

**A NOTE ON THE UPPER BOUNDS ON THE SIZE OF
BIPARTITE AND TRIPARTITE 1-EMBEDDABLE
GRAPHS ON SURFACES**

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Abstract

In this note, we show sharp upper bounds of the size of simple bipartite and tripartite 1-embeddable graphs on closed surfaces.

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1. INTRODUCTION

In this note, we denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. A graph G is *1-embeddable* into a closed surface F^2 if it can be drawn on F^2 so that each of its edges crosses at most one other edge at a point. We consider only *proper* drawings such that (i) all the vertices are put on different points on the surface, (ii) edges are simple arcs not containing any vertex in its interior (except its end vertices), (iii) two adjacent edges do not cross, (iv) two non-adjacent edges having an intersection always cross transversally at the point (i.e., don't touch tangentially), and (v) no more than two edges cross at a

single point. We can also regard the drawing as a continuous map $f : G \rightarrow F^2$ which may not be injective. In this note, we call the above map f a *1-embedding* of G into F^2 . We often consider that a given 1-embeddable graph G is already mapped on a closed surface F^2 , and we denote its image by G itself to simplify the notation. In this case, we say that G is a *1-embedded graph* on F^2 . In particular, when F^2 is the sphere, a 1-embeddable graph on F^2 is called *1-planar*. The notion of 1-planar graphs was first introduced by Ringel [5] and this class of graphs has been widely studied in literature (e.g., see the survey paper [2]).

An edge of a 1-embedded graph G is *crossing* if it crosses another edge, and is *non-crossing* otherwise. If an edge v_0v_2 of a 1-embedded graph G crosses an edge v_1v_3 at a crossing point z , then we say that the arc v_iz is a *half-edge* of G for each $i \in \{0, 1, 2, 3\}$. A connected component D of $F^2 - G$ whose boundary contains no crossing point is a *face* of the 1-embedded graph G ; the boundary of the face D is a set of closed walks consisting of non-crossing edges only. A *k-gonal* face is a 2-cell face bounded by a closed walk of length exactly k . A 1-embedded graph is *topologically simple* if G does not have any k -gonal face for $k \leq 2$. A connected component D of $F^2 - G$ whose boundary contains a crossing point is a *fake face*; note that a fake face is not a face of G , and vice-versa.

Let G be a simple 1-embedded graph on a closed surface F^2 except K_1 and K_2 on the sphere. Then, it was proved in [6] that $|E(G)| \leq 4|V(G)| - 4\chi(F^2)$ where $\chi(F^2)$ is the Euler characteristic of F^2 . (The simple proof was also given in [4]. The proof works for topologically simple 1-embedded graphs.) A topologically simple 1-embedded graph G is *optimal* if it satisfies the equality in the above inequality. It is known that every simple optimal 1-embedded graph is obtained from a polyhedral, i.e., 3-connected and 3-representative, quadrangulation H by adding a pair of crossing edges in each face of H (see [4]).

In this note, we discuss the upper bounds of the size of 1-embeddable multipartite graphs on closed surfaces. For bipartite 1-planar graphs, the following result is known.

Theorem 1 (Karpov [1]). *Every simple bipartite 1-planar graph with n vertices has at most $3n - 8$ edges for even $n \neq 6$ and at most $3n - 9$ for odd n and for $n = 6$.*

Note that “the planarity” (or “the orientability”) was used in the proof of Theorem 1, and hence the strategy cannot be applied at least to graphs on nonorientable closed surfaces. In the next section, we extend the above result to simple bipartite 1-embeddable graphs on nonspherical closed surfaces and show the sharp upper bounds of the size of such graphs. In Section 3, we discuss the upper bounds for simple tripartite 1-embeddable graphs on closed surfaces. Note that the results in [3] guarantee the existence of simple 4-colorable optimal 1-embedded graphs on any closed surface. This implies that for any $k \geq 4$, the

upper bounds of the size of simple k -partite 1-embeddable graphs on a closed surface F^2 cannot be less than $4n - 4\chi(F^2)$.

2. BIPARTITE CASE

Let G be a simple bipartite 1-embeddable graph on a closed surface F^2 . We add *additional edges* to the 1-embedded graph G as much as possible to create a new multigraph 1-embedded on F^2 , denoted by G' , which satisfies the followings: (a) G' is topologically simple, and (b) G' has no new crossing points other than those of G . We call the above G' an *expansion* of G . (See the center of Figure 1, which represents a graph on the torus. To obtain the torus, identify two horizontal sides and vertical sides of the square in the figure, respectively.) By the maximality, every crossing point z of G' created by two edges b_1w_1 and b_2w_2 is surrounded by a 4-cycle $b_1b_2w_1w_2$ such that $zb_1b_2, zb_2w_1, zw_1w_2$ and zw_2b_1 are fake faces; since we can draw those edges, say b_1b_2 here, almost along two half-edges zb_1 and zb_2 if it does not exist in G . Moreover, G' is connected since the boundary of any connected component of $F^2 - G$ has at least two vertices of G ; observe that crossing points are not consecutive on the boundary.

