Discussiones Mathematicae
Graph Theory 43 (2023) 233–244
https://doi.org/10.7151/dmgt.2364

SOME RESULTS ON PATH-FACTOR CRITICAL AVOIDABLE GRAPHS

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Abstract

A path factor is a spanning subgraph F of G such that every component of F is a path with at least two vertices. We write $P_{\geq k} = \{P_i : i \geq k\}$. Then a $P_{\geq k}$-factor of G means a path factor in which every component admits at least k vertices, where $k \geq 2$ is an integer. A graph G is called a $P_{\geq k}$-factor avoidable graph if for any $e \in E(G)$, G admits a $P_{\geq k}$-factor excluding e. A graph G is called a $(P_{\geq k}, n)$-factor critical avoidable graph if for any $Q \subseteq V(G)$ with $|Q| = n$, $G - Q$ is a $P_{\geq k}$-factor avoidable graph. Let G be an $(n+2)$-connected graph. In this paper, we demonstrate that (i) G is a $(P_{\geq 2}, n)$-factor critical avoidable graph if $tough(G) > \frac{n+2}{4}$; (ii) G is a $(P_{\geq 3}, n)$-factor critical avoidable graph if $tough(G) > \frac{n+1}{2}$; (iii) G is a $(P_{\geq 2}, n)$-factor critical avoidable graph if $I(G) > \frac{n+2}{3}$; (iv) G is a $(P_{\geq 3}, n)$-factor critical avoidable graph if $I(G) > \frac{n+3}{2}$. Furthermore, we claim that these conditions are sharp.

Keywords: graph, toughness, isolated toughness, $P_{\geq k}$-factor, $(P_{\geq k}, n)$-factor critical avoidable graph.

2010 Mathematics Subject Classification: 05C70, 05C38, 90B10.

1. Introduction

In this paper, we discuss only finite undirected simple graphs. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ denotes the vertex set of G and $E(G)$ denotes the edge set of G. For $x \in V(G)$, the degree of x in G is denoted by $d_G(x)$. For a set $X \subseteq V(G)$, we use $G[X]$ to denote the subgraph of G induced by X and write $G - X$ for $G[V(G) \setminus X]$. We let $i(G)$ and $\omega(G)$ denote the number of isolated vertices and the number of connected components of G, respectively. Let P_n and

K_n denote the path and the complete graph of order n, respectively. The join $G+H$ denotes the graph with vertex set $V(G)\cup V(H)$ and edge set

$$E(G+H)=E(G)\cup E(H)\cup\{xy : x\in V(G) \text{ and } y\in V(H)\}.$$

Chvátal [2] first introduced the toughness of a graph G, denoted by $tough(G)$, namely,

$$tough(G)=\min\left\{\frac{|X|}{\omega(G-X)} : X\subseteq V(G),\, \omega(G-X)\geq 2\right\},$$

if G is not complete; otherwise, $tough(G)=+\infty$.

Yang, Ma and Liu [13] first posed isolated toughness of a graph G, denoted by $I(G)$, namely,

$$I(G)=\min\left\{\frac{|X|}{i(G-X)} : X\subseteq V(G),\, i(G-X)\geq 2\right\}$$

if G is not a complete graph; otherwise, $I(G)=+\infty$.

A path factor is a spanning subgraph F of G such that every component of F is a path with at least two vertices. We write $P_{\geq k}=\{P_i : i\geq k\}$. Then a $P_{\geq k}$-factor of G means a path factor in which every component admits at least k vertices, where $k\geq 2$ is an integer. A $\{P_k\}$-factor F of G is simply called a P_k-factor if every component of F is isomorphic to P_k.

A 1-factor of G is a spanning subgraph F of G such that $d_F(x)=1$ holds for any $x\in V(G)$. A graph R is a factor-critical graph if for any $x\in V(R)$, $R-\{x\}$ admits a 1-factor. Let R be a factor-critical graph with $V(R)=\{x_1,x_2,\ldots,x_n\}$. n new vertices $y_1,y_2,\ldots,y_n$ together with new edges $x_1y_1,x_2y_2,\ldots,x_ny_n$ are added to R. Then the resulting graph is said to be a sun. By Kaneko [7], K_1 and K_2 are also suns. A big sun is a sun of order at least 6. We use $sun(G)$ to denote the number of sun components of G.

