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ON THE SIZES OF (k,l)-EDGE-MAXIMAL r-UNIFORM HYPERGRAPHS

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Abstract

Let $H = (V, E)$ be a hypergraph, where V is a set of vertices and E is a set of non-empty subsets of V called edges. If all edges of H have the same cardinality r, then H is an r-uniform hypergraph; if E consists of all r-subsets of V, then H is a complete r-uniform hypergraph, denoted by K_n^r, where $n = |V|$. An r-uniform hypergraph $H = (V, E)$ is (k, l)-edge-maximal if every subhypergraph H' of H with $|V(H')| \geq l$ has edge-connectivity at most k, but for any edge $e \in E(K_n^r) \setminus E(H)$, $H + e$ contains at least one subhypergraph H'' with $|V(H'')| \geq l$ and edge-connectivity at least $k+1$. In this paper, we obtain the lower bounds and the upper bounds of the sizes of (k, l)-edge-maximal hypergraphs. Furthermore, we show that these bounds are best possible.

Keywords: edge-connectivity, (k, l)-edge-maximal hypergraphs, r-uniform hypergraphs.

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1. Introduction

In this paper, we consider finite simple graphs. For graph-theoretical terminologies and notation not defined here, we follow [3]. For a graph G, we use $\kappa'(G)$ to denote the *edge-connectivity* of G. The *complement* of a graph G is denoted by G^c. For $X \subseteq E(G^c)$, $G+X$ is the graph with vertex set $V(G)$ and edge set $E(G) \cup X$. We will use $G+e$ for $G+\{e\}$. The *floor* of a real number x, denoted by $\lfloor x \rfloor$, is the greatest integer not larger than x; the *ceiling* of a real number x, denoted by $\lceil x \rceil$, is the least integer greater than or equal to x. For two integers n and k, we define $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ when $k \le n$ and $\binom{n}{k} = 0$ when $k > n$.

For generalizing a prior result of Mader [7], Boesch and McHugh [2] introduced the following definitions. For integers k and l with $l > k \ge 2$, a graph G with $n = |V(G)| \ge l$ is a *(k,l)-graph* if $\kappa'(G') \le k$ for any $G' \subseteq G$ with $|V(G')| \ge l$. A (k,l)-graph G is *(k,l)-edge-maximal* if, for any $e \in E(G^c)$, $G+e$ has a subgraph G' with $|V(G')| \ge l$ and $\kappa'(G') \ge k+1$. $(k,k+1)$-edge-maximal graphs have been studied in [5, 7, 8], among others.

Theorem 1.1. *Let $k \ge 2$ be an integer, and G be a $(k,k+1)$-edge-maximal graph on $n > k+1$ vertices. Each of the following holds.*

(i) (*Mader* [7]) $|E(G)| \le (n-k)k + \binom{k}{2}$. *Furthermore, this bound is best possible.*

(ii) (*Lai* [5]) $|E(G)| \ge (n-1)k - \binom{k}{2}\left\lfloor \frac{n}{k+2} \right\rfloor$. *Furthermore, this bound is best possible.*

In [2], Boesch and McHugh extended Theorem 1(i) to (k,l)-edge-maximal graphs.

Theorem 1.2 (Boesch and McHugh [2]). *Let G be a graph of order n and let $n \ge l \ge k+1$. Let $p, q \ge 0$ be integers such that $n = p(l-1)+q$ with $0 \le q < l-1$. If G is a (k,l)-edge-maximal graph, then*

$$|E(G)| \le \begin{cases} \frac{p(l-1)(l-2)}{2} + (p-1+q)k, & l-1 > 2k \text{ and } q \le 2k, \\ \frac{p(l-1)(l-2)}{2} + pk + \frac{q(q-1)}{2}, & l-1 > 2k \text{ and } q > 2k, \\ \frac{(l-1)(l-2)}{2} + (n-l+1)k, & l-1 \le 2k. \end{cases}$$

Furthermore, these bounds are best possible.

In [6], Lai and Zhang extended Theorem 1(ii) to (k,l)-edge-maximal graphs.

Theorem 1.3 (Lai and Zhang [6]). *Let G be a graph of order n and let $n \ge l \ge k+3 \ge 5$. Let $p, q \ge 0$ be integers such that $n = p(l-1)+q$ with $0 \le q < l-1$.*

If G is a (k,l)-edge-maximal graph, then

$$|E(G)| \geq \begin{cases} \frac{(l-1)(l-2)}{2} + (n-l+1)k, & l \leq n < 2k+4, \\ (n-1)k - \lfloor \frac{n}{k+2} \rfloor \frac{(k+1)^2-3(k+1)}{2}, & l \leq 2k+4 \leq n, \\ (n-2a+1)k + a(a-1) - \lfloor \frac{n-2a}{k+2} \rfloor \frac{(k+1)^2-3(k+1)}{2}, & n \geq l = 2a \geq 2k+5, \\ (n-2b)k + b^2 - \lfloor \frac{n-2b-1}{k+2} \rfloor \frac{(k+1)^2-3(k+1)}{2}, & n \geq l = 2b+1 \geq 2k+5. \end{cases}$$

Furthermore, these bounds are best possible.

Let $H = (V, E)$ be a *hypergraph*, where V is a finite set and E is a set of non-empty subsets of V, called *edges*. An edge of cardinality 2 is just a graph edge. For a vertex $u \in V$ and an edge $e \in E$, we say u is *incident with* e or e is *incident with* u if $u \in e$. If all edges of H have the same cardinality r, then H is an *r-uniform hypergraph*; if E consists of all r-subsets of V, then H is a *complete r-uniform hypergraph*, denoted by K_n^r, where $n = |V|$. For $n < r$, the complete r-uniform hypergraph K_n^r is just the hypergraph with n vertices and no edges. The *complement* of an r-uniform hypergraph $H = (V, E)$, denoted by H^c, is the r-uniform hypergraph with vertex set V and edge set consisting of all r-subsets of V not in E. A hypergraph $H' = (V', E')$ is called a *subhypergraph* of $H = (V, E)$, denoted by $H' \subseteq H$, if $V' \subseteq V$ and $E' \subseteq E$. For $X \subseteq E(H^c)$, $H + X$ is the hypergraph with vertex set $V(H)$ and edge set $E(H) \cup X$; for $X' \subseteq E(H)$, $H - X'$ is the hypergraph with vertex set $V(H)$ and edge set $E(H) \setminus X'$. We use $H + e$ for $H + \{e\}$ and $H - e'$ for $H - \{e'\}$ when $e \in E(H^c)$ and $e' \in E(H)$. For $Y \subseteq V(H)$, we use $H[Y]$ to denote the hypergraph *induced* by Y, where $V(H[Y]) = Y$ and $E(H[Y]) = \{e \in E(H) : e \subseteq Y\}$. $H - Y$ is the hypergraph induced by $V(H) \setminus Y$.

