{B}_0$ is intersecting;*

(ii) *if $|N_G(u)\backslash(\{v\} \cup N_G(v))| \leq 1$, then $\mathcal{B}_1'$ is intersecting;*

(iii) *if $N_G(u)\backslash(\{v\} \cup N_G(v)) = \emptyset$, then $\mathcal{B}_0 \cup \mathcal{B}_1'$ is intersecting.*

For a vertex v of a graph G, let $G - v$ denote the graph obtained by deleting v (that is, $G - v = (V(G)\backslash\{v\}, \{xy \in E(G) \colon x, y \notin \{v\}\})$), and let $G \downarrow v$ be the graph obtained by deleting v and the vertices adjacent to v (that is, $G \downarrow v = (V(G)\backslash(\{v\} \cup N_G(v)), \{xy \in E(G) \colon x, y \notin \{v\} \cup N_G(v)\})$).

***Proof of Theorem* 4.** We use induction on $|V(P)|$. If P^k is a complete graph, then the result is given by Lemma 6. Note that this captures the base case $|V(P)| = 1$. Now suppose that P^k is not a complete graph. Then, $|V(P)| \geq k+2$. If $r = 1$, then the result is trivial. Suppose $r > 1$. Let $\mathcal{A}$ be an intersecting subfamily of ${\mathcal{I}_G}^{(r)}$. Let $u = p-1$ and $v = p$. Let $\mathcal{B} = \Delta_{u,v}(\mathcal{A})$, $\mathcal{B}_0 = \{B \in \mathcal{B} \colon v \notin B\}$, $\mathcal{B}_1 = \{B \in \mathcal{B} \colon v \in B\}$, and $\mathcal{B}_1' = \{B\backslash\{v\} \colon B \in \mathcal{B}_1\}$. By Lemma 8(i), $\mathcal{B}_0$ is intersecting. We have $N_G(u)\backslash(\{v\} \cup N_G(v)) = \{p-k-1\}$, so, by Lemma 8(ii), $\mathcal{B}_1'$ is intersecting. Let $H_0 = P_{p-1}$ and $H_1 = P_{p-k-1}$. Clearly, $\mathcal{B}_0 \subseteq {\mathcal{I}_{G-v}}^{(r)}$, $\mathcal{B}_1' \subseteq {\mathcal{I}_{G\downarrow v}}^{(r)}$, $G-v$ is the union of ${H_0}^k$ and ${}_1C^{k_1}, \ldots, {}_sC^{k_s}$, and $G \downarrow v$ is the union of ${H_1}^k$ and ${}_1C^{k_1}, \ldots, {}_sC^{k_s}$. The condition $\min\big(\omega({}_1C^{k_1}), \ldots, \omega({}_sC^{k_s})\big) > \omega(P^k)$

in the theorem gives us $\min\big(\omega({}_1C^{k_1}),\dots,\omega({}_sC^{k_s})\big) > \omega({H_0}^k) \geq \omega({H_1}^k)$. By the induction hypothesis, $|\mathcal{B}_0| \leq \big|\mathcal{I}_{G-v}{}^{(r)}(1)\big|$ and $|\mathcal{B}_1'| \leq \big|\mathcal{I}_{G\downarrow v}{}^{(r-1)}(1)\big|$. We have

$$\begin{aligned}|\mathcal{A}| = |\mathcal{B}| = |\mathcal{B}_0| + |\mathcal{B}_1'| &\leq \big|\mathcal{I}_{G-v}{}^{(r)}(1)\big| + \big|\mathcal{I}_{G\downarrow v}{}^{(r-1)}(1)\big| \\ &= \big|\big\{A \in \mathcal{I}_G{}^{(r)} \colon 1 \in A, v \notin A\big\}\big| + \big|\big\{A \in \mathcal{I}_G{}^{(r)} \colon 1, v \in A\big\}\big| = \big|\mathcal{I}_G{}^{(r)}(1)\big|,\end{aligned}$$

as required. ∎

***Proof of Theorem* 5.** We use induction on $c = |V(C)|$. We may assume that $C = C_c$. If C^k is a complete graph, then the result is given by Lemma 6. Note that this captures the base case $c = 2$. Now suppose that C^k is not a complete graph. Then, $c \geq 2k+2$. If $r = 1$, then the result is trivial. Suppose $r > 1$. Let $\mathcal{A}$ be an intersecting subfamily of $\mathcal{I}_G{}^{(r)}$.