Las Vergnas [11] posed a criterion for the existence of $P_{\geq 2}$-factors in graphs.

Theorem 1 [11]**.** *A graph G admits a $P_{\geq 2}$-factor if and only if $i(G-X)\leq 2|X|$ for every $X\subseteq V(G)$.*

Kaneko [7] derived a characterization for the existence of $P_{\geq 3}$-factors in graphs.

Theorem 2 [7]**.** *A graph G admits a $P_{\geq 3}$-factor if and only if $sun(G-X)\leq 2|X|$ for every $X\subseteq V(G)$.*

A graph G is called a $P_{\geq k}$-factor avoidable graph if for any $e\in E(G)$, G admits a $P_{\geq k}$-factor excluding e. A graph G is called a $(P_{\geq k},n)$-factor critical avoidable graph if for any $Q\subseteq V(G)$ with $|Q|=n$, $G-Q$ is a $P_{\geq k}$-factor avoidable graph. Obviously, a $(P_{\geq k}, 0)$-factor critical avoidable graph is simply called a $P_{\geq k}$-factor avoidable graph.

Kelmas [10] claimed a result on the existence of path factors in subgraphs.

Theorem 3 [10]**.** *Let G be a 3-connected claw-free graph and $|V(G)| \equiv 1 \pmod 3$. Then for any $x \in V(G)$ and any $e \in E(G)$, $G - \{x, e\}$ has a $\{P_3\}$-factor, namely, $G - \{x\}$ is a $\{P_3\}$-factor avoidable graph.*

Motivated by Theorem 3, we consider a more general problem.

Problem 1. Find sufficient conditions for a graph to be a $(P_{\geq k}, n)$-factor critical avoidable graph.

Kano, Lu and Yu [8] verified that a graph G has a $\{P_3\}$-factor if $i(G-S) \leq \frac{2}{3}|S|$ for every $S \subset V(G)$. Zhou, Yang and Xu [22] proved that an n-connected graph G is $(P_{\geq 3}, n)$-factor critical if its toughness $tough(G) \geq \frac{n+1}{2}$. Some other results on path factors can be found in [3, 15, 17, 18]. Lots of authors derived some toughness conditions for the existence of graph factors [4, 5, 9, 20]. Some results on the relationships between isolated toughness and graph factors are obtained by Gao, Liang and Chen [6]. For many other results on graph factors, see [1, 12, 14, 16, 19, 21, 23]. In this paper, we study $(P_{\geq k}, n)$-factor critical avoidable graphs and get some sufficient conditions for graphs to be $(P_{\geq k}, n)$-factor critical avoidable graphs depending on toughness and isolated toughness, which are given in Sections 2 and 3.

2. Toughness and $(P_{\geq k}, n)$-Factor Critical Avoidable Graphs

In this section, we explore the relationship between toughness and $(P_{\geq k}, n)$-factor critical avoidable graphs, and derive two toughness conditions for the existence of $(P_{\geq k}, n)$-factor critical avoidable graphs for $k = 2, 3$.

Theorem 4. *Let G be an $(n+2)$-connected graph, where $n \geq 0$ is an integer. If its toughness $tough(G) > \frac{n+2}{4}$, then G is a $(P_{\geq 2}, n)$-factor critical avoidable graph.*

Proof. Theorem 4 obviously holds for a complete graph. Next, we assume that G is not complete. Let $Q \subset V(G)$ with $|Q| = n$, and $G' = G - Q$, and let $e \in E(G')$ and $H = G' - e$. Since G is $(n+2)$-connected, H is connected. To prove Theorem 4, it suffices to show that H admits a $P_{\geq 2}$-factor. On the contrary, suppose that H has no $P_{\geq 2}$-factor. Then by Theorem 1, there exists a set $X \subset V(H)$ such that

$$i(H - X) \geq 2|X| + 1. \tag{1}$$

Since H is connected, we have $X \neq \emptyset$. Thus,

$$i(H - X) \geq 2|X| + 1 \geq 3. \tag{2}$$

Note that $\omega(G - (Q \cup X)) \geq \omega(G - (Q \cup X) - e) - 1$. Combining this with (2), we derive

$$\omega(G-(Q\cup X)) \geq \omega(G-(Q\cup X)-e)-1 = \omega(H-X)-1 \geq i(H-X)-1 \geq 2. \tag{3}$$

Claim 1. $|X| \geq 2$.

