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SOME RESULTS ON PATH-FACTOR CRITICAL AVOIDABLE GRAPHS

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Abstract

A path factor is a spanning subgraph F of G such that every component of F is a path with at least two vertices. We write $P_{\geq k} = \{P_i : i \geq k\}$. Then a $P_{\geq k}$ -factor of G means a path factor in which every component admits at least k vertices, where $k \geq 2$ is an integer. A graph G is called a $P_{\geq k}$ -factor avoidable graph if for any $e \in E(G)$, G admits a $P_{\geq k}$ -factor excluding e. A graph G is called a $(P_{\geq k}, n)$ -factor critical avoidable graph if for any $Q \subseteq V(G)$ with |Q| = n, G - Q is a $P_{\geq k}$ -factor avoidable graph. Let G be an (n + 2)-connected graph. In this paper, we demonstrate that (i) G is a $(P_{\geq 2}, n)$ -factor critical avoidable graph if $tough(G) > \frac{n+2}{4}$; (ii) Gis a $(P_{\geq 2}, n)$ -factor critical avoidable graph if $U(G) > \frac{n+2}{2}$; (iii) G is a $(P_{\geq 2}, n)$ -factor critical avoidable graph if $I(G) > \frac{n+2}{3}$; (iv) G is a $(P_{\geq 3}, n)$ factor critical avoidable graph if $I(G) > \frac{n+3}{2}$. Furthermore, we claim that these conditions are sharp.

Keywords: graph, toughness, isolated toughness, $P_{\geq k}$ -factor, $(P_{\geq k}, n)$ -factor critical avoidable graph.

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1. INTRODUCTION

In this paper, we discuss only finite undirected simple graphs. Let G = (V(G), E(G)) be a graph, where V(G) denotes the vertex set of G and E(G) denotes the edge set of G. For $x \in V(G)$, the degree of x in G is denoted by $d_G(x)$. For a set $X \subseteq V(G)$, we use G[X] to denote the subgraph of G induced by X and write G - X for $G[V(G) \setminus X]$. We let i(G) and $\omega(G)$ denote the number of isolated vertices and the number of connected components of G, respectively. Let P_n and

 K_n denote the path and the complete graph of order n, respectively. The join G + H denotes the graph with vertex set $V(G) \cup V(H)$ and edge set

$$E(G+H) = E(G) \cup E(H) \cup \{xy : x \in V(G) \text{ and } y \in V(H)\}.$$

Chvátal [2] first introduced the toughness of a graph G, denoted by tough(G), namely,

$$tough(G) = \min\left\{\frac{|X|}{\omega(G-X)} : X \subseteq V(G), \, \omega(G-X) \ge 2\right\},\$$

if G is not complete; otherwise, $tough(G) = +\infty$.

Yang, Ma and Liu [13] first posed isolated toughness of a graph G, denoted by I(G), namely,

$$I(G) = \min\left\{\frac{|X|}{i(G-X)} : X \subseteq V(G), \, i(G-X) \ge 2\right\}$$

if G is not a complete graph; otherwise, $I(G) = +\infty$.

A path factor is a spanning subgraph F of G such that every component of F is a path with at least two vertices. We write $P_{\geq k} = \{P_i : i \geq k\}$. Then a $P_{\geq k}$ -factor of G means a path factor in which every component admits at least k vertices, where $k \geq 2$ is an integer. A $\{P_k\}$ -factor F of G is simply called a P_k -factor if every component of F is isomorphic to P_k .

A 1-factor of G is a spanning subgraph F of G such that $d_F(x) = 1$ holds for any $x \in V(G)$. A graph R is a factor-critical graph if for any $x \in V(R)$, $R - \{x\}$ admits a 1-factor. Let R be a factor-critical graph with $V(R) = \{x_1, x_2, \ldots, x_n\}$. n new vertices y_1, y_2, \ldots, y_n together with new edges $x_1y_1, x_2y_2, \ldots, x_ny_n$ are added to R. Then the resulting graph is said to be a sun. By Kaneko [7], K_1 and K_2 are also suns. A big sun is a sun of order at least 6. We use sun(G) to denote the number of sun components of G.

