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IMPROVED BOUNDS FOR SOME FACIALLY CONSTRAINED COLORINGS

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Abstract

A facial-parity edge-coloring of a 2-edge-connected plane graph is a facially-proper edge-coloring in which every face is incident with zero or an odd number of edges of each color. A facial-parity vertex-coloring of a 2-connected plane graph is a proper vertex-coloring in which every face is incident with zero or an odd number of vertices of each color. Czap and Jendroľ in [*Facially-constrained colorings of plane graphs: A survey*, Discrete Math. 340 (2017) 2691–2703], conjectured that 10 colors suffice in both colorings. We present an infinite family of counterexamples to both conjectures.

A facial (P_k, P_ℓ)-WORM coloring of a plane graph G is a vertex-coloring such that G contains neither rainbow facial k-path nor monochromatic facial ℓ-path. Czap, Jendroľ and Valiska in [*WORM colorings of planar graphs*, Discuss. Math. Graph Theory 37 (2017) 353–368], proved that for any integer $n \geq 12$ there exists a connected plane graph on n vertices, with maximum degree at least 6, having no facial (P_3, P_3)-WORM coloring. They also asked whether there exists a graph with maximum degree 4 having the same property. We prove that for any integer $n \geq 18$, there exists a connected plane graph, with maximum degree 4, with no facial (P_3, P_3)-WORM coloring.

Keywords: plane graph, facial coloring, facial-parity edge-coloring, facial-parity vertex-coloring, WORM coloring.

2010 Mathematics Subject Classification: 05C15, 05C10.

1. Introduction

Historically, the Four Color Problem became a great motivation for researchers to study many different types of colorings restricted to planar graphs. Among some of the recent studies are those that study colorings of plane graphs where certain constraints are given by the faces. Czap and Jendroľ [7] wrote a survey devoted to presenting many of such colorings, in which they also presented a number of open problems. In this note, we consider three types of facially constrained colorings of plane graphs.

All graphs considered are finite and planar with a fixed embedding, where V denotes the set of vertices, E denotes the set of edges and F denotes the set of faces of G. With P_t we denote a path on t vertices. Let C be a set of colors. For simplicity we take $C = \mathbb{N}$, where each positive integer denotes a single color. A *vertex-coloring* of a graph G is a function $f : V(G) \mapsto \mathbb{N}$, assigning to each vertex of G a color $c \in \mathbb{N}$. An *edge-coloring* of a graph G is a function $g : E(G) \mapsto \mathbb{N}$, assigning to each edge of G a color $c \in \mathbb{N}$. We say that a vertex-coloring (edge-coloring) is *proper* if every two adjacent vertices (edges) receive distinct colors. An edge-coloring is *facially-proper* if for every face $\alpha \in F$, any two incident edges appearing consecutively on the boundary of the face α receive distinct colors.

2. Facial-Parity Edge-Coloring

A *facial-parity edge-coloring* of a 2-edge-connected plane graph G, is a facially-proper edge-coloring of G such that every face is incident with zero or an odd number of edges of each color. We denote with $\chi'_{fp}(G)$ the minimum number k, for which there exists a facial-parity edge-coloring of G with k colors. Facial-parity edge-coloring was first studied by Czap, Jendroľ and Kardoš in [8], where they proved that 92 colors suffice to color every 2-edge-connected plane graph. This bound was later improved by Czap *et al.* [9] to 20 colors. The best known upper bound so far is 16 colors, due to Lužar and Škrekovski [15]. In [5], an example of an outerplane graph is presented, namely two cycles C_5 sharing a single vertex, which needs 10 colors. Later Czap and Jendroľ [7] proposed the following conjecture.

Conjecture 1. *If G is a 2-edge-connected plane graph, then $\chi'_{fp}(G) \leq 10$.*

The *Theta graph*, $\Theta_{i,j,k}$, is the graph consisting of two distinct vertices joined by three internally vertex-disjoint paths of lengths i, j, and k. We present the following result, which disproves Conjecture 1 and shows that the general upper bound for $\chi'_{fp}(G)$ is at least 12.

Theorem 2. *For any integer $k \geq 3$, there exists a 2-edge-connected plane graph G with $4k$ edges and $\chi'_{fp}(G) = 12$.*

Proof. Let G be a Theta graph. Fix some plane embedding of G (e.g., see Figure 1). Clearly, G is 2-edge-connected and it can be edge decomposed into three

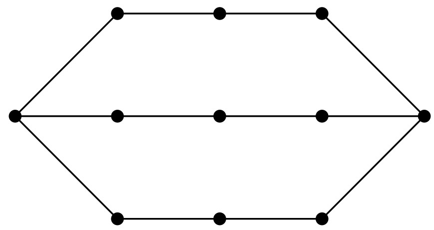

Figure 1. The graph $\Theta_{4,4,4}$ with 12 edges and $\chi'_{fp}(G) = 12$.