Furthermore, we remove all crossing edges of G' to obtain a multigraph G'' embedded on F^2 , which is called an *associated mosaic* of G (see the right-hand side of Figure 1). By the maximality of G' , it is easy to see that every face of G'' is either triangular or quadrangular. Note that there is one-to-one correspondence between quadrangular faces of G'' and crossing points of G (or G') by the above conditions (a) and (b).

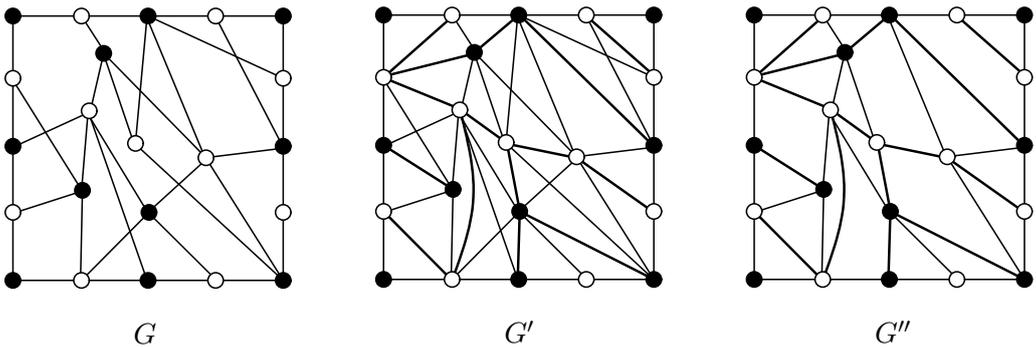


Figure 1. Expansion and associated mosaic of G .

Theorem 2. *Let G be a simple bipartite 1-embeddable graph on a nonspherical closed surface F^2 with n vertices. Then the inequality $|E(G)| \leq 3n - 3\chi(F^2)$ holds. In particular, if F^2 is the projective plane, then $|E(G)| \leq 3n - 4$.*

Proof. Let c denote the number of crossing points of the 1-embedded graph G . Further, let f_k denote the number of k -gonal faces of the associated mosaic G'' of G for $k \in \{3, 4\}$. Thus, we have $c = f_4$ by the definition. By Euler's formula, $f_3 + 2f_4 = 2n - 2\chi(F^2)$ holds, and hence we have $f_4 \leq n - \chi(F^2)$. Now, we consider a bipartite graph H embedded on F^2 which can be obtained from G by removing a crossing edge from every pair of crossing edges. Since H has no face bounded by a closed walk of odd length, we have $|E(H)| \leq 2n - 2\chi(F^2)$ by Euler's formula. Hence, it follows from $|E(G)| = |E(H)| + c$ that $|E(G)| \leq 3n - 3\chi(F^2)$.

In particular, when $f_4 = n - \chi(F^2)$ in the above argument, the expansion G' of G is optimal, that is, we have $|E(G')| = 4n - 4\chi(F^2)$ and every face of G'' is quadrangular, which contains a single crossing point of G . Here, we consider the graph \tilde{G} obtained from G'' by removing edges joining two vertices belonging to the same partite set; \tilde{G} might be disconnected. By the way to construct \tilde{G} , each face (or each connected component of $F^2 - G$) is homeomorphic to an annulus; observe that the subgraph of the dual of G'' induced by edges which cross edges of G'' joining two vertices belonging to the same partite set is 2-regular. Let F_A denote the number of such annular faces of \tilde{G} . By Euler's formula again, we obtain $n - (|\tilde{G}| + F_A) + F_A = \chi(F^2)$; since every annular face becomes a face homeomorphic to a 2-cell by adding an edge joining two vertices on different boundaries. By the above equality, the average degree of \tilde{G} is $2 - \frac{2\chi(F^2)}{n}$. This implies that if $\chi(F^2)$ is the positive integer, i.e., F^2 is either the sphere or the projective plane, then \tilde{G} has a vertex of degree 1. However, this is not the case, since \tilde{G} would have the configuration as shown in Figure 2, a contradiction; observe that every crossing edge is an edge of G , and recall that G is simple.

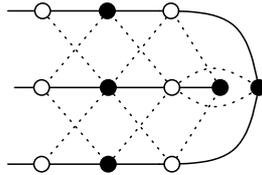


Figure 2. Vertex of degree 1 in \tilde{G} .