Las Vergnas [11] posed a criterion for the existence of $P_{\geq 2}$ -factors in graphs.

Theorem 1 [11]. A graph G admits a $P_{\geq 2}$ -factor if and only if $i(G-X) \leq 2|X|$ for every $X \subseteq V(G)$.

Kaneko [7] derived a characterization for the existence of $P_{\geq 3}$ -factors in graphs.

Theorem 2 [7]. A graph G admits a $P_{\geq 3}$ -factor if and only if $sun(G-X) \leq 2|X|$ for every $X \subseteq V(G)$.

A graph G is called a $P_{\geq k}$ -factor avoidable graph if for any $e \in E(G)$, G admits a $P_{\geq k}$ -factor excluding e. A graph G is called a $(P_{\geq k}, n)$ -factor critical avoidable graph if for any $Q \subseteq V(G)$ with |Q| = n, G - Q is a $P_{\geq k}$ -factor

avoidable graph. Obviously, a $(P_{\geq k}, 0)$ -factor critical avoidable graph is simply called a $P_{>k}$ -factor avoidable graph.

Kelmas [10] claimed a result on the existence of path factors in subgraphs.

Theorem 3 [10]. Let G be a 3-connected claw-free graph and $|V(G)| \equiv 1 \pmod{3}$. Then for any $x \in V(G)$ and any $e \in E(G)$, $G - \{x, e\}$ has a $\{P_3\}$ -factor, namely, $G - \{x\}$ is a $\{P_3\}$ -factor avoidable graph.

Motivated by Theorem 3, we consider a more general problem.

Problem 1. Find sufficient conditions for a graph to be a $(P_{\geq k}, n)$ -factor critical avoidable graph.

Kano, Lu and Yu [8] verified that a graph G has a $\{P_3\}$ -factor if $i(G-S) \leq \frac{2}{3}|S|$ for every $S \subset V(G)$. Zhou, Yang and Xu [22] proved that an n-connected graph G is $(P_{\geq 3}, n)$ -factor critical if its toughness $tough(G) \geq \frac{n+1}{2}$. Some other results on path factors can be found in [3, 15, 17, 18]. Lots of authors derived some toughness conditions for the existence of graph factors [4, 5, 9, 20]. Some results on the relationships between isolated toughness and graph factors are obtained by Gao, Liang and Chen [6]. For many other results on graph factors, see [1, 12, 14, 16, 19, 21, 23]. In this paper, we study $(P_{\geq k}, n)$ -factor critical avoidable graphs and get some sufficient conditions for graphs to be $(P_{\geq k}, n)$ -factor critical avoidable graphs depending on toughness and isolated toughness, which are given in Sections 2 and 3.

2. Toughness and $(P_{>k}, n)$ -Factor Critical Avoidable Graphs

In this section, we explore the relationship between toughness and $(P_{\geq k}, n)$ -factor critical avoidable graphs, and derive two toughness conditions for the existence of $(P_{\geq k}, n)$ -factor critical avoidable graphs for k = 2, 3.

Theorem 4. Let G be an (n + 2)-connected graph, where $n \ge 0$ is an integer. If its toughness tough $(G) > \frac{n+2}{4}$, then G is a $(P_{\ge 2}, n)$ -factor critical avoidable graph.

Proof. Theorem 4 obviously holds for a complete graph. Next, we assume that G is not complete. Let $Q \subset V(G)$ with |Q| = n, and G' = G - Q, and let $e \in E(G')$ and H = G' - e. Since G is (n+2)-connected, H is connected. To prove Theorem 4, it suffices to show that H admits a $P_{\geq 2}$ -factor. On the contrary, suppose that H has no $P_{\geq 2}$ -factor. Then by Theorem 1, there exists a set $X \subset V(H)$ such that

(1)
$$i(H-X) \ge 2|X|+1.$$

Since H is connected, we have $X \neq \emptyset$. Thus,

(2)
$$i(H-X) \ge 2|X| + 1 \ge 3.$$

Note that $\omega(G - (Q \cup X)) \ge \omega(G - (Q \cup X) - e) - 1$. Combining this with (2), we derive

(3)
$$\omega(G - (Q \cup X)) \ge \omega(G - (Q \cup X) - e) - 1 = \omega(H - X) - 1 \ge i(H - X) - 1 \ge 2.$$

Claim 1. $|X| \ge 2$.