For a hypergraph $H = (V, E)$ and two disjoint vertex subsets $X, Y \subseteq V$, let $E_H[X, Y]$ be the set of edges intersecting both X and Y and $d_H(X, Y) = |E_H[X, Y]|$. We use $E_H(X)$ and $d_H(X)$ for $E_H[X, V \setminus X]$ and $d_H(X, V \setminus X)$, respectively. If $X = \{u\}$, we use $E_H(u)$ and $d_H(u)$ for $E_H(\{u\})$ and $d_H(\{u\})$, respectively. We call $d_H(u)$ the *degree* of u in H. The *minimum degree* $\delta(H)$ of H is defined as $\min\{d_H(u) : u \in V\}$; the *maximum degree* $\Delta(H)$ of H is defined as $\max\{d_H(u) : u \in V\}$. When $\delta(H) = \Delta(H) = k$, we call H *k-regular*.

For a nonempty proper vertex subset X of a hypergraph H, we call $E_H(X)$ an *edge-cut* of H. The *edge-connectivity* $\kappa'(H)$ of a hypergraph H is $\min\{d_H(X) : \emptyset \neq X \subsetneqq V(H)\}$. By definition, $\kappa'(H) \leq \delta(H)$. We call a hypergraph H *k-edge-connected* if $\kappa'(H) \geq k$. A hypergraph is connected if it is 1-edge-connected. A maximal connected subhypergraph of H is called a *component* of H. An r-uniform hypergraph $H = (V, E)$ is *(k,l)-edge-maximal* if every subhypergraph H' of H with $|V(H')| \geq l$ has edge-connectivity at most k, but for any edge $e \in E(H^c)$, $H + e$ contains at least one subhypergraph H'' with $|V(H'')| \geq l$

and edge-connectivity at least $k+1$. If H is a (k,l)-edge-maximal r-uniform hypergraph with $n = |V(H)| < l$, then $H \cong K_n^r$. For results on the connectivity of hypergraphs, see [1, 4] for references.

In order to construct the complete r-uniform hypergraph with the maximum number of vertices and degree at most k, we introduce the parameter $t = t(k,r)$, which is determined by k and r.

Definition 1.4. For two integers k and r with $k, r \geq 2$, define $t = t(k,r)$ to be the largest integer such that $\binom{t-1}{r-1} \leq k$. That is, t is the integer satisfying $\binom{t-1}{r-1} \leq k < \binom{t}{r-1}$.

In [9], the authors determined, for given integers n, k and r, the extremal sizes of (k,t)-edge-maximal r-uniform hypergraphs on n vertices.

Theorem 1.5 (Tian, Xu, Lai and Meng [9]). *Let H be a (k,t)-edge-maximal r-uniform hypergraph such that $n \geq t$ and $k, r \geq 2$, where $n = |V(H)|$ and $t = t(k,r)$. Then each of the following holds.*

(i) $|E(H)| \leq \binom{t}{r} + (n-t)k$. *Furthermore, this bound is best possible.*

(ii) $|E(H)| \geq (n-1)k - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n}{t} \right\rfloor$. *Furthermore, this bound is best possible.*

The main goal of this research is to extend these results in [9]. For given integers n, k and r, the extremal sizes of a (k,l)-edge-maximal r-uniform hypergraph on n vertices are determined, where $l \geq t+1$. Section 2 below is devoted to the study of some properties of (k,l)-edge-maximal r-uniform hypergraphs. In section 3, we give the upper bounds of the sizes of (k,l)-edge-maximal r-uniform hypergraphs and illustrate that these bounds are best possible. We obtain the lower bounds of the sizes of (k,l)-edge-maximal r-uniform hypergraphs and show that these bounds are best possible in section 4.

2. PROPERTIES OF (k,l)-EDGE-MAXIMAL r-UNIFORM HYPERGRAPHS

Because of Theorem 1.5, we assume $l \geq t+1$ in this paper.

Lemma 2.1. *Let $H = (V,E)$ be a (k,l)-edge-maximal r-uniform hypergraph such that $n \geq l \geq t+1$ and $k, r \geq 2$, where $n = |V(H)|$ and $t = t(k,r)$. Assume X is a proper nonempty subset of $V(H)$ such that $\kappa'(H) = |E_H(X)|$. Then each of the following holds.*

(i) $E_{H^c}(X) \neq \emptyset$.

(ii) $\kappa'(H) = |E_H(X)| = k$.

Proof. Let $n_1 = |X|$ and $n_2 = |V(H) \setminus X|$. Then $n = n_1 + n_2$. Since H is (k,l)-edge-maximal, we have $\kappa'(H) \le k$.

(i) Assume $E_{H^c}(X) = \emptyset$. Then $E_H(X)$ consists of all r-subsets of $V(H)$ intersecting both X and $V(H) \setminus X$. Thus

$$|E_H(X)| = \sum_{s=1}^{r-1} \binom{n_1}{s}\binom{n_2}{r-s} = \binom{n}{r} - \binom{n_1}{r} - \binom{n_2}{r}.$$

Let $g(x) = \binom{x}{r} + \binom{n-x}{r}$. It is routine to verify that $g(x)$ is a decreasing function when $1 \le x \le n/2$. If $\min\{n_1, n_2\} \ge 2$, then by $\min\{n_1, n_2\} \le n/2$, we have

$$\begin{aligned}\kappa'(H) &= |E_H(X)| = \tbinom{n}{r} - \tbinom{n_1}{r} - \tbinom{n_2}{r} \ge \tbinom{n}{r} - \tbinom{2}{r} - \tbinom{n-2}{r} > \tbinom{n-1}{r-1} \\ &\ge \delta(H),\end{aligned} \tag{1}$$

which contradicts to $\kappa'(H) \le \delta(H)$. Now we assume $\min\{n_1, n_2\} = 1$.

Then

$$\begin{aligned}\kappa'(H) &= |E_H(X)| = \tbinom{n}{r} - \tbinom{n_1}{r} - \tbinom{n_2}{r} = \tbinom{n}{r} - \tbinom{1}{r} - \tbinom{n-1}{r} = \tbinom{n-1}{r-1} \\ &\ge \delta(H),\end{aligned}$$

which implies $\kappa'(H) = \delta(H) = \binom{n-1}{r-1}$ and so H is a complete r-uniform hypergraph. Thus $\kappa'(H) = \binom{n-1}{r-1} \ge \binom{l-1}{r-1} \ge \binom{t}{r-1} > k$, contrary to $\kappa'(H) \le k$. Therefore $E_{H^c}(X) \ne \emptyset$ holds.

