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EXTENDING POTOČNIK AND ŠAJNA'S CONDITIONS ON THE EXISTENCE OF VERTEX-TRANSITIVE SELF-COMPLEMENTARY k-HYPERGRAPHS

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Abstract

Let ℓ be a positive integer, $k = 2^\ell$ or $k = 2^\ell + 1$, and let n be a positive integer with $n \equiv 1 \pmod{2^{\ell+1}}$. For a prime p, $n_{(p)}$ denotes the largest integer i such that p^i divides n. Potočnik and Šajna showed that if there exists a vertex-transitive self-complementary k-hypergraph of order n, then for every prime p we have $p^{n_{(p)}} \equiv 1 \pmod{2^{\ell+1}}$. Here we extend their result to a larger class of integers k.

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1. Introduction

In 1985 Rao [6] determined a sufficient condition on the order n of a vertex-transitive self-complementary graph. Following many partial results, Muzychuk [2] showed in 1999, in an elegant proof, that Rao's sufficient condition was, indeed, also necessary.

For a prime p and a positive integer n, let $n_{(p)}$ denote the largest integer i for which p^i divides n. Using this notation, we combine the theorems of Rao and Muzychuk as follows.

Theorem 1.1 (Rao/Muzychuk)**.** *For a positive integer n, there exists a vertex-transitive self-complementary graph of order n if and only if $p^{n_{(p)}} \equiv 1 \pmod 4$ for every prime p.*

For an interesting discussion of the history of the vertex-transitive self-complementary graph problem, see [1].

For every integer $k \geq 2$, a *k-uniform hypergraph*, or *k-hypergraph*, for short, is a pair $(V; E)$ consisting of a vertex set V and edge set $E \subseteq \binom{V}{k}$, where $\binom{V}{k}$ denotes the set of all k-subsets of V. Clearly a 2-hypergraph is just a simple graph. A hypergraph H is called *vertex-transitive* if for every two vertices u, v of H there is an automorphism ϕ of H for which $u = \phi(v)$. A k-hypergraph $H = (V; E)$ is called *self-complementary* if there is a permutation σ of the set V, called a *self-complementing permutation*, such that for every k-subset e of V, $e \in E$ if and only if $\sigma\ (e) \notin E$. In other words, H is isomorphic to $\overline{H} = (V; \binom{V}{k} \setminus E)$. In 2009, Potočnik and Šajna [5] proposed studying the problem analogous to the previous theorem for k-hypergraphs. In particular, they extended Muzychuk's necessary condition to k-hypergraphs when $k = 2^\ell$ or $k = 2^\ell + 1$ for some positive integer ℓ. Shortly after, Gosselin [3] established the sufficiency of the Potočnik and Šajna result.

Theorem 1.2 (Potočnik-Šajna/Gosselin)**.** *Let m be a positive integer, $k = 2^m$ or $k = 2^m + 1$, and let n be a positive integer with $n \equiv 1 \pmod{2^{m+1}}$. Then there exists a vertex-transitive self-complementary k-hypergraph of order n if and only if for every prime p we have $p^{n_{(p)}} \equiv 1 \pmod{2^{m+1}}$.*

In Theorem 1.2, the only considered values of k are of the form $k = 2^m$ or $k = 2^m + 1$, for some positive integer m. We now consider any integer $k \geq 2$ and look at the binary expansion of k. Then there are positive integers ℓ and m such that $k = \sum_{\ell \leq i < m} k_i 2^i + 2^m$ or $k = 1 + \sum_{\ell \leq i < m} k_i 2^i + 2^m$, where $k_i \in \{0, 1\}$, for every i. In Theorem 1.2, each such $k_i = 0$. Furthermore, in Theorem 1.2, $n \equiv 1 \pmod{2^{m+1}}$. This suggests our next theorem which extends the necessary condition of Potočnik and Šajna for more values of k.

Theorem 1.3. *Let ℓ, k, n and m be positive integers such that $1 < k < n$, $1 \leq \ell \leq m$ and $n \equiv 1 \pmod{2^{m+1}}$, $k = \sum_{\ell \leq j \leq m} k_j 2^j$ or $k = \sum_{\ell \leq j \leq m} k_j 2^j + 1$, where $k_j \in \{0, 1\}$ for every j, $\ell \leq j \leq m$. If there exists a vertex-transitive self-complementary k-hypergraph of order n, then for every prime p we have $p^{n_{(p)}} \equiv 1 \pmod{2^{\ell+1}}$.*

2. Proof of Theorem 1.3

If H is a self-complementary k-hypergraph, then the set of all self-complementing permutations of H will be denoted by $C(H)$. In [7] the following characterization of self-complementing permutations for k-hypergraphs was given. Here $|c|$ denotes the order of a cycle c.