By the above argument, if F^2 is the projective plane, then $c \leq n - \chi(F^2) - 1$, and hence we obtain the upper bound in the theorem. ■

The upper bound in Theorem 2 is the best possible. See Figure 3, which exhibits bipartite 1-embeddings on the projective plane and on the torus, respectively. (To obtain the projective plane, identify antipodal pairs of points of the outermost circle in the left-hand side of the figure.) It is not difficult to check that each of those graphs attains the upper bounds. By inserting multilayer an-

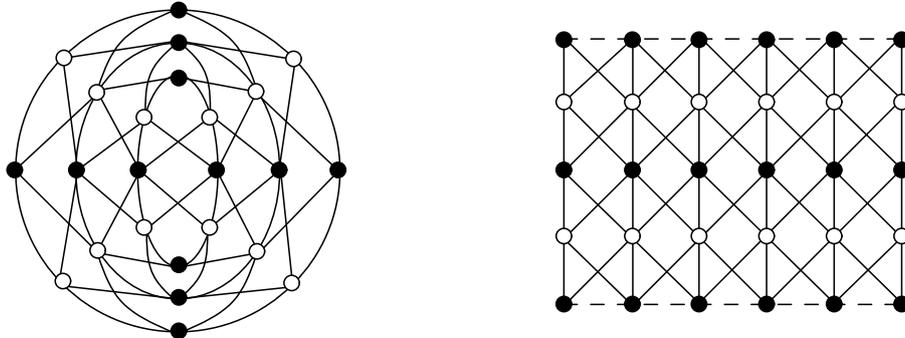


Figure 3. Bipartite 1-embeddings on the projective plane and the torus.

mular faces to \tilde{G} , we can construct our desired example with $n > n_0$ vertices for any natural number n_0 . Furthermore, for other closed surfaces, it is not difficult to divide the surface into annular faces, which is of \tilde{G} , and obtain our desired bipartite 1-embeddings as well as the toroidal case.

3. TRIPARTITE CASE

Theorem 3. *Let G be a simple tripartite 1-embeddable graph on a closed surface F^2 with n vertices. Then we have $|E(G)| \leq \frac{7}{2}n - \frac{7}{2}\chi(F^2)$.*

Proof. Let c be the number of crossing points of G . For every pair of crossing edges $\{v_0v_2, v_1v_3\}$ of G , we perform the following operation. Note that there is a pair of vertices $\{v_i, v_{i+1}\}$, say $\{v_0, v_1\}$ without loss of generality, such that v_0 and v_1 belong to the same partite set. We remove an edge v_0v_2 from G and add an edge v_0v_1 so that $v_0v_1v_3$ forms a corner of a face (see Figure 4). Now denote the resulting multigraph by H . Observe that H is probably not tripartite. If there exists a pair of multiple edges forming a 2-gonal face of H , then such edges come from left and right pairs of crossing edges of G ; note that such edges do not exist in G since each of them joins vertices in the same partite set (see Figure 4 again). Therefore, H has at most $\frac{c}{2}$ such pairs of multiple edges. If we remove an edge from every pair of multiple edges forming a 2-gonal face of H , then we obtain a topologically simple multigraph \hat{H} embedded on F^2 . Hence the number of edges of \hat{H} is at most $3n - 3\chi(F^2)$. Furthermore, since the number of faces of quadrangulations of F^2 with n vertices equals $n - \chi(F^2)$ by Euler's formula, we have $c \leq n - \chi(F^2)$; recall the argument in the proof of Theorem 2. Therefore, we obtain the following.

$$|E(G)| = |E(H)| \leq |E(\hat{H})| + \frac{c}{2} \leq 3n - 3\chi(F^2) + \frac{n - \chi(F^2)}{2} = \frac{7}{2}n - \frac{7}{2}\chi(F^2).$$

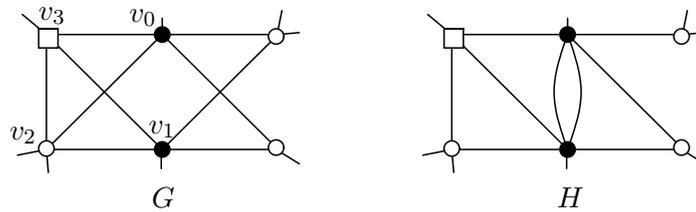


Figure 4. Operation in the proof of Theorem 3.

Therefore, the theorem follows. ■

The upper bound in the above theorem is sharp. See Figure 5. If we identify two horizontal sides and vertical sides of the square in the figure, then we obtain a tripartite 1-embedded graph on the torus attaining the upper bound of the theorem. On the other hand, if we identify antipodal pairs of points of the square, then we obtain one on the projective plane. Furthermore, taking a double cover of the above example on the projective plane, we obtain one on the sphere. For other closed surfaces, it is not difficult to construct such examples attaining the upper bounds by using the above examples; use above examples as “parts” and paste those suitably to obtain examples on a closed surface of higher genus.

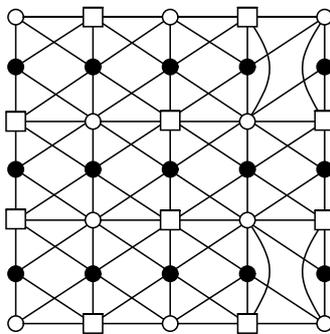


Figure 5. Tripartite 1-embedded graph on F^2 with $\frac{7}{2}n - \frac{7}{2}\chi(F^2)$ edges.

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