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THE HILTON-SPENCER CYCLE THEOREMS VIA KATONA'S SHADOW INTERSECTION THEOREM

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Abstract

A family $\mathcal{A}$ of sets is said to be *intersecting* if every two sets in $\mathcal{A}$ intersect. An intersecting family is said to be *trivial* if its sets have a common element. A graph G is said to be *r-EKR* if at least one of the largest intersecting families of independent r-element sets of G is trivial. Let $\alpha(G)$ and $\omega(G)$ denote the independence number and the clique number of G, respectively. Hilton and Spencer recently showed that if G is the vertex-disjoint union of a cycle C raised to the power k and s cycles ${}_1C, \ldots, {}_sC$ raised to the powers $k_1, \ldots, k_s$, respectively, $1 \leq r \leq \alpha(G)$, and

$$\min\left(\omega\left({}_1C^{k_1}\right), \ldots, \omega\left({}_sC^{k_s}\right)\right) \geq \omega\left(C^k\right),$$

then G is r-EKR. They had shown that the same holds if C is replaced by a path P and the condition on the clique numbers is relaxed to

$$\min\left(\omega\left({}_1C^{k_1}\right), \ldots, \omega\left({}_sC^{k_s}\right)\right) \geq \omega\left(P^k\right).$$

We use the classical Shadow Intersection Theorem of Katona to obtain a significantly shorter proof of each result for the case where the inequality for the minimum clique number is strict.

Keywords: cycle, independent set, intersecting family, Erdős-Ko-Rado theorem, Hilton-Spencer theorem, Katona's shadow intersection theorem.

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1. Introduction

Unless stated otherwise, we shall use small letters such as x to denote non-negative integers or elements of a set, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (sets whose members are sets themselves). The set of positive integers is denoted by $\mathbb{N}$. The set $\{i \in \mathbb{N} \colon m \leq i \leq n\}$ is denoted by $[m, n]$, $[1, n]$ is abbreviated to $[n]$, and $[0]$ is taken to be the empty set $\emptyset$. For a set X, the *power set of* X (that is, $\{A \colon A \subseteq X\}$) is denoted by 2^X. The family of r-element subsets of X is denoted by $\binom{X}{r}$. The family of r-element sets in a family $\mathcal{F}$ is denoted by $\mathcal{F}^{(r)}$. If $\mathcal{F} \subseteq 2^X$ and $x \in X$, then the family $\{A \in \mathcal{F} \colon x \in A\}$ is denoted by $\mathcal{F}(x)$ and called the *star of* $\mathcal{F}$ with *centre* x.

A family $\mathcal{A}$ is said to be *intersecting* if for every $A, B \in \mathcal{A}$, A and B intersect (that is, $A \cap B \neq \emptyset$). The stars of a family $\mathcal{F}$ are among the simplest intersecting subfamilies of $\mathcal{F}$. We say that $\mathcal{F}$ has the *star property* if at least one of the largest intersecting subfamilies of $\mathcal{F}$ is a star of $\mathcal{F}$.

Determining the size of a largest intersecting subfamily of a given family $\mathcal{F}$ is one of the most popular endeavours in extremal set theory. This started in [11], which features the classical result known as the Erdős-Ko-Rado (EKR) Theorem. The EKR Theorem states that if $r \leq n/2$ and $\mathcal{A}$ is an intersecting subfamily of $\binom{[n]}{r}$, then $|\mathcal{A}| \leq \binom{n-1}{r-1}$. Thus, $\binom{[n]}{r}$ has the star property for $r \leq n/2$ (clearly, for $n/2 < r \leq n$, $\binom{[n]}{r}$ itself is intersecting). There are various proofs of the EKR Theorem (see [9, 16, 24, 25, 27]), two of which are particularly short and beautiful: Katona's [25], which introduced the elegant cycle method, and Daykin's [9], using the fundamental Kruskal-Katona Theorem [26, 28]. The EKR Theorem gave rise to some of the highlights in extremal set theory [1, 14, 27, 30] and inspired many variants and generalizations; see [4, 10, 13, 15, 17, 21, 22].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We may represent an edge $\{v, w\}$ by vw. A subset I of $V(G)$ is an *independent set of* G if $vw \notin E(G)$ for every $v, w \in I$. Let $\mathcal{I}_G$ denote the family of independent sets of G. An independent set J of G is *maximal* if $J \nsubseteq I$ for each independent set I of G such that $I \neq J$. The size of a smallest maximal independent set of G is denoted by $\mu(G)$. The size of a largest independent set of G is denoted by $\alpha(G)$. A subset X of $V(G)$ is a *clique of* G if $vw \in E(G)$ for every $v, w \in X$ with $v \neq w$. The size of a largest clique of G is called the *clique number of* G and denoted by $\omega(G)$.