Let $g : V(G) \to V(G)$ be the Talbot compression [20, 29] given by

$$\begin{aligned} g(v) &= v \quad \text{for } v \in V(G)\backslash V(C), \\ g(1) &= 1, \quad \text{and} \\ g(1+j) &= 1+j-1 \quad \text{for } 1 \leq j \leq c-1. \end{aligned}$$

For $X \in \mathcal{I}_G$ and $\mathcal{X} \subseteq \mathcal{I}_G$, we use the notation $g^t(X)$ and $g^t(\mathcal{X})$ similarly to the way it is used above for f. Let F be the union of ${C_{c-1}}^k$ and ${}_1C^{k_1},\dots,{}_sC^{k_s}$. Let K be the union of ${C_{c-k-1}}^k$ and ${}_1C^{k_1},\dots,{}_sC^{k_s}$. Let

$$\begin{aligned} \mathcal{B} &= \big\{A \in \mathcal{A} : 1 \notin A,\ g(A) \in \mathcal{I}_F{}^{(r)}\big\}, \\ \mathcal{C} &= \big\{A \in \mathcal{A} : 1 \in A,\ g(A) \in \mathcal{I}_F{}^{(r)}\big\}, \\ \mathcal{D}_0 &= \{A \in \mathcal{A} : 1,\ k+2 \in A\}, \\ \mathcal{D}_i &= \{A \in \mathcal{A} : 1+c-i,\ k+2-i \in A\} \quad \text{for } 1 \leq i \leq k. \end{aligned}$$

Note that these families partition $\mathcal{A}$. Let

$$\mathcal{F} = \Big(g^{k-1}(\mathcal{E}) - \{1\}\Big) \cup \bigcup_{i=0}^{k} \Big(g^k(\mathcal{D}_i) - \{1\}\Big),$$

where $\mathcal{E} = g(\mathcal{B}) \cap g(\mathcal{C})$ and, for any family $\mathcal{G}$, $\mathcal{G} - \{1\} = \{G\backslash\{1\} : G \in \mathcal{G}\}$.

Claim 9 (See [20, 29])**.** *The following hold*

(i) $|\mathcal{A}| = |g(\mathcal{B} \cup \mathcal{C})| + |\mathcal{F}|$;
(ii) $g(\mathcal{B} \cup \mathcal{C})$ *is an intersecting subfamily of* $\mathcal{I}_F{}^{(r)}$;
(iii) $g(\mathcal{F})$ *is an intersecting subfamily of* $\mathcal{I}_K{}^{(r-1)}$ *of size* $|\mathcal{F}|$;
(iv) $\big|\mathcal{I}_G{}^{(r)}(1)\big| = \big|\mathcal{I}_F{}^{(r)}(1)\big| + \big|\mathcal{I}_K{}^{(r-1)}(1)\big|$.

By the induction hypothesis and Claim 9(ii)–(iii), $|g(\mathcal{B}\cup\mathcal{C})| \le \left|\mathcal{I}_F{}^{(r)}(1)\right|$ and $|\mathcal{F}| = |g(\mathcal{F})| \le \left|\mathcal{I}_K{}^{(r-1)}(1)\right|$. Thus, by Claim 9(i) and Claim 9(iv), we have

$$|\mathcal{A}| = |g(\mathcal{B}\cup\mathcal{C})| + |\mathcal{F}| \le \left|\mathcal{I}_F{}^{(r)}(1)\right| + \left|\mathcal{I}_K{}^{(r-1)}(1)\right| = \left|\mathcal{I}_G{}^{(r)}(1)\right|,$$

and the theorem is proved. ■

Acknowledgements

The authors are grateful to the referees for reading the paper carefully and for providing many valuable comments on improving the presentation. Carl Feghali was supported in part by grant 249994 of the Research Council of Norway via the project CLASSIS and by grant 19-21082S of the Czech Science Foundation.

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Received 14 January 2020
Revised 8 September 2020
Accepted 19 September 2020