Proof. Assume $|X| = 1$. Since $H = G' - e$, we easily know that $i(H-X) = i(G'-e-X) \leq i(G'-X)+2$. Then by (2), we derive $i(G'-X) \geq i(H-X)-2 \geq 1$, which implies that there exists an isolated vertex u in $G' - X$, i.e., $d_{G'-X}(u) = 0$. Thus, we have $d_G(u) \leq d_{G'}(u)+|Q| = d_{G'}(u)+n \leq d_{G'-X}(u)+|X|+n = 0+1+n = n+1$, contradicting that G is $(n+2)$-connected. Therefore, $|X| \geq 2$. $\square$

According to (1), (2), (3), Claim 1 and the definition of $tough(G)$, we have

$$\begin{aligned} tough(G) &\leq \frac{|Q \cup X|}{\omega(G - (Q \cup X))} \leq \frac{|Q| + |X|}{i(H - X) - 1} \\ &= \frac{n + |X|}{i(H - X) - 1} \leq \frac{n + |X|}{2|X|} = \frac{1}{2} + \frac{n}{2|X|} \leq \frac{1}{2} + \frac{n}{4} = \frac{n+2}{4}, \end{aligned}$$

which contradicts $tough(G) > \frac{n+2}{4}$. Theorem 4 is verified. $\blacksquare$

Remark 5. Now, we claim that the result in Theorem 4 is sharp. To see this, we construct the graph $G = K_{n+2} + (3K_1 \cup K_2)$. Clearly, G is $(n+2)$-connected and $tough(G) = \frac{n+2}{4}$. Let $Q \subset V(K_{n+2}) \subseteq V(G)$ with $|Q| = n$ and e be the edge of K_2. Then $G - Q - e$ is a graph isomorphic to $K_2 + (5K_1)$, and it obviously has no $P_{\geq 2}$-factor. Thus, G is not a $(P_{\geq 2}, n)$-factor critical avoidable graph.

Theorem 6. *Let G be an $(n + 2)$-connected graph, where $n \geq 0$ is an integer. If its toughness $tough(G) > \frac{n+1}{2}$, then G is a $(P_{\geq 3}, n)$-factor critical avoidable graph.*

Proof. Theorem 6 obviously holds for a complete graph. In the following, we assume that G is not complete. Let $Q \subset V(G)$ with $|Q| = n$, and $G' = G - Q$, and let $e \in E(G')$ and $H = G' - e$. Since G is $(n+2)$-connected, H is connected. To prove Theorem 6, it suffices to show that H admits a $P_{\geq 3}$-factor. On the contrary, suppose that H has no $P_{\geq 3}$-factor. Then by Theorem 2, there exists a set $X \subset V(H)$ such that

$$sun(H - X) \geq 2|X| + 1. \tag{4}$$

Claim 1. $X \neq \emptyset$.

Proof. Assume that $X = \emptyset$. Then it follows from (4) that

$$sun(H) \geq 1. \tag{5}$$

Since H is connected, we have $sun(H) = 1$ and H itself is a sun.

Since G is $(n+2)$-connected, $|V(G)| \geq n+3$. Thus, $|V(H)| = |V(G)|-n \geq 3$, which implies that H is a big sun. Hence, $|V(H)| \geq 6$. Let R be the factor-critical graph of H. Then $|V(R)| \geq 3$ and there exists $w \in V(R)$ such that $\omega(G' - \{w\}) = \omega(H - \{w\}) = 2$. Thus, we have

$$\omega(G - Q - \{w\}) = \omega(G' - \{w\}) = 2. \tag{6}$$

In terms of (6) and the definition of $tough(G)$, we get

$$tough(G) \leq \frac{|Q \cup \{w\}|}{\omega(G - (Q \cup \{w\}))} = \frac{n+1}{2},$$