Holroyd and Talbot introduced the problem of determining whether ${\mathcal{I}_G}^{(r)}$ has the star property for a given graph G and an integer $r \geq 1$. Following their terminology, a graph G is said to be *r-EKR* if ${\mathcal{I}_G}^{(r)}$ has the star property. The Holroyd-Talbot (HT) Conjecture [22, Conjecture 7] claims that G is r-EKR if $\mu(G) \geq 2r$. This was verified by Borg [2] for $\mu(G)$ sufficiently large depending on r (see also [6, Lemma 4.4 and Theorem 1.4]). By the EKR Theorem, the conjecture is true if G has no edges. The HT Conjecture has been verified for several classes of graphs [2, 3, 7, 8, 12, 18–23, 29, 31]. As demonstrated in [8], for $r > \mu(G)/2$, whether G is r-EKR or not depends on G and r (both cases are possible). Naturally, graphs G of particular interest are those that are r-EKR for all $r \leq \alpha(G)$.

For $n \geq 1$, the graphs $\left([n], \binom{[n]}{2}\right)$ and $([n], \{\{i, i+1\} \colon i \in [n-1]\})$ are denoted by K_n and P_n, respectively. For $n \geq 3$, $([n], E(P_n) \cup \{\{n, 1\}\})$ is denoted by C_n. A copy of K_n is called a *complete graph*. A copy P of P_n is called an *n-path* or simply a *path*, and a vertex of P is called an *end-vertex* if it is not adjacent to more than one vertex. A copy of C_n is called an *n-cycle* or simply a *cycle* (normally, this terminology is used for $n \geq 3$, but we may include the case $n = 2$). If H is a subgraph of a graph G (that is, $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$), then we say that G *contains* H. For $v, w \in V(G)$, the *distance* $d_G(v, w)$ is $\min\{k \colon v, w \in V(P)$ for some $(k+1)$-path P contained by $G\}$. The k^{th} *power of* G, denoted by G^k, is the graph with vertex set $V(G)$ and edge set $\{vw \colon v, w \in V(G),\ 1 \leq d_G(v, w) \leq k\}$; G^k is also referred to as G *raised to the power* k.

Note that ${P_n}^k = K_n$ for $n \leq k+1$, and ${C_n}^k = K_n$ for $n \leq 2k+1$. Also note that $\omega({P_n}^k) = k+1$ if $n \geq k+1$, $\omega({P_n}^k) = n$ if $n \leq k$, $\omega({C_n}^k) = k+1$ if $n \geq 2k+2$, $\omega({C_n}^k) = n$ if $n \leq 2k+1$, $\alpha({P_n}^k) = \lceil n/(k+1) \rceil$, $\alpha({C_n}^k) = \lfloor n/(k+1) \rfloor$ if $n \geq k+1$, and $\alpha({C_n}^k) = 1$ if $2 \leq n \leq k+1$.

The following remarkable analogue of the EKR theorem was obtained by Talbot [29].

Theorem 1 [29]**.** *For $1 \leq r \leq \alpha({C_n}^k)$, ${C_n}^k$ is r-EKR.*

Talbot introduced a compression technique to prove Theorem 1. In vague terms, his compression technique rotates anticlockwise the elements of the independent sets of the intersecting family which are distinct from a specified vertex (see Section 2).

