

MORE TALES OF HOFFMAN: BOUNDS FOR THE VECTOR CHROMATIC NUMBER OF A GRAPH

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Abstract

Let $\chi(G)$ denote the chromatic number of a graph and $\chi_v(G)$ denote the vector chromatic number. For all graphs $\chi_v(G) \leq \chi(G)$ and for some graphs $\chi_v(G) \ll \chi(G)$. Galtman proved that Hoffman's well-known lower bound for $\chi(G)$ is in fact a lower bound for $\chi_v(G)$. We prove that two more spectral lower bounds for $\chi(G)$ are also lower bounds for $\chi_v(G)$. We then use one of these bounds to derive a new characterization of $\chi_v(G)$.

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1. INTRODUCTION

For any graph G let V denote the set of vertices where $|V| = n$, E denote the set of edges where $|E| = m$, A denote the adjacency matrix, $\chi(G)$ denote the chromatic number and $\omega(G)$ the clique number. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ denote the eigenvalues of A and let s^+ and s^- denote the sum of the squares

of the positive and negative eigenvalues of A , respectively. Let \overline{G} denote the complement of G .

Let D be the diagonal matrix of vertex degrees, and let $L = D - A$ denote the Laplacian of G and $Q = D + A$ denote the signless Laplacian of G . The eigenvalues of L are $\lambda_1 \geq \dots \geq \lambda_n = 0$ and the eigenvalues of Q are $\delta_1 \geq \dots \geq \delta_n$.

The off-diagonal entries of A , L and Q are zero or one. A weight matrix $W = [w_{ij}]$ has $w_{ij} = 0$ for $i \not\sim j$ but w_{ij} is arbitrary for $i \sim j$.

2. VECTOR CHROMATIC NUMBERS AND THETA FUNCTIONS

In 1979 Lovász [14] defined the theta function, $\vartheta(G)$, that is now named after him, in order to upper bound the Shannon capacity, $c(G)$, of a graph, and proved that $c(C_5) = \vartheta(C_5) = \sqrt{5}$. He also proved that $\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G)$. Schrijver and Szegedy subsequently defined variants of the Lovász theta function, which are denoted $\vartheta'(G)$ and $\vartheta^+(G)$ respectively, where $\vartheta'(G) \leq \vartheta(G) \leq \vartheta^+(G)$. All three theta functions can be approximated to within a fixed ϵ in polynomial time using semidefinite programming (SDP), even though computing $\omega(G)$ and $\chi(G)$ is NP-hard.

In parallel with the use of these theta functions, various vector chromatic numbers were defined. In 1998 Karger *et al.* [11] defined the vector chromatic number, $\chi_v(G)$, and the strict vector chromatic number, $\chi_{sv}(G)$, where $\chi_v(G) \leq \chi_{sv}(G) \leq \chi(G)$. There exist graphs for which $\chi_v(G) \ll \chi(G)$ [6]. Karger *et al.* [11] also proved that $\chi_{sv}(G) = \vartheta(\overline{G})$, and Godsil *et al.* [9] noted that $\chi_v(G) = \vartheta'(\overline{G})$. Finally there is what is called the rigid vector chromatic number, $\chi_{rv}(G)$, and Roberson proved (see Section 6.7 of [17]) that $\chi_{rv}(G) = \vartheta^+(\overline{G})$. So to summarise

$$(1) \quad \omega(G) \leq \chi_v(G) = \vartheta'(\overline{G}) \leq \chi_{sv}(G) = \vartheta(\overline{G}) \leq \chi_{rv}(G) = \vartheta^+(\overline{G}) \leq \chi(G).$$

In this paper we focus on lower bounds for $\chi_v(G)$ so it is only necessary to include the following definition.

Definition (Vector chromatic number $\chi_v(G)$). Given a graph $G = (V, E)$ on n vertices, and a real number $k \geq 2$, a vector k -coloring of G is an assignment of unit vectors $u_i \in \mathbb{R}^n$ to each vertex $i \in V$, such that for any two adjacent vertices i and j

$$(2) \quad \langle u_i, u_j \rangle \leq -\frac{1}{k-1}.$$

The vector chromatic number $\chi_v(G)$ is the smallest real number k for which a vector k -coloring exists. The vector k -coloring can always be assumed to be in dimension n .