Theorem 2.1. *Let n and k be positive integers, $2 \le k \le n$. A permutation σ of $[1,n]$ with cycles $c_1, \dots, c_\lambda$ is a self-complementing permutation of a k-hypergraph of order n if and only if there is a nonnegative integer t such that the following hold.*

(i) $k = a_t 2^t + s_t$, *for some integers a_t and s_t, where a_t is odd and $0 \le s_t < 2^t$;*

(ii) $n = b_t 2^{t+1} + r_t$, *for some integers b_t and r_t, where $0 \le r_t < 2^t + s_t$; and*

(iii) $\sum_{i:|c_i|_{(2)} \le t} |c_i| = r_t$.

In [7], the condition (iii) has the form of inequality $\sum_{i:|c_i|_{(2)} \le t} |c_i| \le r_t$. However, since $r_t \equiv \sum_{i:|c_i|_{(2)} \le t} |c_i| \pmod{2^{t+1}}$ and $r_t < 2^{t+1}$, we have equality (iii).

Theorem 2.1 implies the following corollary.

Corollary 2.2. *Let ℓ, k, n and m be positive integers such that $1 < k < n$, $1 \le \ell \le m$ and $n \equiv 1 \pmod{2^{m+1}}$, $k = \sum_{\ell \le j \le m} k_j 2^j$ or $k = \sum_{\ell \le j \le m} k_j 2^j + 1$, where $k_j \in \{0,1\}$ for every j, $\ell \le j \le m$. Then every cycle of order greater than one of any self-complementing permutation of a self-complementary k-hypergraph of order n has order divisible by $2^{\ell+1}$.*

Note that any such a permutation has exactly one cycle of order one.

Proof. Let σ be a self-complementing permutation of a self-complementary k-hypergraph of order n with cycles $c_1, \dots, c_\lambda$. By Theorem 2.1 there exists a nonnegative integer t such that

1. $k = a_t 2^t + s_t$, where a_t is odd and $0 \le s_t < 2^t$,
2. $n = b_t 2^{t+1} + r_t$, $r_t \in \{0, \dots, 2^t - 1 + s_t\}$, and
3. $\sum_{|c_i|_{(2)} \le t} |c_i| = r_t$.

First observe that $t = 0$ implies $s_t = 0$, and hence $r_t = 0$ and n is even, a contradiction. Thus, $t \ge 1$. Since a_t is odd, it follows that $t \ge \ell$, and since $k < 2^{m+1}$, we have $t \le m$. Consequently, as $n \equiv 1 \pmod{2^{m+1}}$, we have that $n \equiv 1 \pmod{2^{t+1}}$ and $r_t = 1$. Thus, exactly one cycle c_i, necessarily of length 1, satisfies (3). In other words, with exception of a single fixed point, every cycle of σ has order divisible by 2^{t+1}, and hence by $2^{\ell+1}$. ∎

The proof of Theorem 1.3 uses the technique of Muzychuk [2]. The proof also depends on the first two Sylow theorems (see [4], for example). The following theorem is well-known. We give it however with proof, for completeness.

Theorem 2.3. *Let p be a prime and G a finite group. If P is a Sylow p-subgroup of its normalizer in G, then P is a Sylow p-subgroup of the group G.*

Proof. To prove this theorem, we shall use the notion of group action. If we have a group G acting on a set X, we use symbols X_{fix}, G_x, and $\mathcal{O}_x$ to denote the set of all fixed points of X, the stabilizer of a point x in G, and the orbit of x, respectively. Recall that for any point x, the Orbit-Stabilizer Theorem (see, for instance, [4] Section 8.3 Lemma 3) asserts that $|\mathcal{O}_x| = |G/G_x|$, and clearly $\mathcal{O}_x = \{x\}$ if and only if $G_x = G$.

The well-known Orbit Decomposition Theorem (see [4]) states that if a group G acts on a finite set $X \neq \emptyset$, and $x_1, \ldots, x_n \in X$ are representatives of mutually disjoint orbits with at least two elements, then

$$|X| = |X_{fix}| + \sum_{i=1}^{n} |G/G_{x_i}|.$$

Thus, the Orbit Decomposition Theorem implies that if G is a p-group, then

$$|X| \equiv |X_{fix}| \pmod p.$$

By $N_G(H)$ we denote the normalizer of a subgroup H in G; that is the largest subgroup of G in which H is normal, namely $N_G(H) = \{g \in G: gHg^{-1} = H\}$. Now we have the following fact.