If $G, G_1, \ldots, G_k$ are graphs such that the vertex sets of $G_1, \ldots, G_k$ are pairwise disjoint and $G = \left(\bigcup_{i=1}^k V(G_i), \bigcup_{i=1}^k E(G_i)\right)$, then G is said to be the *disjoint union of* $G_1, \ldots, G_k$, and $G_1, \ldots, G_k$ are said to be *vertex-disjoint*.

Inspired by the work of Talbot, Hilton and Spencer [19] went on to prove the following result, which is stated with notation used in [19, 20].

Theorem 2 [19]**.** *If G is the disjoint union of a path P raised to the power k and s cycles ${}_1C, \ldots, {}_sC$ raised to the powers $k_1, \ldots, k_s$, respectively, $1 \leq r \leq \alpha(G)$, and*

$$\min\left(\omega\left({}_1C^{k_1}\right), \ldots, \omega\left({}_sC^{k_s}\right)\right) \geq \omega(P^k), \tag{1}$$

then G is r-EKR. Moreover, for any end-vertex x of P, ${\mathcal{I}_G}^{(r)}(x)$ is a largest intersecting subfamily of ${\mathcal{I}_G}^{(r)}$.

However, it was desired to obtain a generalization of Theorem 1, and this was eventually achieved by Hilton and Spencer [20] with the following theorem.

Theorem 3 [20]**.** *If G is the disjoint union of $s+1$ cycles $C, {}_1C, \ldots, {}_sC$ raised to the powers $k, k_1, \ldots, k_s$, respectively, $1 \le r \le \alpha(G)$, and*

$$\min\left(\omega\left({}_1C^{k_1}\right), \ldots, \omega\left({}_sC^{k_s}\right)\right) \ge \omega(C^k), \tag{2}$$

then G is r-EKR. Moreover, for any $x \in V(C)$, ${\mathcal{I}_G}^{(r)}(x)$ is a largest intersecting subfamily of ${\mathcal{I}_G}^{(r)}$.

Hilton and Spencer [20] conjectured that every disjoint union of powers of cycles is r-EKR.

The proof of Theorem 3 is also inspired by Talbot's proof of Theorem 1. In particular, an essential ingredient in the proof of Theorem 3 is the use of Theorem 2 for the special case where P^k is a complete graph as the base case of an induction argument.

In this paper, we give a significantly shorter and simpler proof of Theorem 2 and of Theorem 3, except for the cases of equality in conditions (1) and (2), respectively. In other words, we prove the following two results.

Theorem 4. *Theorem 2 is true if the inequality in (1) is strict.*

Theorem 5. *Theorem 3 is true if the inequality in (2) is strict.*

Our argument is based on the Shadow Intersection Theorem of Katona [27], hence demonstrating yet another application of this classical and useful result in extremal set theory.

2. The New Proof

Let $P, {}_1C, \ldots, {}_sC$ be as in Theorem 2. Let $p = |V(P)|$ and $c_i = |V({}_iC)|$. For $1 \le i \le s$, we label the vertices of ${}_iC$ by $v_{1,i}, v_{2,i}, \ldots, v_{c_i,i}$, where $E({}_iC) = \{v_{j,i}v_{j+1,i} : j \in [c_i - 1]\} \cup \{v_{c_i,i}v_{1,i}\}$. We may assume that $P = P_p$, that is, $V(P) = [p]$ and $E(P) = \{\{i, i+1\} : i \in [p-1]\}$. Let H be the union of ${}_1C^{k_1}, \ldots, {}_sC^{k_s}$, and let $f : V(H) \to V(H)$ be the bijection given by

$$f(v_{c_i,i}) = v_{1,i} \quad \text{and} \quad f(v_{j,i}) = v_{j+1,i} \quad \text{for} \quad 1 \le i \le s \quad \text{and} \quad 1 \le j \le c_i - 1.$$