contradicting to $tough(G) > \frac{n+1}{2}$. Hence, $X \neq \emptyset$. □

By (4) and Claim 1, we gain $\omega(G-(Q\cup X)) = \omega(G'-X) \geq \omega(G'-X-e)-1 = \omega(H - X) - 1 \geq sun(H - X) - 1 \geq 2|X| \geq 2$. Combining this with Claim 1 and the definition of $tough(G)$, we have

$$tough(G) \leq \frac{|Q \cup X|}{\omega(G - (Q \cup X))} \leq \frac{n + |X|}{2|X|} = \frac{1}{2} + \frac{n}{2|X|} \leq \frac{1}{2} + \frac{n}{2} = \frac{n+1}{2},$$

this contradicts $tough(G) > \frac{n+1}{2}$. This finishes the proof of Theorem 6. ■

Remark 7. Now, we show that the conditions in Theorem 6 are best possible, which cannot be replaced by G being $(n + 1)$-connected and $tough(G) \geq \frac{n+1}{2}$.

Let $G = K_{n+1} + (2K_2)$. We easily see that G is $(n + 1)$-connected and $tough(G) = \frac{n+1}{2}$. Let $Q \subset V(K_{n+1}) \subseteq V(G)$ with $|Q| = n$, and e be an edge of $2K_2$. Then $G-Q-e$ is a graph isomorphic to $K_1 + (2K_1 \cup K_2)$, and it obviously has no $P_{\geq 3}$-factor, and so G is not a $(P_{\geq 3}, n)$-factor critical avoidable graph.

3. Isolated Toughness and $(P_{\geq k}, n)$-Factor Critical Avoidable Graphs

In this section we give two sufficient conditions using isolated toughness for a graph to be a $(P_{\geq k}, n)$-factor critical avoidable graph for $k = 2, 3$.

Theorem 8. *Let G be an $(n + 2)$-connected graph, where $n \geq 0$ is an integer. If its isolated toughness $I(G) > \frac{n+2}{3}$, then G is a $(P_{\geq 2}, n)$-factor critical avoidable graph.*

Proof. Theorem 8 obviously holds for a complete graph. In what follows, we assume that G is not complete. Let $Q \subset V(G)$ with $|Q| = n$, and $G' = G - Q$, and let $e \in E(G')$ and $H = G' - e$. Since G is $(n+2)$-connected, H is connected. To prove Theorem 8, it suffices to show that H admits a $P_{\geq 2}$-factor. On the contrary, suppose that H has no $P_{\geq 2}$-factor. Then by Theorem 1, there exists a set $X \subset V(H)$ such that

$$i(H - X) \geq 2|X| + 1. \tag{7}$$

Claim 1. $|X| \geq 2$.

Proof. If $X = \emptyset$, then by (7) and H being connected, we obtain

$$1 \leq i(H) = 0,$$

which is a contradiction.

Next, we consider $|X| = 1$. Note that $i(H-X) = i(G'-e-X) \leq i(G'-X)+2$. Combining this with (7), we derive $i(G' - X) \geq i(H - X) - 2 \geq 2|X| + 1 - 2 = 2|X| - 1 = 1$, which hints that there exists $w \in V(G') \setminus X$ with $d_{G'-X}(w) = 0$. Therefore, we admit $d_G(w) = d_{G'+Q}(w) \leq d_{G'}(w) + |Q| = d_{G'}(w) + n \leq d_{G'-X}(w) + |X| + n = 0 + 1 + n = n + 1$, which contradicts that G is $(n + 2)$-connected. Thus, we derive $|X| \geq 2$. $\square$

According to (7) and Claim 1, we get

$$i(G - (Q \cup X)) \geq i(G - (Q \cup X) - e) - 2 = i(H - X) - 2 \geq 2|X| - 1 \geq 3. \tag{8}$$