internally vertex-disjoint paths P_1, P_2 and P_3, where P_i and P_j, $1 \leq i < j \leq 3$, are both incident with the unique face α_{ij}. Let $f : E(G) \to \mathbb{N}$ be any facial-parity edge-coloring of G. First suppose that some color c appears an even number of times on the edges of some path P_i. Without loss of generality, we can assume that $i = 1$. Since P_1 is incident with both α_{12} and α_{13}, it follows that the color c must appear an odd number of times on the edges of both P_2 and P_3, but then it appears an even number of times on the edges incident with α_{23}, a contradiction. It follows directly that no color can appear on two distinct paths P_i and P_j at the same time. Therefore, the number of colors needed to color the edges of G is the sum of the number of colors needed to color the edges of each P_i individually. Let us consider again a single path $P \in \{P_1, P_2, P_3\}$ and let the length of P be ℓ. In the case when $\ell = 1$, it is easy to see that we need exactly 1 color to color the single edge of P. Therefore, we need to consider the following remaining four cases.

Case 1. If $\ell = 2m$ for some $m \in \mathbb{N}$, where m is odd, then we can properly color the edges of P with exactly two colors c_1 and c_2, each appearing m times on P.

Case 2. If $\ell = 2m + 1$ for some $m \in \mathbb{N}$, where m is even, then we can color the edges of P with exactly three colors c_1, c_2 and c_3, where each of the colors c_1 and c_2 appears $m - 1$ times on P and the color c_3 appears 3 times on P.

Case 3. If $\ell = 2m + 1$ for some $m \in \mathbb{N}$, where m is odd, then we can color the edges of P with exactly three colors c_1, c_2 and c_3, where each of the colors c_1 and c_2 appears m times on P and the color c_3 appears only once on P.

Case 4. If $\ell = 4m$ for some $m \in \mathbb{N}$, then we can color the edges of P with exactly four colors c_1, c_2, c_3 and c_4, where each of the colors c_1 and c_2 appears $2m - 1$ times on P and each of the colors c_3 and c_4 appears only once on P.

It follows that if each of the paths P_i has length divisible by 4, then $\chi'_{fp}(G) = 12$, thus proving the theorem. The smallest such case is depicted in Figure 1, where all three paths are of length 4 and G has 12 edges. ■

3. Facial-Parity Vertex-Coloring

A *facial-parity vertex-coloring* of a 2-connected plane graph G is a proper vertex-coloring of G such that every face is incident with zero or an odd number of vertices of each color. We denote by $\chi_{fp}(G)$ the minimum number k, for which there exists a facial-parity vertex-coloring of G with k colors. Czap, Jendroľ and Voigt [11] proved that 118 colors are sufficient to color every 2-connected plane graph G. Kaiser *et al.* [14] improved the bound to 97 colors, which is the best known bound so far. Czap [4] showed that there exists an outerplane graph which needs 10 colors. In [18] the authors proved that there exist only two 2-connected outerplane graphs with $\chi_{fp} = 10$. Motivated by that, Czap and Jendroľ [7] proposed the following.

Conjecture 3. *Every* 2*-connected plane graph admits a facial-parity vertex-coloring with at most* 10 *colors.*

With the following theorem, we prove that there exists an infinite family of 2-connected plane graphs with $\chi_{fp}(G) = 12$.

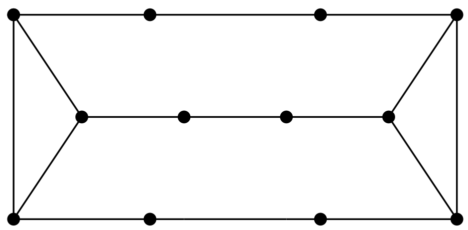

Figure 2. The line graph of the graph $\Theta_{4,4,4}$. It has 12 vertices and $\chi_{fp}(G) = 12$.

Theorem 4. *For any integer* $k \geq 3$*, there exists a* 2*-edge-connected plane graph* G *with* $4k$ *vertices and* $\chi_{fp}(G) = 12$.

Proof. First observe that the line graphs of Theta graphs $\Theta_{i,j,k}$ such that i, j, k are divisible by four are 2-connected and planar. Let H be a Theta graph $\Theta_{i,j,k}$ such that i, j, k are divisible by four and G be its line graph (Figure 2 represents a particular embedding of a line graph of the graph from Figure 1). It is clear that the number of vertices of G is at least 12 and divisible by four. Observe that any facial-parity vertex-coloring of G defines a facial-parity edge-coloring of H and vice-versa. Therefore, from the proof of Theorem 2 follows that $\chi_{fp}(G) = 12$. ∎

4. Facial (P_3, P_3)-WORM Coloring

We say that a vertex-coloring of a graph is *rainbow* (e.g., see [1, 2]), if no two vertices receive the same color. On the other hand, we say that a vertex-coloring of a graph is *monochromatic* (e.g., see [16]), if every vertex receives the same color. Given three graphs G, H and F, an (H, F)-*WORM* coloring of G is a vertex-coloring such that no subgraph of G isomorphic to H is rainbow and no subgraph of G isomorphic to F is monochromatic. The study of WORM colorings was initiated by Voloshin [17] and they have been extensively studied ever since (e.g., see [3, 7, 10]). In [3], the authors studied a special case of (H, F)-WORM coloring, namely an F-WORM coloring, where H and F are isomorphic. The idea of an F-WORM coloring was first introduced by Goddard, Wash, and Xu [12, 13].