Let $f^1 = f$, and for any integer $t \ge 2$, let $f^t = f \circ f^{t-1}$ and $f^{-t} = f^{-1} \circ f^{-(t-1)}$. Note that for $t \ge 1$, one can think of f^t as t clockwise rotations, and of f^{-t} as t anticlockwise rotations. For $I \in \mathcal{I}_H$, we denote the set $\{f^t(x) : x \in I\}$ by $f^t(I)$, and for $\mathcal{A} \subseteq \mathcal{I}_H$, we denote the family $\{f^t(A) : A \in \mathcal{A}\}$ by $f^t(\mathcal{A})$. The notation $f^{-t}(I)$ and $f^{-t}(\mathcal{A})$ is defined similarly.

The new argument presented in this paper lies entirely in the proof of the following important case, which both Theorem 4 and Theorem 5 pivot on.

Lemma 6. *Theorem* 2 *is true if* P^k *is a complete graph and the inequality in* (1) *is strict.*

As shown in this section, Theorem 4 follows from Lemma 6 by applying the compression method in [21], and Theorem 5 follows from Lemma 6 by applying the same compression method of Talbot in [29].

We now start working towards the proof of Lemma 6.

Let $\mathcal{A}$ be a family of r-element sets. The *shadow of* $\mathcal{A}$, denoted by $\partial\mathcal{A}$, is the family $\bigcup_{A\in\mathcal{A}}\binom{A}{r-1}$. A special case of Katona's Shadow Intersection Theorem [27] is that

$$|\mathcal{A}| \le |\partial\mathcal{A}| \quad \text{if } \mathcal{A} \text{ is intersecting.} \tag{3}$$

***Proof of Lemma* 6.** Suppose that P^k is a complete graph. Then, $\omega(P^k) = p$. Suppose $\min\big(\omega({}_1C^{k_1}),\dots,\omega({}_sC^{k_s})\big) > p$. Note that this implies that for every $i \in [s]$, $c_i \ge p+1$ and, for every $v_{h,i}, v_{j,i} \in V({}_iC^{k_i})$,

$$\text{if } v_{j,i} \in \big\{f^{-q}(v_{h,i})\colon q \in [p]\big\} \cup \big\{f^{q}(v_{h,i})\colon q \in [p]\big\}, \text{ then } v_{h,i}v_{j,i} \in E\big({}_iC^{k_i}\big). \tag{4}$$

It is worth pointing out that the strict inequality is only used for (4), from which we obtain Claim 7.

Let $\mathcal{A}$ be an intersecting subfamily of $\mathcal{I}_G{}^{(r)}$. Recall that $V(P^k) = [p]$. Let $\mathcal{A}_0 = \{A \in \mathcal{A}\colon A \cap [p] = \emptyset\}$ and $\mathcal{A}_i = \{A \in \mathcal{A}\colon A \cap [p] = \{i\}\}$ for $1 \le i \le p$. Since P^k is a complete graph, the families $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_p$ partition $\mathcal{A}$. Let $\mathcal{A}_0' = \mathcal{A}_0$ and $\mathcal{A}_i' = \{A\backslash\{i\}\colon A \in \mathcal{A}_i\}$ for $1 \le i \le p$. Since $\mathcal{A}$ is intersecting,

$$\text{for any } i, j \in \{0\} \cup [p] \text{ with } i \ne j, \text{ each set in } \mathcal{A}_i' \text{ intersects each set in } \mathcal{A}_j'. \tag{5}$$

Claim 7. *The families* $\partial\mathcal{A}_0, f^1(\mathcal{A}_1'), f^2(\mathcal{A}_2'), \dots, f^p(\mathcal{A}_p')$ *are pairwise disjoint.*

Proof. Suppose $B \in f^i(\mathcal{A}_i') \cap f^j(\mathcal{A}_j')$ for some $i, j \in [p]$ with $i < j$. Then, $B = f^i(A_i) = f^j(A_j)$ for some $A_i \in \mathcal{A}_i'$ and $A_j \in \mathcal{A}_j'$. Thus, $A_i = f^{j-i}(A_j)$. Since $1 \le j-i < p$, (4) gives us $A_i \cap A_j = \emptyset$, but this contradicts (5). Therefore, $f^1(\mathcal{A}_1'), f^2(\mathcal{A}_2'), \dots, f^p(\mathcal{A}_p')$ are pairwise disjoint.