It follows from (8), Claim 1 and the definition of $I(G)$ that

$$\begin{aligned} I(G) &\leq \frac{|Q \cup X|}{i(G - (Q \cup X))} \leq \frac{|Q| + |X|}{2|X| - 1} \\ &= \frac{n + |X|}{2|X| - 1} = \frac{n + \frac{1}{2}}{2|X| - 1} + \frac{|X| - \frac{1}{2}}{2(|X| - \frac{1}{2})} \\ &= \frac{n + \frac{1}{2}}{2|X| - 1} + \frac{1}{2} \leq \frac{n + \frac{1}{2}}{3} + \frac{1}{2} = \frac{n + 2}{3}, \end{aligned}$$

which contradicts $I(G) > \frac{n+2}{3}$. Theorem 8 is proved. ■

Remark 9. Now, we explain that the result in Theorem 8 is sharp. To see this, we construct the graph $G = K_{n+2} + (3K_1 \cup K_2)$. Obviously, G is $(n+2)$-connected and $I(G) = \frac{n+2}{3}$. Let $Q \subset V(K_{n+2}) \subseteq V(G)$ with $|Q| = n$, and e be the edge of K_2. Then $G - Q - e$ is a graph isomorphic to $K_2 + (5K_1)$, and it obviously has no $P_{\geq 2}$-factor. Thus, G is not a $(P_{\geq 2}, n)$-factor critical avoidable graph.

Theorem 10. *Let G be an $(n+2)$-connected graph, where n is a positive integer. If its isolated toughness $I(G) > \frac{n+3}{2}$, then G is a $(P_{\geq 3}, n)$-factor critical avoidable graph.*

Proof. Theorem 10 obviously holds for a complete graph. Next, we assume that G is not complete. Let $Q \subset V(G)$ with $|Q| = n$, and $G' = G - Q$, and let $e = xy \in E(G')$ and $H = G' - e$. Since G is $(n+2)$-connected, H is connected. To prove Theorem 10, it suffices to show that H admits a $P_{\geq 3}$-factor. On the contrary, suppose that H has no $P_{\geq 3}$-factor. Then by Theorem 2, there exists a set $X \subset V(H)$ such that

$$sun(H - X) \geq 2|X| + 1. \tag{9}$$

Claim 1. $X \neq \emptyset$.

Proof. Assume $X = \emptyset$. Then $sun(H) \geq 1$. This implies $sun(H) = 1$ since H is connected.

Note that G is $(n+2)$-connected. Hence, $|V(G)| \geq n+3$. Thus, $|V(H)| = |V(G)| - n \geq (n+3) - n = 3$, which implies that H is a big sun. Therefore, $|V(H)| \geq 6$. Let R be the factor-critical subgraph of H. Then $i(H - V(R)) = |V(R)| \geq 3$. Next, we consider two cases.

Case 1. $x, y \in V(H) \setminus V(R)$. Clearly, there exists $z \in V(R)$ with $yz \in E(G)$. Thus, we easily see

$$\begin{aligned} i(G - (Q \cup (V(R) \setminus \{z\}) \cup \{y\})) &= i(G' - ((V(R) \setminus \{z\}) \cup \{y\})) \\ &= i(G' - ((V(R) \setminus \{z\}) \cup \{y\}) - e) \\ &= i(H - ((V(R) \setminus \{z\}) \cup \{y\})) \\ &= |V(R)| \geq 3. \end{aligned}$$

Combining this with the definition of $I(G)$ and $I(G) > \frac{n+3}{2}$, we admit. Clearly, there exists $z \in V(R)$ with $yz \in E(G)$. Thus, we easily get

$$\begin{aligned} \frac{n+3}{2} &< I(G) \leq \frac{|Q \cup (V(R) \setminus \{z\}) \cup \{y\}|}{i(G - (Q \cup (V(R) \setminus \{z\}) \cup \{y\}))} \\ &= \frac{|Q| + |V(R)|}{|V(R)|} = \frac{n}{|V(R)|} + 1 \leq \frac{n}{3} + 1 = \frac{n+3}{3}, \end{aligned}$$

which is a contradiction.