Facially constrained WORM colorings were studied in several papers (e.g., see [6]) and in [10] the authors introduce a *facial* (P_k, P_ℓ)-*WORM coloring.* That is a vertex-coloring of a plane graph G, having no rainbow facial P_k and no monochromatic facial P_ℓ. Among others, they proved that the graph G, with $\Delta(G) = 6$, where Δ denotes the maximum degree of a graph, obtained from the graph depicted in Figure 3 by contracting the edges of the 4-cycles, incident with the outer face, has no facial (P_3, P_3)-WORM coloring. They also asked a question whether there exist plane graphs, with maximum degree 4, having no facial (P_3, P_3)-WORM coloring. We answer the question in affirmative.

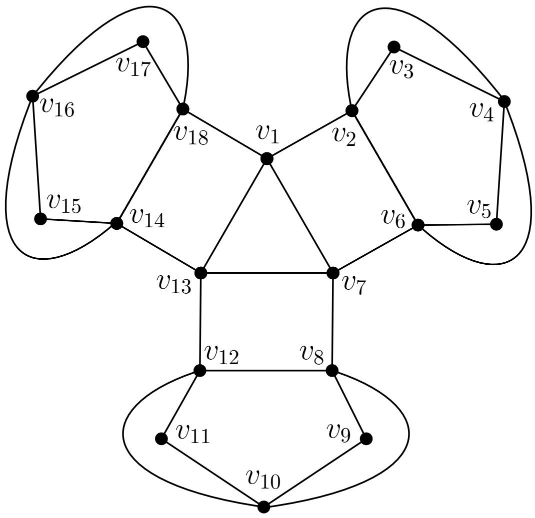

Figure 3. A graph with 18 vertices, with maximum degree 4, having no facial (P_3, P_3)-WORM coloring.

Theorem 5. *For any integer $n \geq 18$, there exists a connected plane graph G on n vertices with $\Delta(G) = 4$, having no facial (P_3, P_3)-WORM coloring.*

Proof. Let G be a planar graph on 18 vertices with its planar embedding as given in Figure 3. Suppose that G admits a facial (P_3, P_3)-WORM coloring. Note that the vertices v_1, v_7 and v_{13} form a face of size 3. Since there is no rainbow facial path P_3 and no monochromatic facial path P_3 in G, it follows that exactly two of the three vertices share a color. Without loss of generality, we can assume that v_1 and v_7 are colored with the same color c_1. Now observe that the vertices v_1, v_2, v_6 and v_7 form a face of size 4. Since the vertices v_1 and v_7 are adjacent and both colored with color c_1, and there is no monochromatic facial P_3, it follows that the vertices v_2 and v_6 are both colored with a color different from c_1. We also know that both v_2 and v_6 are colored with the same color c_2, since there is no rainbow facial P_3 in G. Now observe that none of the vertices v_3, v_4 or v_5 is colored with color c_2, otherwise we would obtain a monochromatic facial P_3. Let c_3 be the color of the vertex v_3. Suppose that the color of the vertex v_4 is different from c_3. Then we obtain a rainbow facial P_3 on the face of size 3, formed by vertices v_2, v_3 and v_4, a contradiction. Thus, v_4 must be colored with the color c_3. Consider now the vertex v_5. If v_5 is colored with the color c_3, then we obtain a monochromatic facial P_3, formed by the vertices v_3, v_4 and v_5. It follows that v_5 must receive a color different from c_2 and c_3, say c_4, but then we obtain a rainbow facial P_3, formed by the vertices v_4, v_5 and v_6, a contradiction.

If $n > 18$, then take the graph G and any connected planar graph H, with $\Delta(H) \leq 4$, having a planar embedding such that the outer face contains a vertex v of degree at most 3. Then place H in any face α of G of size 3, distinct from the face formed by the vertices v_1, v_7 and v_{12}, and add the edge from v to the only vertex of the face α, with degree less than 4. ■

5. Conclusion

Examples presented in the previous sections provide new bounds for their corresponding coloring problems. It is shown that there are 2-edge-connected plane graphs which need 12 colors in any facial-parity edge-coloring. The current upper bound however remains 16, and hence there is still a gap of 4 colors. On the other hand, for the vertex version, even though we presented an example which proves that the upper bound is at least 12, the best known upper bound remains 97.

In regards with facial WORM vertex-coloring of plane graphs, it is known that not all plane graphs have a (P_3, P_3)-WORM coloring. Czap, Jendroľ and Valiska [10] presented some results about a (P_3, P_4)-WORM coloring, yet their conjecture that every connected plane graph admits a (P_3, P_4)-WORM coloring with 2 colors remains open.

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