Proof. Assume |X| = 1. Since H = G' - e, we easily know that $i(H-X) = i(G' - e-X) \le i(G'-X)+2$. Then by (2), we derive $i(G'-X) \ge i(H-X)-2 \ge 1$, which implies that there exists an isolated vertex u in G' - X, i.e., $d_{G'-X}(u) = 0$. Thus, we have $d_G(u) \le d_{G'}(u) + |Q| = d_{G'}(u) + n \le d_{G'-X}(u) + |X| + n = 0 + 1 + n = n + 1$, contradicting that G is (n+2)-connected. Therefore, $|X| \ge 2$.

According to (1), (2), (3), Claim 1 and the definition of tough(G), we have

$$\begin{aligned} tough(G) &\leq \frac{|Q \cup X|}{\omega(G - (Q \cup X))} \leq \frac{|Q| + |X|}{i(H - X) - 1} \\ &= \frac{n + |X|}{i(H - X) - 1} \leq \frac{n + |X|}{2|X|} = \frac{1}{2} + \frac{n}{2|X|} \leq \frac{1}{2} + \frac{n}{4} = \frac{n + 2}{4}, \end{aligned}$$

which contradicts $tough(G) > \frac{n+2}{4}$. Theorem 4 is verified.

Remark 5. Now, we claim that the result in Theorem 4 is sharp. To see this, we construct the graph $G = K_{n+2} + (3K_1 \cup K_2)$. Clearly, G is (n+2)-connected and $tough(G) = \frac{n+2}{4}$. Let $Q \subset V(K_{n+2}) \subseteq V(G)$ with |Q| = n and e be the edge of K_2 . Then G - Q - e is a graph isomorphic to $K_2 + (5K_1)$, and it obviously has no $P_{>2}$ -factor. Thus, G is not a $(P_{>2}, n)$ -factor critical avoidable graph.

Theorem 6. Let G be an (n + 2)-connected graph, where $n \ge 0$ is an integer. If its toughness tough $(G) > \frac{n+1}{2}$, then G is a $(P_{\ge 3}, n)$ -factor critical avoidable graph.

Proof. Theorem 6 obviously holds for a complete graph. In the following, we assume that G is not complete. Let $Q \subset V(G)$ with |Q| = n, and G' = G - Q, and let $e \in E(G')$ and H = G' - e. Since G is (n+2)-connected, H is connected. To prove Theorem 6, it suffices to show that H admits a $P_{\geq 3}$ -factor. On the contrary, suppose that H has no $P_{\geq 3}$ -factor. Then by Theorem 2, there exists a set $X \subset V(H)$ such that

$$(4) \qquad \qquad sun(H-X) \ge 2|X|+1.$$

Claim 1. $X \neq \emptyset$.

Proof. Assume that $X = \emptyset$. Then it follows from (4) that

(5)
$$sun(H) \ge 1.$$

Since H is connected, we have sun(H) = 1 and H itself is a sun.

Since G is (n+2)-connected, $|V(G)| \ge n+3$. Thus, $|V(H)| = |V(G)| - n \ge 3$, which implies that H is a big sun. Hence, $|V(H)| \ge 6$. Let R be the factorcritical graph of H. Then $|V(R)| \ge 3$ and there exists $w \in V(R)$ such that $\omega(G' - \{w\}) = \omega(H - \{w\}) = 2$. Thus, we have

(6)
$$\omega(G - Q - \{w\}) = \omega(G' - \{w\}) = 2.$$

In terms of (6) and the definition of tough(G), we get

$$tough(G) \leq \frac{|Q \cup \{w\}|}{\omega(G - (Q \cup \{w\}))} = \frac{n+1}{2}$$

contradicting to $tough(G) > \frac{n+1}{2}$. Hence, $X \neq \emptyset$.