Fact. *If H is a p-subgroup of G, then* $|N_G(H)/H| \equiv |G/H| \pmod p$.

To prove it, we consider the following action of H on the set G/H of right cosets: for every $a \in H$ and every coset Hb, we define $a(Hb) = Hba^{-1}$. It is straightforward to verify that we are indeed defining a group action. Clearly, for every $a \in H$, and for every $b \in G$, $Hba^{-1} = Hb$ if and only if $bab^{-1} \in H$, and hence, $(G/H)_{fix} = N_G(H)/H$. Since H is a p-group, $|G/H| - |N_G(H)/H| = |G/H| - |(G/H)_{fix}|$ is divisible by p.

If P is a Sylow p-subgroup of $N_G(P)$, then $|N_G(P)/P| \not\equiv 0 \pmod p$, and by our Fact, it follows that P is a Sylow p-subgroup of G. ∎

***Proof of Theorem* 1.3.** Suppose that $H = (V; E)$ is a self-complementary vertex-transitive k-hypergraph of order n, where k and n satisfy the conditions of our theorem. Let p be a prime; if $n_{(p)} = 0$, then the result is clear. Thus assume that $n_{(p)} > 0$. We shall find a self-complementary vertex-transitive k-subhypergraph H' of H of order $p^{n_{(p)}}$ such that the cycles of a self-complementing permutation of H' are cycles of a self-complementing permutation σ of H and the fixed point of σ is one of the vertices of H'. By Corollary 2.2, all cycles of σ have order divisible by $2^{\ell+1}$, with the exception of a single fixed point. Hence the order of H', that is $p^{n_{(p)}}$, is congruent to 1 modulo $2^{\ell+1}$, and the statement of Theorem 1.3 follows.

Let $M = \mathrm{Aut}(H)$ be the automorphism group of H. For any group K, denote the set of the Sylow p-subgroups of K by $\mathrm{Syl}_p(K)$.

Note that for every $\sigma \in C(H)$ we have $\sigma^2 \in \mathrm{Aut}(H)$. Moreover a product of a number of automorphisms and self-complementing permutations is an automorphism of H if the number of self-complementing permutations is even; otherwise, the product is a self-complementing permutation of H. The set $G = \mathrm{Aut}(H) \cup C(H)$ is a group which is generated by $\mathrm{Aut}(H) \cup \{\sigma\}$, where σ is an arbitrary element of $C(H)$.

Define $\mathcal{P}$ to be the set of p-subgroups P of M with the property that there exists a vertex v of H and $\tau \in C(H)$ such that

(1) $\tau(v) = v$;

(2) $\tau P \tau^{-1} = P$ (τ normalizes P);

(3) $P_v \in \mathrm{Syl}_p(M_v)$.

We will show that $\mathcal{P}$ is not empty and any maximal element of $\mathcal{P}$ is, in fact, a Sylow p-subgroup of M.

Since H is self-complementary, $C(H)$ is not empty. Choose any $\sigma \in C(H)$. By Corollary 2.2 there is a fixed point v of σ. Let $P \in \mathrm{Syl}_p(M_v)$.

Note that if p does not divide $|M_v|$, then P is trivial. Since P is a subgroup of M_v, then $P = P_v$, and clearly $\sigma P \sigma^{-1}$ is a subgroup of M_v isomorphic to P. By the second Sylow Theorem, there exists $g \in M_v$ such that $\sigma P \sigma^{-1} = gPg^{-1}$. Set $\tau = g^{-1}\sigma$. Then $\tau \in C(H)$, $\tau(v) = v$, $\tau P \tau^{-1} = P$, and $P_v \in \mathrm{Syl}_p(M_v)$. Hence $P \in \mathcal{P}$ and $\mathcal{P} \neq \emptyset$.

From now on we shall assume that

- $P \in \mathcal{P}$ is a maximal element of $\mathcal{P}$,
- N is the normalizer of P in M,
- Q is a Sylow p-subgroup of N containing P (Q exists by the second Sylow Theorem).

Claim. *P is a Sylow p-subgroup of M.*

Proof. To prove this claim, it suffices to show that $Q \in \mathcal{P}$, and hence $Q = P$ by the maximality of P. It will then follow that P is a Sylow p-subgroup of its own normalizer in M, and hence by Theorem 2.3, it is a Sylow p-subgroup of M.

Since $P \in \mathcal{P}$, there are $\tau \in C(H)$ and a vertex v such that $\tau(v) = v$, $\tau P \tau^{-1} = P$ and $P_v \in \mathrm{Syl}_p(M_v)$. It is straightforward to show that τ normalizes N, that is, $\tau N \tau^{-1} = N$. Thus, $\tau N = N \tau$.