(ii) By (i), we have $E_{H^c}(X) \ne \emptyset$. Pick an edge $e \in E_{H^c}(X)$. Since H is (k,l)-edge-maximal, there is a subhypergraph $H' \subseteq H + e$ such that $|V(H')| \ge l$ and $\kappa'(H') \ge k+1$. We have $e \in H'$ by H is (k,l)-edge-maximal. It follows that $(E_H(X) \cup \{e\}) \cap E(H')$ is an edge-cut of H'. Thus $|E_H(X)|+1 \ge |E_H(X) \cup \{e\}| \ge \kappa'(H') \ge k+1$, implying $\kappa'(H) = |E_H(X)| \ge k$. By $\kappa'(H) \le k$, we obtain $\kappa'(H) = |E_H(X)| = k$. ■

Lemma 2.2. *Let $H = (V,E)$ be a (k,l)-edge-maximal r-uniform hypergraph such that $n \ge l \ge t+1$ and $k, r \ge 2$, where $n = |V(H)|$ and $t = t(k,r)$. Assume X is a proper nonempty subset of $V(H)$ such that $|E_H(X)| = k$. Then each of the following holds.*

- (i) *If $|X| \le r-1$, then $H[X]$ contains no edges in $E(H)$, and each edge of $E_H(X)$ contains X as a subset.*
- (ii) *If $r \le |X| \le l-1$, then $H[X]$ is a complete r-uniform hypergraph and $|X| \ge t$.*
- (iii) *If $|X| \ge l$, then $H[X]$ is also a (k,l)-edge-maximal r-uniform hypergraph.*

Proof. (i) Since H is an r-uniform hypergraph, $H[X]$ contains no edges in $E[H]$ if $|X| \le r-1$. By Lemma 2.1 and $|E_H(X)| = k$, we obtain $k = |E_H(X)| \ge \delta(H) \ge \kappa'(H) = k$, implying each edge of $E_H(X)$ contains X as a subset.

(ii) Assume $r \le |X| \le l-1$. If $H[X]$ is not complete, then there is an edge $e \in E(H[X]^c) \subseteq E(H^c)$ and so $H+e$ has no subhypergraph H' with $|V(H')| \ge l$ and $\kappa'(H') \ge k+1$, contrary to the assumption that H is (k,l)-edge-maximal. Hence $H[X]$ must be complete.

On the contrary, assume $|X| < t$. Since $\delta(H) \ge \kappa'(H) = |E_H(X)| = k$ and $\binom{t-1}{r-1} \le k < \binom{t}{r-1}$, in order to ensure each vertex in X has degree at least k in H, we must have $|X| = t-1$ and $k = \binom{t-1}{r-1}$. Moreover, each vertex in X is incident with exact $\binom{t-2}{r-2}$ edges in $E_H(X)$, and thus $d_H(u) = k$ for each $u \in X$. By Lemma 2.1(i), there is an e intersecting both X and $V(H) \setminus X$ but $e \notin E_H(X)$. Since $|X| \ge r$, there is a vertex $w \in X$ such that w is not incident with e. Then $d_{H+e}(w) = k$. This implies w is not contained in a $(k+1)$-edge-connected subhypergraph of $H+e$. But then each vertex in $X \setminus \{w\}$ has at most degree k in $(H+e)-w$, and thus each vertex in $X \setminus \{w\}$ is not contained in a $(k+1)$-edge-connected subhypergraph of $H+e$. This implies that there is no $(k+1)$-edge-connected subhypergraph with at least l vertices in $H+e$, a contradiction. Thus we have $|X| \ge t$.

(iii) Assume $|X| \ge l$. If $H[X]$ is complete, then $\kappa'(H[X]) = \delta(H[X]) = \binom{|X|-1}{r-1} \ge \binom{l-1}{r-1} \ge \binom{t}{r-1} > k$, contrary to the definition of (k,l)-edge-maximal hypergraph. Thus $H[X]$ is not complete. For any edge $e \in E(H[X]^c) \subseteq E(H^c)$, $H+e$ has a subhypergraph H' with $|V(H')| \ge l$ and $\kappa'(H') \ge k+1$. Since $|E_H(X)| = k$, we have $E_H(X) \cap E(H') = \emptyset$. As $e \in E(H') \cap E(H[X]^c)$, we conclude that H' is a subhypergraph of $H[X]+e$, and so $H[X]$ is a (k,l)-edge-maximal r-uniform hypergraph. ■

3. The Upper Bounds of the Sizes of (k,l)-Edge-Maximal r-Uniform Hypergraphs

We first extend the definition of star-like-(k,l) graphs in [2] to hypergraphs.

Definition 3.1. Let k,l,r be integers such that $k,r \ge 2$ and $l \ge t+1$, where $t = t(k,r)$. Star-like-(k,l) r-uniform hypergraphs are defined constructively as follows. Start with a complete r-uniform hypergraph K^r_{l-1}. Call it the *nucleus*. Attach a single vertex K_1 or a complete r-uniform hypergraph K^r_i $(r \le i \le l-1)$ to this nucleus using k edges joining K_1 or K^r_i to the nucleus. Call this attached hypergraph a *satellite*. Attach an arbitrary number of such satellites to the nucleus in the same manner. We call r-uniform hypergraphs constructed in this manner *star-like-(k,l) r-uniform hypergraphs*. We use $SH(k,l,r)$ to denote the collection of all star-like-(k,l) r-uniform hypergraphs. See Figure 1 for example.

Definition 3.2. For integers $k,r \ge 2$, let $s = s(k,r)$ be the largest integer such that $k + \binom{s}{r} \le ks$.

Remark 3.3. Since $k + \binom{t}{r} = k + \frac{t}{r}\binom{t-1}{r-1} \le k + \frac{t}{r}k \le kt$, we have $t \le s$, where $t = t(k,r)$ and $s = s(k,r)$.

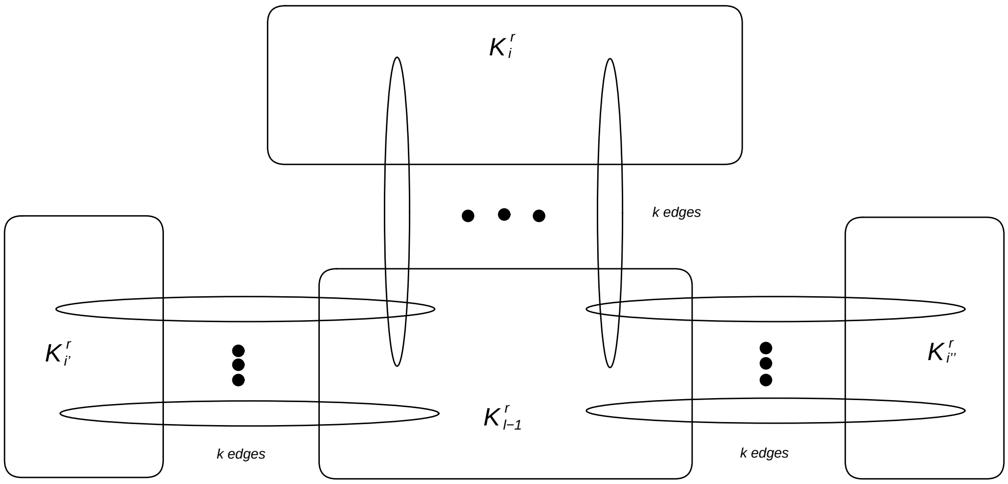

Figure 1. An example of a star-like-(k,l) r-uniform hypergraph.