Suppose $B \in \partial\mathcal{A}_0 \cap f^i(\mathcal{A}_i')$ for some $i \in [p]$. Then, $C\backslash\{x\} = B = f^i(A_i)$ for some $C \in \mathcal{A}_0$, $x \in C$, and $A_i \in \mathcal{A}_i'$. Since $1 \le i \le p$, (4) gives us $C \cap A_i = \emptyset$, but this contradicts (5). The claim follows. □

Let $\mathcal{A}_0^* = \{A \cup \{1\}\colon A \in \partial\mathcal{A}_0\}$ and $\mathcal{A}_i^* = \{A \cup \{1\}\colon A \in f^i(\mathcal{A}_i')\}$ for $1 \le i \le p$. For $0 \le i \le p$, $\mathcal{A}_i^* \subseteq \mathcal{I}_G{}^{(r)}(1)$. By Claim 7, $\sum_{i=0}^{p} |\mathcal{A}_i^*| = \big|\bigcup_{i=0}^{p} \mathcal{A}_i^*\big| \le \big|\mathcal{I}_G{}^{(r)}(1)\big|$. By (3), $|\mathcal{A}_0| \le |\partial(\mathcal{A}_0)| = |\mathcal{A}_0^*|$. We have

$$|\mathcal{A}| = \sum_{i=0}^{p} |\mathcal{A}_i| = |\mathcal{A}_0| + \sum_{i=1}^{p} |\mathcal{A}_i^*| \le \sum_{i=0}^{p} |\mathcal{A}_i^*| \le \big|\mathcal{I}_G{}^{(r)}(1)\big|,$$

and the lemma is proved. ■

The full Theorem 4 is now obtained by the line of argument laid out in [21], hence making use of established facts regarding compressions on independent sets.

For any edge uv of a graph G, let $\delta_{u,v}\colon \mathcal{I}_G \to \mathcal{I}_G$ be defined by

$$\delta_{u,v}(A) = \begin{cases} (A\backslash\{v\}) \cup \{u\} & \text{if } v \in A, u \notin A, \text{ and } (A\backslash\{v\}) \cup \{u\} \in \mathcal{I}_G; \\ A & \text{otherwise,} \end{cases}$$

and let $\Delta_{u,v}\colon 2^{\mathcal{I}_G} \to 2^{\mathcal{I}_G}$ be the *compression operation* (also called a *shifting operation*) defined by

$$\Delta_{u,v}(\mathcal{A}) = \{\delta_{u,v}(A)\colon A \in \mathcal{A}\} \cup \{A \in \mathcal{A}\colon \delta_{u,v}(A) \in \mathcal{A}\}.$$

It is well known, and easy to see, that

$$|\Delta_{u,v}(\mathcal{A})| = |\mathcal{A}|$$

(see [11, 15]). For any $x \in V(G)$, let $N_G(x)$ denote the set $\{y \in V(G)\colon xy \in E(G)\}$. The following is given by [8, Lemma 2.1] (which is actually stated for ${\mathcal{I}_G}^{(r)}$ but proved for $\mathcal{I}_G$) and essentially originated in [21]. We omit the proof.

Lemma 8 [8, 21]**.** *If G is a graph, $uv \in E(G)$, $\mathcal{A}$ is an intersecting subfamily of $\mathcal{I}_G$, $\mathcal{B} = \Delta_{u,v}(\mathcal{A})$, $\mathcal{B}_0 = \{B \in \mathcal{B}\colon v \notin B\}$, $\mathcal{B}_1 = \{B \in \mathcal{B}\colon v \in B\}$, and $\mathcal{B}_1' = \{B\backslash\{v\}\colon B \in \mathcal{B}_1\}$, then*