3. SPECTRAL LOWER BOUNDS FOR CHROMATIC NUMBERS

Most of the known spectral lower bounds for the chromatic number can be summarised as follows:

$$(3) \quad 1 + \max \left(\frac{\mu_1}{|\mu_n|}, \frac{2m}{2m - n\delta_n}, \frac{\mu_1}{\mu_1 - \delta_1 + \lambda_1}, \frac{s^\pm}{s^\mp} \right) \leq \chi(G),$$

where reading from left to right, these bounds are due to Hoffman [10], Lima *et al.* [13], Kolotilina [12], and Ando and Lin [1]. (Note that s^\pm/s^\mp denotes s^+/s^- or s^-/s^+ .) It should be noted that Nikiforov [16] pioneered the use of non-adjacency matrix eigenvalues to bound $\chi(G)$, and his general result implies the Hoffman and Kolotilina bounds.

Note that for regular graphs the first three bounds are equal. Some of these bounds are further generalised in Elphick and Wocjan [4], which for reasons discussed in Section 5 we exclude here. Several of these bounds equal two for all bipartite graphs.

Wocjan and Elphick [5] strengthened (3) by proving that the Ando and Lin bound is a lower bound for the quantum chromatic number, $\chi_q(G)$, with arbitrary Hermitian weight matrices. Wocjan and Elphick [19] further strengthened (3) by proving that the Kolotilina and Lima *et al.* bounds are lower bounds for the vectorial chromatic number, $\chi_{vect}(G) = \lceil \vartheta^+(\overline{G}) \rceil$, again with arbitrary Hermitian weight matrices.

Galtman [7] provides eight characterizations of $\chi_v(G)$. The fifth of these is that:

$$(4) \quad \chi_v(G) = 1 + \max_W \left(\frac{\mu_1(W)}{|\mu_n(W)|} \right),$$

where W is an arbitrary non-negative¹ weight matrix. This shows that the Hoffman bound is a lower bound for the vector chromatic number, $\chi_v(G) = \vartheta'(\overline{G})$, but for non-negative weight matrices only. This bound was also independently obtained by Bilu [3].

We prove below that the bounds due to Lima *et al.* and Kolotilina are also lower bounds for $\chi_v(G)$. It is straightforward to amend our proofs to show that the Lima *et al.* and the Kolotilina bounds remain lower bounds for $\chi_v(G)$ with arbitrary non-negative weight matrices. In the case of the Lima *et al.* bound this involves replacing $2m$ in the numerator with the sum of the off-diagonal entries of the weight matrix and $2m$ in the denominator with the trace of the weight matrix. In Section 4 we use the Lima *et al.* bound to prove a new characterization of the vector chromatic number. As discussed by Galtman [7], removing the non-negativity constraint would provide lower bounds for $\chi_{sv}(G)$, which can exceed $\chi_v(G)$.

¹Non-negative means that all matrix entries are non-negative.

4. PROOF OF THE LIMA BOUND

Theorem 1. *For any graph G*

$$(5) \quad 1 + \frac{2m}{2m - n\delta_n} \leq \chi_v(G).$$

Proof. Let $u_1, \dots, u_n \in \mathbb{R}^n$ be the unit vectors on which the vector chromatic number χ_v is attained. That is $\langle u_i, u_j \rangle \leq -1/(\chi_v - 1)$ for all $ij \in E$.

Let e_1, \dots, e_n denote the standard basis of \mathbb{R}^n . Define the vector

$$(6) \quad v = \sum_{i=1}^n e_i \otimes u_i \in \mathbb{R}^n \otimes \mathbb{R}^n,$$

where \otimes denotes the tensor product.

Let q_{ij} denote the entries of the signless Laplacian Q . We have

$$(7) \quad n \cdot \delta_n = \langle v, v \rangle \cdot \delta_n$$

$$(8) \quad \leq \langle v, (Q \otimes I_n)v \rangle$$

$$(9) \quad = \sum_{i,j=1}^n q_{ij} \cdot \langle u_i, u_j \rangle$$

$$(10) \quad = \sum_{i=1}^n d_i + 2 \sum_{ij \in E} \langle u_i, u_j \rangle$$

$$(11) \quad \leq 2m - 2m \cdot \frac{1}{\chi_v - 1}.$$

This proof uses first the Rayleigh principle $\delta_n \leq \langle v, (Q \otimes I_n)v \rangle / \langle v, v \rangle$. We then use $Q = D + A$, that is, $q_{ii} = d_i$, $q_{ij} = 1$ for all $ij \in E$ and $q_{ij} = 0$ for all $ij \notin E$ and $i \neq j$. We finally use $\langle u_i, u_j \rangle \leq -1/(\chi_v - 1)$ for all $ij \in E$. ■

We also present an alternative proof of Theorem 1. This proof does not make use of the definition of the vector chromatic number in terms of certain vectors as in the definition of $\chi_v(G)$ in Section 2. Instead, we rely on the third characterization of $\chi_v(G)$ in [7, Section 3] which is as follows.

$$(12) \quad \chi_v(G) = \max_B \sum_{i,j=1}^n b_{ij},$$

where $B = (b_{ij})$ is a non-negative symmetric positive semi-definite matrix such that $\text{tr}(B) = 1$ and $b_{ij} = 0$ if i and j are distinct non-adjacent vertices. We can now reformulate the above characterization of $\chi_v(G)$ so that the Lima *et al.* bound arises as a special case.