Definition 3.4. Let n, k, l, r be integers such that $k, r \ge 2$ and $n \ge l \ge t+1$, where $t = t(k,r)$. Let $p, q \ge 0$ be integers such that $n = p(l-1) + q$ with $0 \le q < l-1$. We construct a class of star-like-(k,l) r-uniform hypergraphs on n vertices as follows.

(i) If $l - 1 > s$ (where $s = s(k,r)$), then a star-like-(k,l) r-uniform hypergraph consists of p copies of K_{l-1}^r, one serving as the nucleus, the rest as satellites, together with addition satellites determined as follows: (*i-a*) if $q > s$, the single additional satellite is K_q^r; (*i-b*) if $q \le s$, the additional satellites are q copies of K_1, each attached to the nucleus by k edges.

(ii) If $l - 1 \le s$ (where $s = s(k,r)$), then the nucleus is K_{l-1}^r. The satellites are $n - (l-1)$ copies of K_1, each attached to the nucleus by k edges.

We denote the collection of all star-like-(k,l) r-uniform hypergraphs on n vertices constructed in Definition 3.4 by $MSH(n;k,l,r)$. Note that all hypergraphs in $MSH(n;k,l,r)$ have the same number of edges, denoted this number by $|E(MSH(n;k,l,r))|$ for brevity. By definition, we have

$$|E(MSH(n;k,l,r))| = \begin{cases} p\binom{l-1}{r})) + pk + \binom{q}{r}, & l-1 > s \textit{ and } q > s, \\ p\binom{l-1}{r} + (p-1+q)k, & l-1 > s \textit{ and } q \le s, \\ \binom{l-1}{r} + (n-l+1)k, & l-1 \le s. \end{cases}$$

The following theorem shows that $MSH(n;k,l,r)$ is a class of star-like-(k,l) r-uniform hypergraphs with the maximum number of edges among all star-like-(k,l) r-uniform hypergraphs on n vertices.

Theorem 3.5. *Let n, k, l, r be integers such that $k, r \geq 2$ and $n \geq l \geq t+1$, where $t = t(k, r)$. Let $p, q \geq 0$ be integers such that $n = p(l-1) + q$ with $0 \leq q < l-1$. For each star-like-(k, l) r-uniform hypergraph H on n vertices, we have $|E(H)| \leq |E(MSH(n; k, l, r))|$.*

Proof. The idea of the proof is that we transform H into one of the hypergraph in $MSH(n; k, l, r)$ by appropriate addition and deletion of edges. The edge transformations applied always yield an increase (not necessary net increase) in the number of edges. The proof uses two basic techniques: *splitting* and *grouping*. The splitting operation replaces a single K_i^r $(i \geq r)$ by i K_1-satellites. The grouping operations move vertices from smaller to larger satellites (with corresponding edge additions and deletions), or cluster a set of K_1-satellites into a single large satellite. We define the *satellite spectrum* of a star-like-(k, l) hypergraph H as $(S_1, S_r, \ldots, S_{l-1})$, where S_i, $i \in \{1, r, \ldots, l-1\}$, is the number of satellites of H with i vertices. We consider two cases.

Case 1. $l - 1 > s$, where $s = s(k, r)$. If H contains a K_i^r-satellite, $r \leq i \leq s$, then perform the *splitting operation*. That is, replace K_i^r-satellite (and the k edges connecting it to the nucleus) by i K_1-satellites (together with the k edges that join each of them to the nucleus). The new hypergraph contains at least as many edges as H. The argument is as follows. The number of edges associated with the original K_i^r-satellite is $k + \binom{i}{r}$. On the other hand, the satellites introduced by the splitting operation contribute ik edges. Since $i \leq s$, we have $k + \binom{i}{r} \leq ik$ by Definition 3.2. Thus the hypergraph produced by the splitting operation contains at least as many edges as H. If we repeat this process on any remaining K_i^r-satellites, $r \leq i \leq s$, we eventually obtain a transformed hypergraph with satellite spectrum satisfying $S_i = 0$ for $r \leq i \leq s$.

If there are two satellites K_i^r and K_j^r satisfying $s < i \leq j < l-1$, we perform the following grouping operation (call *Grouping operation* 1). Since $i > s \geq t$ and $\binom{i-1}{r-1} \geq \binom{t}{r-1} > k$, we can assume that there is a vertex u in K_i^r not adjacent to the nucleus. Delete the edges connecting u to K_i^r. Add edges from u to K_j^r such that $V(K_j^r) \cup \{u\}$ induces a complete r-uniform hypergraph. The operation is edge-increasing because while it removes $\binom{i-1}{r-1}$ edges, it adds $\binom{j}{r-1}$ edges. The (K_i^r, K_j^r) pair of satellites become a (K_{i-1}^r, K_{j+1}^r) pair. If $i - 1 \leq s$, then we apply the splitting operation to K_{i-1}^r. We repeat the grouping operation until at most one satellite remains in the range $s < i < l-1$. Now the satellite spectrum of the resulting hypergraph is simple. Except $S_1 \geq 0$, $S_{l-1} \geq 0$, and some S_{i_0} $(s < i_0 < l-1)$ may be 1 or 0, all other entries must be zero.

If S_1 and S_{i_0} are positive, then we apply a second grouping operation (call *Grouping operation* 2). Select one K_1-satellite. Delete the k edges connecting the K_1-satellite to the nucleus. Add $\binom{i_0}{r-1}$ edges connecting the K_1-satellite to $K_{i_0}^r$. This transformation is edge-increasing since $\binom{i_0}{r-1} > \binom{s}{r-1} \geq \binom{t}{r-1} > k$. We repeat this operation until the supply of K_1-satellites has been exhausted or the original $K^r_{i_0}$ has been augmented to K^r_{l-1}-satellite.