Since Q is a subgroup of N and $\tau N \tau^{-1} = N$, we have that $\tau Q \tau^{-1}$ is a subgroup of N and since $|\tau Q \tau^{-1}| = |Q|$, we conclude that $\tau Q \tau^{-1}$ is a Sylow p-subgroup of N.

Recall that v is a fixed point of τ, and let $U = N(v)$, where $N(v) = \{h(v): h \in N\}$. Then we have $\tau(U) = \tau(N(v)) = (\tau N)(v) = (N\tau)(v)$, since $\tau N = N\tau$ by our previous argument. This implies that $\tau(U) = N(\tau(v)) = N(v) = U$.

By Corollary 2.2, every cycle c of the self-complementing permutation τ has length divisible by $2^{\ell+1}$, with the exception of one fixed point. Since $\tau(U) = U$, for every cycle c of the permutation τ we know that either all the vertices of c are in U or else, the set of vertices of c is disjoint with U. Therefore, U is a set of vertices of a self-complementary vertex-transitive k-hypergraph $H' = (U; E \cap \binom{U}{k})$ with self-complementing permutation τ (restricted to U) and vertex-transitive group of automorphisms containing N. Moreover, vertex v, the fixed point of τ, is in U. Hence we have

$$|U| \equiv 1 \left(\bmod 2^{\ell+1}\right).$$

Since $\tau Q \tau^{-1}$ and Q are two Sylow p-subgroups of the group N, by the second Sylow Theorem, there is $g \in N$ such that $\tau Q \tau^{-1} = gQg^{-1}$.

Hence $(g^{-1}\tau)Q(g^{-1}\tau)^{-1} = Q$.

Write $\sigma = \tau^{-1}g$. By the definition of U and since $g \in N$, we have $g(U) = U$, and hence, $\sigma(U) = U$. We have $\sigma Q \sigma^{-1} = Q$, and the restriction of $\sigma \in C(H)$ to the set U is also a self-complementing permutation of H'.

By Corollary 2.2, the permutation σ has a fixed point u, and all remaining cycles are of lengths congruent to 1 (mod $2^{\ell+1}$). Since $|U| \equiv 1$ (mod $2^{\ell+1}$) and the cycles of the restriction of σ to U are the cycles of σ, we have $u \in U$.

Since the group N is transitive on the set U, there is $h \in N$ such that $h(v) = u$. Thus the subgroups M_v and M_u are conjugate, that is,

- $M_u = hM_vh^{-1}$.

  Moreover, we also have

- $P_u = hP_vh^{-1}$.

Hence $|M_u| = |M_v|$ and $|P_u| = |P_v|$, and therefore P_u is a Sylow p-subgroup of M_u. Since $P_u \leq Q_u \leq M_u$ and Q_u is a p-subgroup of M_u, it follows that $Q_u = P_u$ and Q_u is a Sylow p-subgroup of M_u. Finally, we have $Q \in \mathcal{P}$. This completes the proof of the claim. □

Now we shall show that the orbit $P(v)$ induces a self-complementary vertex-transitive k-hypergraph of order p^r, where $r = n_{(p)}$. Note first that since $\tau P = P\tau$ and $\tau(v) = v$, we have

$$\tau(P(v)) = P(\tau(v)) = P(v)$$

and therefore the k-subhypergraph of H induced by $P(v)$ is self-complementary and vertex-transitive.

Write $|M| = p^d q$, where q and p are relatively prime. Then $|P| = p^d$ by the Claim. Since M acts transitively on V we have

$$|M_v| = \frac{|M|}{|M(v)|} = \frac{p^d q}{p^r m} = p^{d-r} s,$$

for some positive integers m and s both relatively prime with p.

Since $P_v \in \mathrm{Syl}_p(M_v)$, it follows that $|P_v| = p^{d-r}$. On the other hand, since $P \in \mathrm{Syl}_p(M)$ and $P_v \in \mathrm{Syl}_p(M_v)$ we have

$$p^{d-r} = |P_v| = \frac{|P|}{|P(v)|} = \frac{p^d}{|P(v)|}.$$

This implies $|P(v)| = p^r$. Since τ is a self-complementing permutation of H, by Corollary 2.2, the length of every cycle of τ, with exception of a single fixed point, is divisible by $2^{\ell+1}$. Since $\tau(P(v)) = P(v)$, we know that $P(v)$ is the union of orbits of τ, including the fixed point v. Hence $p^r \equiv 1 \pmod{2^{\ell+1}}$ as claimed. ■

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