By (4) and Claim 1, we gain $\omega(G - (Q \cup X)) = \omega(G' - X) \ge \omega(G' - X - e) - 1 = \omega(H - X) - 1 \ge sun(H - X) - 1 \ge 2|X| \ge 2$. Combining this with Claim 1 and the definition of tough(G), we have

$$tough(G) \le \frac{|Q \cup X|}{\omega(G - (Q \cup X))} \le \frac{n + |X|}{2|X|} = \frac{1}{2} + \frac{n}{2|X|} \le \frac{1}{2} + \frac{n}{2} = \frac{n + 1}{2},$$

this contradicts $tough(G) > \frac{n+1}{2}$. This finishes the proof of Theorem 6.

Remark 7. Now, we show that the conditions in Theorem 6 are best possible, which cannot be replaced by G being (n + 1)-connected and $tough(G) \ge \frac{n+1}{2}$.

Let $G = K_{n+1} + (2K_2)$. We easily see that G is (n + 1)-connected and $tough(G) = \frac{n+1}{2}$. Let $Q \subset V(K_{n+1}) \subseteq V(G)$ with |Q| = n, and e be an edge of $2K_2$. Then G - Q - e is a graph isomorphic to $K_1 + (2K_1 \cup K_2)$, and it obviously has no $P_{\geq 3}$ -factor, and so G is not a $(P_{\geq 3}, n)$ -factor critical avoidable graph.

3. Isolated Toughness and $(P_{\geq k}, n)$ -Factor Critical Avoidable Graphs

In this section we give two sufficient conditions using isolated toughness for a graph to be a $(P_{>k}, n)$ -factor critical avoidable graph for k = 2, 3.

Theorem 8. Let G be an (n+2)-connected graph, where $n \ge 0$ is an integer. If its isolated toughness $I(G) > \frac{n+2}{3}$, then G is a $(P_{\ge 2}, n)$ -factor critical avoidable graph.

Proof. Theorem 8 obviously holds for a complete graph. In what follows, we assume that G is not complete. Let $Q \subset V(G)$ with |Q| = n, and G' = G - Q, and let $e \in E(G')$ and H = G' - e. Since G is (n+2)-connected, H is connected. To prove Theorem 8, it suffices to show that H admits a $P_{\geq 2}$ -factor. On the contrary, suppose that H has no $P_{\geq 2}$ -factor. Then by Theorem 1, there exists a set $X \subset V(H)$ such that

(7)
$$i(H-X) \ge 2|X|+1.$$

Claim 1. $|X| \ge 2$.

Proof. If $X = \emptyset$, then by (7) and H being connected, we obtain

$$1 \le i(H) = 0,$$

which is a contradiction.

Next, we consider |X| = 1. Note that $i(H-X) = i(G'-e-X) \le i(G'-X)+2$. Combining this with (7), we derive $i(G'-X) \ge i(H-X) - 2 \ge 2|X| + 1 - 2 = 2|X| - 1 = 1$, which hints that there exists $w \in V(G') \setminus X$ with $d_{G'-X}(w) = 0$. Therefore, we admit $d_G(w) = d_{G'+Q}(w) \le d_{G'}(w) + |Q| = d_{G'}(w) + n \le d_{G'-X}(w) + |X| + n = 0 + 1 + n = n + 1$, which contradicts that G is (n + 2)-connected. Thus, we derive $|X| \ge 2$.

According to (7) and Claim 1, we get

$$(8) \quad i(G - (Q \cup X)) \ge i(G - (Q \cup X) - e) - 2 = i(H - X) - 2 \ge 2|X| - 1 \ge 3.$$

It follows from (8), Claim 1 and the definition of I(G) that

$$\begin{split} I(G) &\leq \frac{|Q \cup X|}{i(G - (Q \cup X))} \leq \frac{|Q| + |X|}{2|X| - 1} \\ &= \frac{n + |X|}{2|X| - 1} = \frac{n + \frac{1}{2}}{2|X| - 1} + \frac{|X| - \frac{1}{2}}{2(|X| - \frac{1}{2})} \\ &= \frac{n + \frac{1}{2}}{2|X| - 1} + \frac{1}{2} \leq \frac{n + \frac{1}{2}}{3} + \frac{1}{2} = \frac{n + 2}{3}, \end{split}$$

which contradicts $I(G) > \frac{n+2}{3}$. Theorem 8 is proved.