If $S_1 > s$, we apply one further grouping operation (call *Grouping operation* 3). Let $S_1 = p_1(l-1) + q_1$, $0 \le q_1 < l-1$. Replace the S_1 K_1-satellites by p_1 K^r_{l-1}-satellites, and by either q_1 K_1-satellites or 1 K_{q_1}-satellite, as $q_1 \le s$ or not. By Definiton 3.2, this operation is edge increasing.

This completes the edge transformation of the original hypergraph in Case 1. For the resultant hypergraph, either $S_{i_0} = 1$ for some $s < i_0 < l-1$ and $S_1 = 0$, or all S_i for $r \le i \le l-2$ are zero, which correspond to the case (*i-a*) or the case (*i-b*) in Definition 3.4.

Case 2. $l-1 \le s$, where $s = s(k,r)$. We apply splitting operation to all satellites. By Definition 3.2, the resultant hypergraph, which correspond to the case (ii) in Definition 3.4, is edge increasing.

This completes the proof of the theorem. ∎

Theorem 3.6. *Let n,k,l,r be integers such that $k,r \ge 2$ and $n \ge l \ge t+1$, where $t = t(k,r)$. Let $p,q \ge 0$ be integers such that $n = p(l-1)+q$ with $0 \le q < l-1$. If H is a (k,l)-edge-maximal r-uniform hypergraph on n vertices, then*

$$|E(H)| \le |E(MSH(n;k,l,r))| = \begin{cases} p\binom{l-1}{r} + pk + \binom{q}{r}, & l-1 > s \text{ and } q > s, \\ p\binom{l-1}{r} + (p-1+q)k, & l-1 > s \text{ and } q \le s, \\ \binom{l-1}{r} + (n-l+1)k, & l-1 \le s, \end{cases}$$

where $s = s(k,r)$.

Proof. By Theorem 3.5, we only need to prove that there is a star-like-(k,l) r-uniform hypergraph H' on n vertices such that $|E(H)| \le |E(H')|$. The proof is by induction on n.

If $n = l$, let H' be a star-like-(k,l) r-uniform hypergraph with the nucleus K^r_{l-1} and a single K_1-satellite. Since $\kappa'(K^r_l) = \binom{l-1}{r-1}$, we need to delete at least $\binom{l-1}{r-1} - k$ edges such that the remaining hypergraph have edge-connectivity at most k. Since $\kappa'(H) = k$ (by Lemma 2.1), we have $|E(K^r_l)| - |E(H)| \ge \binom{l-1}{r-1} - k$, implying $|E(H)| \le |E(H')|$.

Now we assume $n > l$, and assume that for any (k,l)-edge-maximal r-uniform hypergraph with less than n vertices, there is a star-like-(k,l) r-uniform hypergraph having the same number of vertices and at least as many edges as the given hypergraph.

Let F be a minimum edge-cut H. By Lemma 2.1, we have $|F| = k$. We consider two cases in the following.

Case 1. There is a component, say H_1, of $H - F$ such that $|V(H_1)| = 1$. Let $H_2 = H - V(H_1)$. Then $|V(H_2)| = n-1 \ge l$. By Lemma 2.2(iii), H_2 is

(k,l)-edge-maximal. By induction assumption, there is a star-like-(k,l) r-uniform hypergraph, say H_2', such that $|V(H_2')| = |V(H_2)|$ and $|E(H_2')| \geq |E(H_2)|$. Let H' be the star-like-(k,l) r-uniform hypergraph obtained from H_2' by adding a K_1-satellite. Since $|E(H)| = k + |E(H_2)|$, we have $|E(H)| \leq |E(H')|$.

Case 2. Each component of $H-F$ has at least two vertices. Then, by Lemma 2.2, each component of $H-F$ is either a complete r-uniform hypergraph with at least t vertices, or a (k,l)-edge-maximal r-uniform hypergraph with at least l vertices.

Let H_1 be a component of $H-F$ and $H_2 = H - V(H_1)$. Assume $n_1 = |V(H_1)|$ and $n_2 = |V(H_2)|$. Then $n_1 + n_2 = n$.

Subcase 2.1. $n_1 \geq l$ and $n_2 \geq l$. By Lemma 2.2, both H_1 and H_2 are (k,l)-edge-maximal r-uniform hypergraphs. By induction assumption, there are two star-like-(k,l) r-uniform hypergraphs, say H_1' and H_2', such that $|V(H_i')| = |V(H_i)|$ and $|E(H_i')| \geq |E(H_i)|$ for $i = 1, 2$. Let H' be a star-like-(k,l) r-uniform hypergraph obtained from H_1' and H_2' by moving the satellites of H_1' to H_2' and changing the nucleus of H_1' to be a satellite of H_2'. Then $|E(H')| = |E(H_1')| + |E(H_2')| + k$, and thus $|E(H)| \leq |E(H')|$ holds.

Subcase 2.2. $n_1 \geq l$ and $n_2 < l$. By Lemma 2.2, H_1 is a (k,l)-edge-maximal r-uniform hypergraphs and H_2 is a complete r-uniform hypergraph with at least t vertices. By induction assumption, there is star-like-(k,l) r-uniform hypergraphs, say H_1', such that $|V(H_1')| = |V(H_1)|$ and $|E(H_1')| \geq |E(H_1)|$. Let H' be a star-like-(k,l) r-uniform hypergraph obtained from H_1' by adding H_2 to be a satellite of H_1. Then $|E(H')| = |E(H_1')| + |E(H_2)| + k$, and thus $|E(H)| \leq |E(H')|$ holds.

Subcase 2.3. $n_1 < l$ and $n_2 < l$. By Lemma 2.2, both H_1 and H_2 are complete r-uniform hypergraphs with at least t vertices. Since $|E(H)| = \binom{n_1}{r} + \binom{n_2}{r} + k \leq \binom{l-1}{r} + \binom{n-l+1}{r} + k$, we obtain that each star-like-(k,l) r-uniform hypergraph on n vertices having at least as many edges as H in this case. Therefore, the proof of this case follows.

This completes the proof of Theorem 3.6. ∎

In the following theorem, we will show that each hypergraph in $MSH(n;k,l,r)$ is (k,l)-edge-maximal. So the upper bounds given in Theorem 3.6 are best possible.

Theorem 3.7. *Let n,k,l,r be integers such that $k,r \geq 2$ and $n \geq l \geq t+1$, where $t = t(k,r)$. If $H \in MSH(n;k,l,r)$, then H is (k,l)-edge-maximal.*

Proof. If $l-1 = t$, then all satellites of H are K_1-satellites by $s \geq t$, where $s = s(k,r)$. By Lemma 3.1 in [9], H is (k,l)-edge-maximal. Thus, in the following, we assume $l-1 > t$.