Theorem 2. *For any graph G*

$$(13) \quad \chi_v(G) = 1 + \max_W \left(\frac{\sum_{i \neq j} w_{ij}}{\text{tr}(W) - n\lambda_{\min}(W)} \right),$$

where $W = (w_{ij})$ is a non-negative weight matrix and $\lambda_{\min}(W)$ denotes the minimum eigenvalue of W .

Proof. Let W be an arbitrary non-negative symmetric matrix. Then, the matrix

$$(14) \quad B = \frac{W - \lambda_{\min}(W)I}{\text{tr}(W) - n\lambda_{\min}(W)}$$

is positive semidefinite and $\text{tr}(B) = 1$. Substituting B into (12) yields the characterization. Setting W equal to the signless Laplacian Q yields the Lima *et al.* bound as a special case. ■

5. PROOF OF THE KOLOTILINA BOUND

We briefly recall some standard concepts and results that are needed to prove that the Kolotilina bound is a lower bound for the vector chromatic number. Let $X, Y \in \mathbb{C}^{n \times n}$ be two arbitrary Hermitian matrices with eigenvalues $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \dots \geq \beta_n$, respectively. We say that X majorizes Y , denoted by $X \succeq Y$, if

$$(15) \quad \sum_{i=1}^{\ell} \alpha_i \geq \sum_{i=1}^{\ell} \beta_i$$

for all $\ell \in \{1, \dots, n-1\}$ and

$$(16) \quad \sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i.$$

Recall that the Schur product of two matrices $M, N \in \mathbb{C}^{n \times n}$, denoted by $M \circ N$ is defined to be the matrix whose entries are the products of the corresponding entries of M and N . We say that a Hermitian matrix $M \in \mathbb{C}^{n \times n}$ is positive semidefinite if all its eigenvalues are non-negative.

The Schur product $M \circ N$ of any two positive semidefinite matrices M and N is positive semidefinite. Let $u_1, \dots, u_n \in \mathbb{C}^n$ be a collection of n arbitrary unit vectors. Their Gram matrix $\Phi = (\Phi_{ij}) \in \mathbb{C}^{n \times n}$, whose entries Φ_{ij} are the inner products $\langle u_i, u_j \rangle$, is positive semidefinite.

We say that a matrix $M \in \mathbb{R}^{n \times n}$ is non-negative if all its entries are non-negative. Similarly, we say a vector $v \in \mathbb{R}^n$ is non-negative if all its entries are

non-negative. Note that Mv is non-negative whenever M and v are non-negative. For two matrices $M, N \in \mathbb{R}^{n \times n}$, we write $M \geq N$ to indicate that $M - N$ is non-negative. For any two non-negative matrices $M, N \in \mathbb{R}^{n \times n}$ and non-negative vector $v \in \mathbb{R}^n$, $M \geq N$ implies $Mv \geq Nv$.

Let $M \in \mathbb{R}^{n \times n}$ be an arbitrary symmetric, non-negative, and irreducible matrix. Then, the eigenvector corresponding to the largest eigenvalue can be chosen to have positive entries. This follows from the proof of the Perron-Frobenius theorem for non-negative irreducible matrices [15, Chapter 8].

For our purposes, it is useful to reformulate the defining condition of a k -vector coloring as follows.

Remark 3. Note that condition (2) in the definition of a k -vector coloring of $G = (V, E)$ can be equivalently formulated as

$$(17) \quad \Phi \circ (D - A) \geq D + \frac{1}{k-1} A,$$

where D denotes the diagonal matrix of vertex degrees d_i , $A = (a_{ij})$ the adjacency matrix, and $\Phi = (\Phi_{ij})$ the Gram matrix of the n unit vectors $u_i \in \mathbb{R}^n$ of the k -vector coloring, that is, $\Phi_{ij} = \langle u_i, u_j \rangle \leq -1/(k-1)$ for all $ij \in E$.