Remark 9. Now, we explain that the result in Theorem 8 is sharp. To see this, we construct the graph $G = K_{n+2} + (3K_1 \cup K_2)$. Obviously, G is (n+2)-connected and $I(G) = \frac{n+2}{3}$. Let $Q \subset V(K_{n+2}) \subseteq V(G)$ with |Q| = n, and e be the edge of K_2 . Then G - Q - e is a graph isomorphic to $K_2 + (5K_1)$, and it obviously has no $P_{\geq 2}$ -factor. Thus, G is not a $(P_{\geq 2}, n)$ -factor critical avoidable graph.

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Theorem 10. Let G be an (n+2)-connected graph, where n is a positive integer. If its isolated toughness $I(G) > \frac{n+3}{2}$, then G is a $(P_{\geq 3}, n)$ -factor critical avoidable graph.

Proof. Theorem 10 obviously holds for a complete graph. Next, we assume that G is not complete. Let $Q \subset V(G)$ with |Q| = n, and G' = G - Q, and let $e = xy \in E(G')$ and H = G' - e. Since G is (n + 2)-connected, H is connected. To prove Theorem 10, it suffices to show that H admits a $P_{\geq 3}$ -factor. On the contrary, suppose that H has no $P_{\geq 3}$ -factor. Then by Theorem 2, there exists a set $X \subset V(H)$ such that

(9)
$$sun(H-X) \ge 2|X|+1.$$

Claim 1. $X \neq \emptyset$.

Proof. Assume $X = \emptyset$. Then $sun(H) \ge 1$. This implies sun(H) = 1 since H is connected.

Note that G is (n + 2)-connected. Hence, $|V(G)| \ge n + 3$. Thus, $|V(H)| = |V(G)| - n \ge (n + 3) - n = 3$, which implies that H is a big sun. Therefore, $|V(H)| \ge 6$. Let R be the factor-critical subgraph of H. Then $i(H - V(R)) = |V(R)| \ge 3$. Next, we consider two cases.

Case 1. $x, y \in V(H) \setminus V(R)$. Clearly, there exists $z \in V(R)$ with $yz \in E(G)$. Thus, we easily see

$$\begin{split} i(G - (Q \cup (V(R) \setminus \{z\}) \cup \{y\})) &= i(G' - ((V(R) \setminus \{z\}) \cup \{y\})) \\ &= i(G' - ((V(R) \setminus \{z\}) \cup \{y\}) - e) \\ &= i(H - ((V(R) \setminus \{z\}) \cup \{y\})) \\ &= |V(R)| \ge 3. \end{split}$$

Combining this with the definition of I(G) and $I(G) > \frac{n+3}{2}$, we admit. Clearly, there exists $z \in V(R)$ with $yz \in E(G)$. Thus, we easily get

$$\begin{aligned} \frac{n+3}{2} &< I(G) \leq \frac{|Q \cup (V(R) \setminus \{z\}) \cup \{y\}|}{i(G - (Q \cup (V(R) \setminus \{z\}) \cup \{y\}))} \\ &= \frac{|Q| + |V(R)|}{|V(R)|} = \frac{n}{|V(R)|} + 1 \leq \frac{n}{3} + 1 = \frac{n+3}{3}, \end{aligned}$$

which is a contradiction.

Case 2. $x \in V(R)$ or $y \in V(R)$. In this case, $i(G - (Q \cup (V(R))) = i(G' - V(R)) = i(G' - V(R) - e) = i(H - V(R)) = |V(R)| \ge 3$. Thus, we get

$$I(G) \le \frac{|Q \cup V(R)|}{i(G - (Q \cup V(R)))} = \frac{|Q| + |V(R)|}{|V(R)|} = \frac{n}{|V(R)|} + 1 \le \frac{n}{3} + 1 = \frac{n+3}{3},$$

which contradicts $I(G) > \frac{n+3}{2}$. Hence, $X \neq \emptyset$.