By definition, there is no subhypergraph H' of H such that $|V(H')| \geq l$ and $\kappa'(H') > k$. We will prove the theorem by induction on n. If $n = l$, then H is a star-like-(k,l) r-uniform hypergraph with the nucleus K^r_{l-1} and a K_1-satellite. Since $\kappa'(K^r_{l-1}) = \binom{l-2}{r-1} \geq \binom{t}{r-1} > k$, for any $e \in E(H^c)$, we have $\kappa'(H+e) > k$. Thus H is (k,l)-edge-maximal.

Now suppose $n > l$. We assume that each hypergraph in $MSH(n';k,l,r)$, where $n' < n$, is (k,l)-edge-maximal. In the following, we will show that each H in $MSH(n;k,l,r)$ is also (k,l)-edge-maximal.

By contradiction, assume that there is an edge $e \in E(H^c)$ such that $H+e$ contains no subhypergraph H' satisfying $|V(H')| \geq l$ and $\kappa'(H') > k$. Let F be an edge-cut in $H+e$ with cardinality at most k. Since $\kappa'(K^r_{l-1}) = \binom{l-2}{r-1} \geq \binom{t}{r-1} > k$ and $\kappa'(K^r_q) = \binom{q-1}{r-1} \geq \binom{s}{r-1} \geq \binom{t}{r-1} > k$ when $q > s$, we obtain that F is exact the edge-cut joining some satellite and the nucleus. Thus there is a component, say H_1, of $H-F$, such that H_1 is the hypergraph obtained from H by deleting one satellite and $e \in H_1^c$. By induction assumption, H_1+e contains a subhypergraph H_1' such that $|V(H_1')| \geq l$ and $\kappa'(H_1') > k$. But H_1' is also a subhypergraph of $H+e$, a contradiction. ■

4. The Lower Bounds of the Sizes of (k,l)-Edge-Maximal r-Uniform Hypergraphs

The following lemma will be needed in proving the main result in this section.

Lemma 4.1. *Let n, a, k, r be integers such that $k, r \geq 2$ and $n \geq a \geq t$, where $t = t(k,r)$. We have the following two inequalities.*

(i) $\binom{n}{r} \geq (n-1)k - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n}{t} \right\rfloor$.

(ii) $\binom{n}{r} \geq (n-a)k + \binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n-a}{t} \right\rfloor$.

Proof. Since $|E(K^r_n)| = \binom{n}{r}$, the lemma will hold if we can construct two r-uniform hypergraphs H and H' on n vertices such that $|E(H)| = (n-1)k - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n}{t} \right\rfloor$ and $|E(H')| = (n-a)k + \binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n-a}{t} \right\rfloor$.

Let H be a r-uniform star-like hypergraph with the nucleus K^r_t, $\left\lfloor \frac{n}{t} \right\rfloor - 1$ K^r_t-satellites and $n - t\left\lfloor \frac{n}{t} \right\rfloor$ K_1-satellites, adding k edges joining each satellite to the nucleus. It is routine to count that $|E(H)| = (n-1)k - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n}{t} \right\rfloor$.

Let H' be a r-uniform star-like hypergraph with the nucleus K^r_a, $\left\lfloor \frac{n-a}{t} \right\rfloor$ K^r_t-satellites and $n - a - t\left\lfloor \frac{n-a}{t} \right\rfloor$ K_1-satellites, adding k edges joining each satellite to the nucleus. Then $|E(H')| = (n-a)k + \binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n-a}{t} \right\rfloor$. ■

It is routine to verify the following lemma.

Lemma 4.2. *For given integers n and r with $n \geq r \geq 2$, the function $g(x) = \binom{x}{r} + \binom{n-x}{r}$ is decreasing in the range $1 \leq x \leq n/2$.*

Theorem 4.3. *Let n, k, l, r be integers such that $k, r \geq 2$ and $n \geq l \geq t+1$, where $t = t(k,r)$. If H is a (k,l)-edge-maximal r-uniform hypergraph on n vertices, then*

$$|E(H)| \geq \begin{cases} \binom{l-1}{r} + (n-l+1)k, & l \leq n < 2t, \\ (n-1)k - \left((t-1)k - \binom{t}{r}\right)\lfloor \frac{n}{t} \rfloor, & l \leq 2t \leq n, \\ (n-2a+1)k + 2\binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\lfloor \frac{n-2a}{t} \rfloor, & n \geq l = 2a \geq 2t+1, \\ (n-2b)k + \binom{b}{r} + \binom{b+1}{r} - \left((t-1)k - \binom{t}{r}\right)\lfloor \frac{n-2b-1}{t} \rfloor, & n \geq l = 2b+1 \geq 2t+1. \end{cases}$$

Proof. Let F be a minimum edge-cut of H. By Lemma 2.1, $|F| = k$. Assume H_1 is a minimum component of $H - F$ and $H_2 = H - V(H_1)$. Let $n_1 = |V(H_1)|$ and $n_2 = |V(H_2)|$. Then $n = n_1 + n_2$ and $n_1 \leq n_2$.

(i) For $l \leq n < 2t$, we have $n_1 < t$, and then $n_1 = 1$ by Lemma 2.2. If $n = l$, then, by Lemma 2.2, H_2 is a complete r-uniform hypergraph on $l-1$ vertices. Thus $|E(H)| = k + \binom{l-1}{r} = \binom{l-1}{r} + (n-l+1)k$. If $n > l$, by Lemma 2.2, H_2 is (k,l)-edge-maximal. By induction on n, assume $|E(H_2)| \geq \binom{l-1}{r} + (n_2 - l + 1)k$. Therefore, $|E(H)| = |F| + |E(H_2)| \geq k + \binom{l-1}{r} + (n_2 - l+1)k = \binom{l-1}{r} + (n-l+1)k$.

(ii) We now assume $l \leq 2t \leq n$. We shall prove this case by induction on n.

If $n = 2t$, then either $n_1 = n_2 = t$, or $n_1 = 1$ and $n_2 = n-1$ by Lemma 2.2. When $n_1 = n_2 = t$, then H_1 and H_2 are complete, and thus $|E(H)| = \binom{t}{r} + \binom{t}{r} + k = (n-1)k - \left((t-1)k - \binom{t}{r}\right)\lfloor \frac{n}{t} \rfloor$. Assume $n_1 = 1$ and $n_2 = n-1$. If $n = l$, then, by Lemma 2.2, H_2 is a complete r-uniform hypergraph. By Lemma 4.2, $\binom{n-1}{r} = \binom{1}{r} + \binom{n-1}{r} \geq \binom{t}{r} + \binom{t}{r}$. Thus $|E(H)| = k + \binom{n-1}{r} \geq k + 2\binom{t}{r} = (n-1)k - \left((t-1)k - \binom{t}{r}\right)\lfloor \frac{n}{t} \rfloor$. If $n > l$, by Lemma 2.2, H_2 is (k,l)-edge-maximal. Assume, by induction hypothesis, $|E(H_2)| \geq (n_2-1)k - \left((t-1)k - \binom{t}{r}\right)\lfloor \frac{n_2}{t} \rfloor$. Therefore, $|E(H)| = |E(H_2)| + k \geq (n-1)k - \left((t-1)k - \binom{t}{r}\right)\lfloor \frac{n-1}{t} \rfloor \geq (n-1)k - \left((t-1)k - \binom{t}{r}\right)\lfloor \frac{n}{t} \rfloor$, the last inequality holds because $(t-1)k - \binom{t}{r} \geq (t-1)\binom{t-1}{r-1} - \frac{t}{r}\binom{t-1}{r-1} \geq 0$.