This reformulation enables us to leverage the well-known Perron-Frobenius theorem because the entries of the matrix on the right hand side of (17) are all non-negative.

In the book [20], correlation matrices are positive semidefinite matrices with ones along the diagonal. It is easy to see that a Gram matrix (of unit vectors) is a correlation matrix and vice versa.² Besides the Perron-Frobenius theorem, the result in [20, Corollary 2.15] plays a central role in showing that the spectral bounds are also lower bounds on the vector chromatic number. We decided to include a proof of this key result.

Lemma 4. *Let $\Phi \in \mathbb{C}^{n \times n}$ be an arbitrary correlation matrix. Then, for any Hermitian matrix $X \in \mathbb{C}^{n \times n}$*

$$(18) \quad X \succeq \Phi \circ X.$$

Proof. Set $Y = \Phi \circ X$. Let

$$(19) \quad X = \sum_{j=1}^n \alpha_j P_j, \quad \text{and} \quad Y = \sum_{i=1}^n \beta_i Q_i$$

²Every positive semidefinite matrix Φ can be written as $\Phi = B^* B$ for some B [2, Exercise I.2.2]. When Φ is a correlation matrix, then the columns of B of this decomposition are the desired unit vectors whose pairwise inner products form Φ . The other direction is obvious because a Gram matrix is positive semidefinite and its diagonal entries are all one when the corresponding vectors are unit vectors.

denote the spectral decompositions of X and Y , respectively. We assume that the eigenvalues of X and Y in the above decompositions are ordered in non-increasing order, that is, $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \dots \geq \beta_n$, respectively. We also assume that the orthogonal projectors P_j and Q_i are one-dimensional. Note that $\sum_{j=1}^n P_j = \sum_{i=1}^n Q_i = I$.

For an arbitrary $i \in \{1, \dots, n\}$, we can use the spectral decompositions to write β_i as follows

$$(20) \quad \beta_i = \sum_{j=1}^n \text{Tr}(Q_i(\Phi \circ P_j)) \alpha_j.$$

For $i, j \in \{1, \dots, n\}$, define the values $p_{ij} = \text{Tr}(Q_i(\Phi \circ P_j))$ so that $\beta_i = \sum_{j=1}^n p_{ij} \alpha_j$. Note that equivalently $p_{ij} = v_i^\dagger (\Phi \circ P_j) v_i \geq 0$, where $v_i \in \mathbb{C}^n$ with $Q_i = v_i v_i^\dagger$. Therefore, these values are non-negative because the Schur product $\Phi \circ P_j$ of the two positive semidefinite matrices Φ and P_j is positive semidefinite.

We now show that the matrix $P = (p_{ij})$ is doubly stochastic, that is, all row and column sums are equal to 1. We have $\text{Tr}(\Phi \circ M) = \text{Tr}(M)$ for all matrices $M \in \mathbb{C}^{n \times n}$ and $\Phi \circ I = I$ because Φ has ones along the diagonal. These two simple observations and the properties of spectral decompositions imply that

$$(21) \quad \sum_{i=1}^n p_{ij} = \text{Tr}(\Phi \circ P_j) = \text{Tr}(P_j) = 1$$

and

$$(22) \quad \sum_{j=1}^n p_{ij} = \text{Tr}(Q_i(\Phi \circ I)) = \text{Tr}(Q_i) = 1.$$

Hence $(\beta_1, \dots, \beta_n)^T = P(\alpha_1, \dots, \alpha_n)^T$ for some doubly stochastic matrix P . The Hardy-Littlewood-Pólya theorem [2, Theorem II.1.10] now implies that the spectrum of X majorizes the spectrum of $\Phi \circ X$, that is, $(\alpha_1, \dots, \alpha_n) \succeq (\beta_1, \dots, \beta_n)$. ■

Using the above results, we establish the following theorem.

Theorem 5. *Assume that $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is irreducible, that D is a diagonal matrix with non-negative entries and that there exists a correlation matrix $\Phi \in \mathbb{R}^{n \times n}$ such that*

$$(23) \quad \Phi \circ (D - A) \geq D + \frac{1}{k-1} A,$$

which is the condition (17) in Remark 3. Then, we have

$$(24) \quad \lambda_{\max}(D - A) \geq \lambda_{\max}\left(D + \frac{1}{k-1} A\right).$$