Let Sun(H-X) denote the union of sun components of H-X, which consists of a isolated vertices, $b \ K_2$ -components and c big sun components S_1, S_2, \ldots, S_c . Let R_i be the factor-critical subgraph of S_i for $1 \le i \le c$, and write $Z = \bigcup_{1 \le i \le c} V(R_i)$. We select one vertex from every K_2 component of H-X, and the set of such vertices is denoted by Y. Clearly, |Y| = b. Then $i(H - (X \cup Y \cup Z)) = a + b + |Z|$ and it follows from (9) and Claim 1 that

(10)
$$sun(H-X) = a + b + c \ge 2|X| + 1 \ge 3.$$

Claim 2. $0 \le a \le 1$.

Proof. Assume that $a \ge 2$. By (10), $c \ge 0$ and $|V(R_i)| \ge 3$, we derive

$$i(G - (Q \cup X \cup Y \cup Z \cup \{x\})) = i(G' - (X \cup Y \cup Z \cup \{x\}) - e)$$

= $i(H - (X \cup Y \cup Z \cup \{x\}))$
 $\ge i(H - (X \cup Y \cup Z)) - 1$
= $a + b + |Z| - 1 \ge a + b + 3c - 1$
 $\ge a + b + c - 1 \ge 2.$

Combining this with the definition of I(G) and $I(G) > \frac{n+3}{2}$, we derive

$$\frac{n+3}{2} < I(G) \le \frac{|Q \cup X \cup Y \cup Z \cup \{x\}|}{i(G - (Q \cup X \cup Y \cup Z \cup \{x\}))} \le \frac{n+|X|+b+|Z|+1}{a+b+|Z|-1},$$

namely,

(11)
$$0 > \frac{n+1}{2}(a+b+|Z|) + a - |X| - \frac{3n+5}{2}.$$

It follows from (10), (11), $a \ge 2$, $c \ge 0$, $|Z| = \sum_{i=1}^{c} |V(R_i)| \ge 3c$ and Claim 1 that

$$\begin{aligned} 0 &> \frac{n+1}{2}(a+b+|Z|) + a - |X| - \frac{3n+5}{2} \\ &\ge \frac{n+1}{2}(a+b+3c) + 2 - |X| - \frac{3n+5}{2} \\ &\ge \frac{n+1}{2}(a+b+c) - |X| - \frac{3n+1}{2} \\ &\ge \frac{n+1}{2}(2|X|+1) - |X| - \frac{3n+1}{2} \\ &= n(|X|-1) \ge 0, \end{aligned}$$

which is a contradiction. Therefore, $0 \le a \le 1$.

We easily see that $x \notin V(aK_1)$ or $y \notin V(aK_1)$ since $0 \le a \le 1$ (by Claim 2).

Claim 3. $x \in V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)$ or $y \in V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)$.

Proof. Assume that $x, y \notin V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)$. Note that $x \notin V(aK_1)$ or $y \notin V(aK_1)$. Hence, there is at least one vertex in $\{x, y\}$ such that the vertex does not belong $V(aK_1) \cup V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)$. Without loss of generality, we let $x \notin V(aK_1) \cup V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)$. Then $x \in$ $V(G) \setminus (Q \cup V(aK_1) \cup V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c))$. Thus, we easily deduce

$$i(G - (Q \cup X \cup Y \cup Z \cup \{x\})) \ge a + b + |Z| \ge a + b + 3c \ge 3$$

by (10), $c \ge 0$ and $|Z| = \sum_{i=1}^{c} |V(R_i)| \ge 3c$. In terms of the definition of I(G), we derive

$$(12) \qquad I(G) \le \frac{|Q \cup X \cup Y \cup Z \cup \{x\}|}{i(G - (Q \cup X \cup Y \cup Z \cup \{x\}))} \le \frac{n + |X| + b + |Z| + 1}{a + b + |Z|}$$