(i) *$\mathcal{B}_0$ is intersecting;*

(ii) *if $|N_G(u)\backslash(\{v\} \cup N_G(v))| \leq 1$, then $\mathcal{B}_1'$ is intersecting;*

(iii) *if $N_G(u)\backslash(\{v\} \cup N_G(v)) = \emptyset$, then $\mathcal{B}_0 \cup \mathcal{B}_1'$ is intersecting.*

For a vertex v of a graph G, let $G - v$ denote the graph obtained by deleting v (that is, $G - v = (V(G)\backslash\{v\}, \{xy \in E(G)\colon x, y \notin \{v\}\})$), and let $G \downarrow v$ be the graph obtained by deleting v and the vertices adjacent to v (that is, $G \downarrow v = (V(G)\backslash(\{v\} \cup N_G(v)), \{xy \in E(G)\colon x, y \notin \{v\} \cup N_G(v)\})$).

***Proof of Theorem* 4.** We use induction on $|V(P)|$. If P^k is a complete graph, then the result is given by Lemma 6. Note that this captures the base case $|V(P)| = 1$. Now suppose that P^k is not a complete graph. Then, $|V(P)| \geq k+2$. If $r = 1$, then the result is trivial. Suppose $r > 1$. Let $\mathcal{A}$ be an intersecting subfamily of ${\mathcal{I}_G}^{(r)}$. Let $u = p-1$ and $v = p$. Let $\mathcal{B} = \Delta_{u,v}(\mathcal{A})$, $\mathcal{B}_0 = \{B \in \mathcal{B}\colon v \notin B\}$, $\mathcal{B}_1 = \{B \in \mathcal{B}\colon v \in B\}$, and $\mathcal{B}_1' = \{B\backslash\{v\}\colon B \in \mathcal{B}_1\}$. By Lemma 8(i), $\mathcal{B}_0$ is intersecting. We have $N_G(u)\backslash(\{v\} \cup N_G(v)) = \{p - k - 1\}$, so, by Lemma 8(ii), $\mathcal{B}_1'$ is intersecting. Let $H_0 = P_{p-1}$ and $H_1 = P_{p-k-1}$. Clearly, $\mathcal{B}_0 \subseteq {\mathcal{I}_{G-v}}^{(r)}$, $\mathcal{B}_1' \subseteq {\mathcal{I}_{G\downarrow v}}^{(r)}$, $G - v$ is the union of ${H_0}^k$ and ${}_1C^{k_1}, \dots, {}_sC^{k_s}$, and $G \downarrow v$ is the union of ${H_1}^k$ and ${}_1C^{k_1}, \dots, {}_sC^{k_s}$. The condition $\min\big(\omega({}_1C^{k_1}), \dots, \omega({}_sC^{k_s})\big) > \omega(P^k)$

in the theorem gives us $\min\big(\omega({}_1C^{k_1}),\dots,\omega({}_sC^{k_s})\big) > \omega({H_0}^k) \geq \omega({H_1}^k)$. By the induction hypothesis, $|\mathcal{B}_0| \leq \big|\mathcal{I}_{G-v}{}^{(r)}(1)\big|$ and $|\mathcal{B}_1'| \leq \big|\mathcal{I}_{G\downarrow v}{}^{(r-1)}(1)\big|$. We have

$$\begin{aligned}|\mathcal{A}| = |\mathcal{B}| = |\mathcal{B}_0| + |\mathcal{B}_1'| &\leq \big|\mathcal{I}_{G-v}{}^{(r)}(1)\big| + \big|\mathcal{I}_{G\downarrow v}{}^{(r-1)}(1)\big| \\ &= \big|\big\{A \in \mathcal{I}_G{}^{(r)} \colon 1 \in A, v \notin A\big\}\big| + \big|\big\{A \in \mathcal{I}_G{}^{(r)} \colon 1, v \in A\big\}\big| = \big|\mathcal{I}_G{}^{(r)}(1)\big|,\end{aligned}$$

as required. ■

***Proof of Theorem* 5.** We use induction on $c = |V(C)|$. We may assume that $C = C_c$. If C^k is a complete graph, then the result is given by Lemma 6. Note that this captures the base case $c = 2$. Now suppose that C^k is not a complete graph. Then, $c \geq 2k+2$. If $r = 1$, then the result is trivial. Suppose $r > 1$. Let $\mathcal{A}$ be an intersecting subfamily of $\mathcal{I}_G{}^{(r)}$.