Case 2. $x \in V(R)$ or $y \in V(R)$. In this case, $i(G - (Q \cup (V(R))) = i(G' - V(R)) = i(G' - V(R) - e) = i(H - V(R)) = |V(R)| \geq 3$. Thus, we get

$$I(G) \leq \frac{|Q \cup V(R)|}{i(G - (Q \cup V(R)))} = \frac{|Q| + |V(R)|}{|V(R)|} = \frac{n}{|V(R)|} + 1 \leq \frac{n}{3} + 1 = \frac{n+3}{3},$$

which contradicts $I(G) > \frac{n+3}{2}$. Hence, $X \neq \emptyset$. $\square$

Let $Sun(H-X)$ denote the union of sun components of $H-X$, which consists of a isolated vertices, b K_2-components and c big sun components $S_1, S_2, \dots, S_c$. Let R_i be the factor-critical subgraph of S_i for $1 \le i \le c$, and write $Z = \bigcup_{1\le i\le c} V(R_i)$. We select one vertex from every K_2 component of $H-X$, and the set of such vertices is denoted by Y. Clearly, $|Y| = b$. Then $i(H-(X\cup Y\cup Z)) = a+b+|Z|$ and it follows from (9) and Claim 1 that

$$sun(H-X) = a+b+c \ge 2|X|+1 \ge 3. \tag{10}$$

Claim 2. $0 \le a \le 1$.

Proof. Assume that $a \ge 2$. By (10), $c \ge 0$ and $|V(R_i)| \ge 3$, we derive

$$\begin{aligned} i(G-(Q\cup X\cup Y\cup Z\cup\{x\})) &= i(G'-(X\cup Y\cup Z\cup\{x\})-e) \\ &= i(H-(X\cup Y\cup Z\cup\{x\})) \\ &\ge i(H-(X\cup Y\cup Z))-1 \\ &= a+b+|Z|-1 \ge a+b+3c-1 \\ &\ge a+b+c-1 \ge 2. \end{aligned}$$

Combining this with the definition of $I(G)$ and $I(G) > \frac{n+3}{2}$, we derive

$$\frac{n+3}{2} < I(G) \le \frac{|Q\cup X\cup Y\cup Z\cup\{x\}|}{i(G-(Q\cup X\cup Y\cup Z\cup\{x\}))} \le \frac{n+|X|+b+|Z|+1}{a+b+|Z|-1},$$

namely,

$$0 > \frac{n+1}{2}(a+b+|Z|)+a-|X|-\frac{3n+5}{2}. \tag{11}$$

It follows from (10), (11), $a \ge 2$, $c \ge 0$, $|Z| = \sum_{i=1}^{c}|V(R_i)| \ge 3c$ and Claim 1 that

$$\begin{aligned} 0 &> \frac{n+1}{2}(a+b+|Z|)+a-|X|-\frac{3n+5}{2} \\ &\ge \frac{n+1}{2}(a+b+3c)+2-|X|-\frac{3n+5}{2} \\ &\ge \frac{n+1}{2}(a+b+c)-|X|-\frac{3n+1}{2} \\ &\ge \frac{n+1}{2}(2|X|+1)-|X|-\frac{3n+1}{2} \\ &= n(|X|-1) \ge 0, \end{aligned}$$

which is a contradiction. Therefore, $0 \le a \le 1$. □

We easily see that $x \notin V(aK_1)$ or $y \notin V(aK_1)$ since $0 \le a \le 1$ (by Claim 2).

Claim 3. $x \in V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)$ or $y \in V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)$.

Proof. Assume that $x, y \notin V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)$. Note that $x \notin V(aK_1)$ or $y \notin V(aK_1)$. Hence, there is at least one vertex in $\{x, y\}$ such that the vertex does not belong $V(aK_1) \cup V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)$. Without loss of generality, we let $x \notin V(aK_1) \cup V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)$. Then $x \in V(G) \setminus (Q \cup V(aK_1) \cup V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c))$. Thus, we easily deduce

$$i(G - (Q \cup X \cup Y \cup Z \cup \{x\})) \geq a + b + |Z| \geq a + b + 3c \geq 3$$

by (10), $c \geq 0$ and $|Z| = \sum_{i=1}^{c} |V(R_i)| \geq 3c$. In terms of the definition of $I(G)$, we derive

$$I(G) \leq \frac{|Q \cup X \cup Y \cup Z \cup \{x\}|}{i(G - (Q \cup X \cup Y \cup Z \cup \{x\}))} \leq \frac{n + |X| + b + |Z| + 1}{a + b + |Z|}. \tag{12}$$