Assume that $n > 2t$. If $n_1 = 1$, then $n_2 = n-1 \geq 2t$. By induction assumption, $|E(H_2)| \geq (n_2-1)k - \left((t-1)k - \binom{t}{r}\right)\lfloor \frac{n_2}{t} \rfloor$. Thus $|E(H)| = |E(H_2)| + k \geq (n-1)k - \left((t-1)k - \binom{t}{r}\right)\lfloor \frac{n-1}{t} \rfloor \geq (n-1)k - \left((t-1)k - \binom{t}{r}\right)\lfloor \frac{n}{t} \rfloor$. So we assume $n_1 \geq t$.

Claim. $|E(H_i)| \geq (n_i - 1)k - \left((t-1)k - \binom{t}{r}\right)\lfloor \frac{n_i}{t} \rfloor$ *for* $i \in \{1, 2\}$.

If $|V(H_i)| \geq l$, then by induction assumption, we have $|E(H_i)| \geq (n_i - 1)k - \left((t-1)k - \binom{t}{r}\right)\lfloor \frac{n_i}{t} \rfloor$. If $t \leq |V(H_i)| \leq l-1$, then by Lemma 2.2, H_i is a complete r-uniform hypergrpah. Thus, by Lemma 4.1(i), $|E(H_i)| = \binom{n_i}{r} \geq (n_i-1)k - \left((t-1)k - \binom{t}{r}\right)\lfloor \frac{n_i}{t} \rfloor$.

By this claim, we have

$$\begin{aligned} |E(H)| &= |E(H_1)| + |E(H_2)| + k \\ &\geq (n_1-1)k - \left((t-1)k - \binom{t}{r}\right)\lfloor \frac{n_1}{t} \rfloor + (n_2-1)k - \left((t-1)k - \binom{t}{r}\right)\lfloor \frac{n_2}{t} \rfloor + k \\ &= (n-1)k - \left((t-1)k - \binom{t}{r}\right)\left(\lfloor \frac{n_1}{t} \rfloor + \lfloor \frac{n_2}{t} \rfloor\right) \end{aligned}$$

$$\geq (n-1)k - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n_1+n_2}{t} \right\rfloor \text{ (By } (t-1)k - \tbinom{t}{r} \geq 0\text{)}$$
$$= (n-1)k - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n}{t} \right\rfloor.$$

(iii) We then assume $n \geq l = 2a \geq 2t+1$. By induction on n, we will prove this case.

If $n = l$, then either $n_1 = 1, n_2 = n-1$ or $t \leq n_1 \leq n_2 \leq n-t$. When $n_1 = 1$ and $n_2 = n-1 = l-1$, then H_2 is complete by Lemma 2.2. Since $\binom{l-1}{r} = \binom{1}{r} + \binom{l-1}{r} = \binom{1}{r} + \binom{2a-1}{r} \geq \binom{a}{r} + \binom{a}{r}$ (by Lemma 4.2), we have $|E(H)| = |E(H_2)| + k = k + \binom{n-1}{r} \geq k + 2\binom{a}{r} = (n-2a+1)k + 2\binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n-2a}{t} \right\rfloor$. When $t \leq n_1 \leq n_2 \leq l-t$, by Lemma 2.2, both H_1 and H_2 are complete. Since $n_1 \leq n/2 = a$, we have $\binom{n_1}{r} + \binom{n_2}{r} \geq \binom{a}{r} + \binom{a}{r}$ by Lemma 4.2. Thus $|E(H)| = |E(H_1)| + |E(H_2)| + k = \binom{n_1}{r} + \binom{n_2}{r} + k \geq k + 2\binom{a}{r} = (n-2a+1)k + 2\binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n-2a}{t} \right\rfloor$.

Thus we assume $n > l$. If $n_1 = 1$, then $n_2 = n-1 \geq l$. By induction assumption, $|E(H_2)| \geq (n_2 - 2a + 1)k + 2\binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n_2-2a}{t} \right\rfloor$. Thus $|E(H)| = |E(H_2)| + k \geq (n_2 - 2a + 1)k + 2\binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n_2-2a}{t} \right\rfloor + k \geq (n-2a+1)k + 2\binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n-2a}{t} \right\rfloor$. So we assume $n_1 \geq t$ and consider three cases in the following.

Case 1. $n_1 \geq l$. By induction assumption, we have $|E(H_i)| \geq (n_i - 2a + 1)$ $k + 2\binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n_i-2a}{t} \right\rfloor$ for $i = 1, 2$. By setting n to be a in Lemma 4.1(i), we have $\binom{a}{r} \geq (a-1)k - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{a}{t} \right\rfloor$. Thus

$$\begin{aligned}
|E(H)| &= |E(H_1)| + |E(H_2)| + k \\
&\geq (n_1 - 2a + 1)k + 2\binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n_1-2a}{t} \right\rfloor \\
&\quad + (n_2 - 2a + 1)k + 2\binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n_2-2a}{t} \right\rfloor + k \\
&\geq (n_1 - 2a + 1)k + 2\binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n_1-2a}{t} \right\rfloor + (n_2 - 2a + 1)k \\
&\quad + 2\left((a-1)k - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{a}{t} \right\rfloor\right) - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n_2-2a}{t} \right\rfloor + k \\
&\geq (n - 2a + 1)k + 2\binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n-2a}{t} \right\rfloor.
\end{aligned}$$

Case 2. $n_1 < l$ and $n_2 \geq l$. By $n_1 < l$ and $n_2 \geq l$, we have H_1 is complete and H_1 is k-edge-maximal by Lemma 2.2. By induction assumption, $|E(H_2)| \geq (n_2 - 2a + 1)k + 2\binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n_2-2a}{t} \right\rfloor$. Setting n to be n_1 in Lemma 4.1(i), we have $\binom{n_1}{r} \geq (n_1-1)k - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n_1}{t} \right\rfloor$ by $n_1 \geq t$. Thus

$$\begin{aligned}
|E(H)| &= |E(H_1)| + |E(H_2)| + k \\
&\geq \binom{n_1}{r} + (n_2 - 2a + 1)k + 2\binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n_2-2a}{t} \right\rfloor + k \\
&\geq (n_1 - 1)k - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n_1}{t} \right\rfloor \\
&\quad + (n_2 - 2a + 1)k + 2\binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n_2-2a}{t} \right\rfloor + k \\
&\geq (n - 2a + 1)k + 2\binom{a}{r} - \left((t-1)k - \binom{t}{r}\right)\left\lfloor \frac{n-2a}{t} \right\rfloor.
\end{aligned}$$