Proof. We have the following facts:

$$(25) \quad D - A \succeq \Phi \circ (D - A),$$

$$(26) \quad \Phi \circ (D - A) \geq D + \frac{1}{k-1}A.$$

Observe that the matrix $D + 1/(k-1)A$ is symmetric, non-negative, and irreducible because A has these properties and D is a diagonal matrix with non-negative entries. As discussed at the beginning of this section, the Perron-Frobenius theorem implies that the eigenvector corresponding to the maximum eigenvalue can be chosen to have non-negative entries. Denote this eigenvector by w . Using (26) and $w \geq 0$, we obtain

$$(27) \quad \langle w, (\Phi \circ (D - A))w \rangle \geq \left\langle w, \left(D + \frac{1}{k-1}A\right)w \right\rangle = \lambda_{\max}\left(D + \frac{1}{k-1}A\right).$$

Using the Rayleigh principle, we obtain

$$(28) \quad \lambda_{\max}(\Phi \circ (D - A)) \geq \langle w, (\Phi \circ (D - A))w \rangle.$$

Finally, (25) implies $\lambda_{\max}(D - A) \geq \lambda_{\max}(\Phi \circ (D - A))$. Combining all the inequalities yields the proof. ■

Note that it is essential that the eigenvector corresponding to the maximum eigenvalue has non-negative entries. Otherwise, we cannot establish the inequality in (27). Therefore, it does not seem to be possible to generalize these proof techniques to include other eigenvalues besides the maximum eigenvalue as in [4].

Note also that (24) is the same as the result proved in [16], except that Nikiforov has A as a Hermitian matrix and D as a real diagonal matrix in order to bound the chromatic number, whereas we have A as an irreducible adjacency matrix and D as a non-negative matrix in order to bound the vector chromatic number.

We can now prove that the Kolotilina bound is a lower bound for $\chi_v(G)$.

Theorem 6. *For any³ graph G*

$$(29) \quad 1 + \frac{\mu_1}{\mu_1 - \delta_1 + \lambda_1} \leq \chi_v(G).$$

Proof. The matrix on the right hand side of (24) is equal to $D + A - \frac{k-2}{k-1}A$. It is easy to see that $\lambda_{\max}(X - Y) \geq \lambda_{\max}(X) - \lambda_{\max}(Y)$ holds for arbitrary Hermitian matrices. In particular, this inequality holds for $X = D + A$ and $Y = \frac{k-2}{k-1}A$. The Kolotilina bound for $\chi_v(G)$ therefore follows immediately when A is the adjacency matrix and D is the diagonal matrix of vertex degrees. ■

³We may assume without loss of generality that the adjacency matrix is irreducible, which is equivalent to the graph being connected. The result is true for each connected component.

The Hoffman bound for $\chi_v(G)$, proved by Galtman and Bilu, follows when D is the zero matrix.

6. EXTREMAL GRAPHS

A graph, G , is said to have a *Hoffman coloring* if $\chi(G)$ equals the Hoffman bound. We have investigated graphs with $\chi_v(G) < \chi(G)$ and $\chi_v(G)$ equal to one or more of the bounds proved in this paper. We have found no irregular graph meeting these criteria.

For regular graphs, the Kolotilina and Lima *et al.* bounds equal the Hoffman bound, and there are numerous regular graphs for which $\chi_v(G) < \chi(G)$ and $\chi_v(G)$ equals the Hoffman bound. Such graphs can be said to have a *Hoffman vector coloring*. For example the Clebsch graph has $\chi = 5$ and $\chi_v = \text{Hoffman bound} = 8/3$; and the Kneser graph $K_{p,k}$ has $\chi = p - 2k + 2$ and $\chi_v = \text{Hoffman bound} = p/k$. The orthogonality graph, $\Omega(n)$, has $\chi_v = \text{Hoffman bound} = n$ and, for large enough n , χ is exponential in n .

Godsil *et al.* [8] proved that any 1-homogeneous graph has $\chi_v = \text{Hoffman bound}$. 1-homogeneous graphs are always regular and include distance regular (and thus strongly regular) and non-bipartite edge transitive graphs; and graphs which are both vertex and edge transitive.

7. AN OPEN QUESTION

As discussed in Section 3, Ando and Lin [1] proved a conjecture due to two of the authors [18] that:

$$1 + \max\left(\frac{s^+}{s^-}, \frac{s^-}{s^+}\right) \leq \chi(G).$$

We have been unable to prove that this bound is also a lower bound for $\chi_v(G)$. We have, however, tested this question, using that $\chi_v(G) = \vartheta'(\overline{G})$ and SDP, against thousands of named graphs in the Wolfram Mathematica database and found no counter-example. We have also tested 10,000s of circulant graphs and found no counter-example.

Our code for testing this question is available in the GitHub repository [21].

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