It follows from (10), (12), $a \ge 0$, $c \ge 0$, $|Z| = \sum_{i=1}^{c} |V(R_i)| \ge 3c$ and $I(G) > \frac{n+3}{2}$ that

$$\begin{split} 0 &\geq (I(G)-1)(a+b+|Z|)+a-n-|X|-1\\ &\geq (I(G)-1)(a+b+3c)-n-|X|-1\\ &\geq (I(G)-1)(a+b+c)-n-|X|-1\\ &\geq (I(G)-1)(2|X|+1)-n-|X|-1\\ &= I(G)(2|X|+1)-n-3|X|-2, \end{split}$$

which implies

(13)
$$I(G) \le \frac{3|X| + n + 2}{2|X| + 1}$$

From (13), Claim 1 and $n \ge 1$, we have

$$I(G) \leq \frac{3|X|+n+2}{2|X|+1} = \frac{3}{2} + \frac{n+\frac{1}{2}}{2|X|+1} \leq \frac{3}{2} + \frac{n+\frac{1}{2}}{3} = \frac{n+3}{2} + \frac{1-n}{6} \leq \frac{n+3}{2},$$

which contradicts $I(G) > \frac{n+3}{2}$. Claim 3 is verified.

Without loss of generality, we let $x \in V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)$ by Claim 3. Then there exists $z \in V(bK_2) \cup V(S_1) \cup \cdots \cup V(S_c)$ such that $xz \in E(G)$ and there is at least one vertex of $\{x, z\}$ with degree 1 in the subgraph $(bK_2) \cup S_1 \cup \cdots \cup S_c$. Thus, we obtain

$$i(G - (Q \cup X \cup ((Y \cup Z) \setminus \{z\}) \cup \{x\})) = a + b + |Z| \ge a + b + 3c \ge 3$$

by (10), $c \ge 0$ and $|Z| = \sum_{i=1}^{c} |V(R_i)| \ge 3c$. Combining this with the definition of I(G) and $I(G) > \frac{n+3}{2}$, we obtain

$$\frac{n+3}{2} < I(G) \le \frac{|Q \cup X \cup ((Y \cup Z) \setminus \{z\}) \cup \{x\}|}{i(G - (Q \cup X \cup ((Y \cup Z) \setminus \{z\}) \cup \{x\}))} = \frac{n + |X| + b + |Z|}{a + b + |Z|},$$

that is,

$$0 > \frac{n+1}{2}(a+b+|Z|) - n - |X| + a.$$

Combining this with (10), $a \ge 0$, $c \ge 0$, $n \ge 1$, $|Z| = \sum_{i=1}^{c} |V(R_i)| \ge 3c$ and Claim 1, we derive

$$\begin{split} 0 \, &>\, \frac{n+1}{2}(a+b+|Z|)-n-|X|+a \geq \frac{n+1}{2}(a+b+c)-n-|X|\\ &\geq\, \frac{n+1}{2}(2|X|+1)-n-|X|=n|X|+\frac{1}{2}-\frac{n}{2}\geq n+\frac{1}{2}-\frac{n}{2}=\frac{n+1}{2}\geq 1, \end{split}$$

which is a contradiction. This finishes the proof of Theorem 10.

Remark 11. Next, we elaborate that the conditions in Theorem 10 are best possible, which cannot be replaced by G being (n+1)-connected and $I(G) \ge \frac{n+3}{2}$. Let $G = K_{n+1} + (2K_2)$. It is clear that G is (n+1)-connected and $I(G) = \frac{n+3}{2}$.

Let $G = K_{n+1} + (2K_2)$. It is clear that G is (n+1)-connected and $I(G) = \frac{n+3}{2}$. Let $Q \subset V(K_{n+1}) \subseteq V(G)$ with |Q| = n, and e be an edge of $2K_2$. Then G - Q - e is a graph isomorphic to $K_1 + (2K_1 \cup K_2)$, and it obviously has no $P_{\geq 3}$ -factor. Therefore, G is not a $(P_{\geq 3}, n)$ -factor critical avoidable graph.

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