It follows from (10), (12), $a \geq 0$, $c \geq 0$, $|Z| = \sum_{i=1}^{c} |V(R_i)| \geq 3c$ and $I(G) > \frac{n+3}{2}$ that

$$\begin{aligned} 0 &\geq (I(G) - 1)(a + b + |Z|) + a - n - |X| - 1 \\ &\geq (I(G) - 1)(a + b + 3c) - n - |X| - 1 \\ &\geq (I(G) - 1)(a + b + c) - n - |X| - 1 \\ &\geq (I(G) - 1)(2|X| + 1) - n - |X| - 1 \\ &= I(G)(2|X| + 1) - n - 3|X| - 2, \end{aligned}$$

which implies

$$I(G) \leq \frac{3|X| + n + 2}{2|X| + 1}. \tag{13}$$

From (13), Claim 1 and $n \geq 1$, we have

$$I(G) \leq \frac{3|X| + n + 2}{2|X| + 1} = \frac{3}{2} + \frac{n + \frac{1}{2}}{2|X| + 1} \leq \frac{3}{2} + \frac{n + \frac{1}{2}}{3} = \frac{n+3}{2} + \frac{1-n}{6} \leq \frac{n+3}{2},$$

which contradicts $I(G) > \frac{n+3}{2}$. Claim 3 is verified. □

Without loss of generality, we let $x \in V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)$ by Claim 3. Then there exists $z \in V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)$ such that $xz \in E(G)$ and there is at least one vertex of $\{x, z\}$ with degree 1 in the subgraph $(bK_2) \cup S_1 \cup \cdots \cup S_c$. Thus, we obtain

$$i(G - (Q \cup X \cup ((Y \cup Z) \setminus \{z\}) \cup \{x\})) = a + b + |Z| \geq a + b + 3c \geq 3$$

by (10), $c \geq 0$ and $|Z| = \sum_{i=1}^{c} |V(R_i)| \geq 3c$. Combining this with the definition of $I(G)$ and $I(G) > \frac{n+3}{2}$, we obtain

$$\frac{n+3}{2} < I(G) \leq \frac{|Q \cup X \cup ((Y \cup Z) \setminus \{z\}) \cup \{x\}|}{i(G - (Q \cup X \cup ((Y \cup Z) \setminus \{z\}) \cup \{x\}))} = \frac{n + |X| + b + |Z|}{a + b + |Z|},$$

that is,

$$0 > \frac{n+1}{2}(a + b + |Z|) - n - |X| + a.$$

Combining this with (10), $a \geq 0$, $c \geq 0$, $n \geq 1$, $|Z| = \sum_{i=1}^{c} |V(R_i)| \geq 3c$ and Claim 1, we derive

$$\begin{aligned} 0 &> \frac{n+1}{2}(a + b + |Z|) - n - |X| + a \geq \frac{n+1}{2}(a + b + c) - n - |X| \\ &\geq \frac{n+1}{2}(2|X| + 1) - n - |X| = n|X| + \frac{1}{2} - \frac{n}{2} \geq n + \frac{1}{2} - \frac{n}{2} = \frac{n+1}{2} \geq 1, \end{aligned}$$

which is a contradiction. This finishes the proof of Theorem 10. ■

Remark 11. Next, we elaborate that the conditions in Theorem 10 are best possible, which cannot be replaced by G being $(n+1)$-connected and $I(G) \geq \frac{n+3}{2}$.

Let $G = K_{n+1} + (2K_2)$. It is clear that G is $(n+1)$-connected and $I(G) = \frac{n+3}{2}$. Let $Q \subset V(K_{n+1}) \subseteq V(G)$ with $|Q| = n$, and e be an edge of $2K_2$. Then $G - Q - e$ is a graph isomorphic to $K_1 + (2K_1 \cup K_2)$, and it obviously has no $P_{\geq 3}$-factor. Therefore, G is not a $(P_{\geq 3}, n)$-factor critical avoidable graph.

Acknowledgements

The authors would like to thank the anonymous referees for their comments on this paper. This work is supported by Six Big Talent Peak of Jiangsu Province (Grant No. JY–022).

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Received 9 June 2020
Revised 3 September 2020
Accepted 3 September 2020