Let $g : V(G) \to V(G)$ be the Talbot compression [20, 29] given by

$$\begin{aligned} g(v) &= v \quad \text{for } v \in V(G)\backslash V(C), \\ g(1) &= 1, \quad \text{and} \\ g(1+j) &= 1+j-1 \quad \text{for } 1 \leq j \leq c-1. \end{aligned}$$

For $X \in \mathcal{I}_G$ and $\mathcal{X} \subseteq \mathcal{I}_G$, we use the notation $g^t(X)$ and $g^t(\mathcal{X})$ similarly to the way it is used above for f. Let F be the union of ${C_{c-1}}^k$ and ${}_1C^{k_1},\dots,{}_sC^{k_s}$. Let K be the union of ${C_{c-k-1}}^k$ and ${}_1C^{k_1},\dots,{}_sC^{k_s}$. Let

$$\begin{aligned} \mathcal{B} &= \big\{A \in \mathcal{A} : 1 \notin A,\ g(A) \in \mathcal{I}_F{}^{(r)}\big\}, \\ \mathcal{C} &= \big\{A \in \mathcal{A} : 1 \in A,\ g(A) \in \mathcal{I}_F{}^{(r)}\big\}, \\ \mathcal{D}_0 &= \{A \in \mathcal{A} : 1,\ k+2 \in A\}, \\ \mathcal{D}_i &= \{A \in \mathcal{A} : 1+c-i,\ k+2-i \in A\} \quad \text{for } 1 \leq i \leq k. \end{aligned}$$

Note that these families partition $\mathcal{A}$. Let

$$\mathcal{F} = \Big(g^{k-1}(\mathcal{E}) - \{1\}\Big) \cup \bigcup_{i=0}^{k} \Big(g^k(\mathcal{D}_i) - \{1\}\Big),$$

where $\mathcal{E} = g(\mathcal{B}) \cap g(\mathcal{C})$ and, for any family $\mathcal{G}$, $\mathcal{G} - \{1\} = \{G\backslash\{1\} : G \in \mathcal{G}\}$.

Claim 9 (See [20, 29])**.** *The following hold*

(i) $|\mathcal{A}| = |g(\mathcal{B} \cup \mathcal{C})| + |\mathcal{F}|$;
(ii) $g(\mathcal{B} \cup \mathcal{C})$ *is an intersecting subfamily of* $\mathcal{I}_F{}^{(r)}$;
(iii) $g(\mathcal{F})$ *is an intersecting subfamily of* $\mathcal{I}_K{}^{(r-1)}$ *of size* $|\mathcal{F}|$;
(iv) $\big|\mathcal{I}_G{}^{(r)}(1)\big| = \big|\mathcal{I}_F{}^{(r)}(1)\big| + \big|\mathcal{I}_K{}^{(r-1)}(1)\big|$.

By the induction hypothesis and Claim 9(ii)–(iii), $|g(\mathcal{B}\cup\mathcal{C})| \leq \left|\mathcal{I}_F{}^{(r)}(1)\right|$ and $|\mathcal{F}| = |g(\mathcal{F})| \leq \left|\mathcal{I}_K{}^{(r-1)}(1)\right|$. Thus, by Claim 9(i) and Claim 9(iv), we have

$$|\mathcal{A}| = |g(\mathcal{B}\cup\mathcal{C})| + |\mathcal{F}| \leq \left|\mathcal{I}_F{}^{(r)}(1)\right| + \left|\mathcal{I}_K{}^{(r-1)}(1)\right| = \left|\mathcal{I}_G{}^{(r)}(1)\right|,$$

and the theorem is proved. ■

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