Case 3. $n_1 \le n_2 < l$. By $n_1 \le n_2 < l$, we obtain that both H_1 and H_2 are complete by Lemma 2.2. If $n_1 \ge a$, then by setting n to be n_i in Lemma 4.1(ii), we have $\binom{n_i}{r} \ge (n_i - a)k + \binom{a}{r} - \big((t-1)k - \binom{t}{r}\big)\lfloor \frac{n_i - a}{t} \rfloor$ for $i = 1, 2$. Thus

$$\begin{aligned}|E(H)| &= |E(H_1)| + |E(H_2)| + k = \tbinom{n_1}{r} + \tbinom{n_2}{r} + k\\ &\ge (n_1 - a)k + \tbinom{a}{r} - \big((t-1)k - \tbinom{t}{r}\big)\lfloor \tfrac{n_1 - a}{t} \rfloor\\ &+ (n_2 - a)k + \tbinom{a}{r} - \big((t-1)k - \tbinom{t}{r}\big)\lfloor \tfrac{n_2 - a}{t} \rfloor + k\\ &\ge (n - 2a + 1)k + 2\tbinom{a}{r} - \big((t-1)k - \tbinom{t}{r}\big)\lfloor \tfrac{n - 2a}{t} \rfloor.\end{aligned}$$

If $n_1 \le a$, then $n_2 = n - n_1 \ge 2a - n_1 \ge t$. By setting n to be n_2 and a to be $2a - n_1$ in Lemma 4.1(ii), we have $\binom{n_2}{r} \ge (n_1 + n_2 - 2a)k + \binom{2a - n_1}{r} - ((t-1)k - \binom{t}{r}))\lfloor \frac{n_1 + n_2 - 2a}{t} \rfloor$. Together with $\binom{n_1}{r} + \binom{2a - n_1}{r} \ge 2\binom{a}{r}$ by Lemma 4.2, we have

$$\begin{aligned}|E(H)| &= |E(H_1)| + |E(H_2)| + k = \tbinom{n_1}{r} + \tbinom{n_2}{r} + k\\ &\ge \tbinom{n_1}{r} + (n_1 + n_2 - 2a)k + \tbinom{2a - n_1}{r} - \big((t-1)k - \tbinom{t}{r}\big)\lfloor \tfrac{n_1 + n_2 - 2a}{t} \rfloor + k\\ &\ge (n - 2a + 1)k + 2\tbinom{a}{r} - \big((t-1)k - \tbinom{t}{r}\big)\lfloor \tfrac{n - 2a}{t} \rfloor.\end{aligned}$$

Therefore, the proof for (iii) of Theorem 4.3 is complete.

(iv) The proof for (iv) of Theorem 4.3 is similar to that (iii) of Theorem 4.3, thus we omit the proof here. ■

Remark 4.4. For $l \le n < 2t$, the star-like-(k, l) hypergraphs in $MSH(n; k, l, r)$ show that the bound given in Theorem 4.3(i) is best possible. Hypergraphs constructed in Definition 3 [9] illustrate that the bound given in Theorem 4.3(ii) is best possible. The following example will show that bounds given in Theorem 4.3(iii) and (iv) are also best possible.

Definition 4.5. Let k, t, r be integers such that $t > r > 2$, $k = \binom{t-1}{r-1}$ and $kr \ge 2t$. Assume n and l are integers satisfying $n = l + pt$ and $l \ge 2t + 2$, where $p \ge 0$. Let $a = \lceil \frac{l}{2} \rceil$ and $b = \lfloor \frac{l}{2} \rfloor$. Then $a, b \ge t + 1$.

Let H_0 be an r-uniform hypergraph obtained from the disjoint union of K_a^r and K_b^r by adding k edges joining K_a^r and K_b^r. Let H be a star-like r-uniform hypergraph with the nucleus H_0 and p K_t^r-satellites, adding k edges from each satellite to the nucleus such that (i) each vertex in the satellite adjacent to some added edge (we can do this by $kr \ge 2t$); (ii) not all of the added k edges are incident with the same complete subhypergraph K_a^r or K_b^r.

Theorem 4.6. *Let H be an r-uniform hypergraph constructed in Definition* 4.5. *Then H is (k, l)-edge-maximal.*

Proof. By definition, there is no subhypergraph H' in H such that $|V(H')| \ge l$ and $\kappa'(H') \ge k + 1$. We will prove the theorem by induction on p. If $p = 0$, then $|V(H)| = l$. Since $\kappa'(K_a^r) = \binom{a-1}{r-1} \ge \binom{t}{r-1} > k$ and $\kappa'(K_b^r) = \binom{b-1}{r-1} \ge \binom{t}{r-1} > k$, the only edge-cut with k edges of H is these edges connecting K_a^r and K_b^r. For any $e \in E(H^c)$, we have $e \in E_{H^c}[V(K_a^r), V(K_b^r)]$. Thus every edge-cut of $H+e$ has cardinality at least $k+1$, that is, $\kappa'(H+e) \geq k+1$. Thus H is (k,l)-edge-maximal.

Now suppose $p \geq 1$. We assume that each hypergraph constructed in Definition 4.5 with less than $l+pt$ vertices is (k,l)-edge-maximal. In the following, we will show that each H in Definition 4.5 with $l+pt$ vertices is also (k,l)-edge-maximal.

On the contrary, assume that there is an edge $e \in E(H^c)$ such that $H+e$ contains no subhypergraph H' such that $|V(H')| \geq l$ and $\kappa'(H') \geq k+1$. Let F be an edge-cut in $H+e$ with cardinality at most k. By Definition 4.5(i), we have $\delta(H) \geq k+1$. By (1) in the proof of Lemma 2.1 and Definition 4.5(ii), we obtain that edge-cuts in $H+e$ with cardinality at most k are these k edges joining one satellite to the nucleus. Thus there is a component, say H_1, of $H-F$, such that H_1 is the hypergraph obtained from H by deleting one satellite and $e \in H_1^c$. By induction assumption, H_1+e contains a subhypergraph H_1' such that $|V(H_1')| \geq l$ and $\kappa'(H_1') > k$. But H_1' is also a subhypergraph of $H+e$, a contradiction. ■

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