

EQUIMATCHABLE BIPARTITE GRAPHS ^{*,†}

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Abstract

A graph is called equimatchable if all of its maximal matchings have the same size. Lesk *et al.* [*Equi-matchable graphs*, Graph Theory and Combinatorics (Academic Press, London, 1984) 239–254] has provided a characterization of equimatchable bipartite graphs. Motivated by the fact that this characterization is not structural, Frendrup *et al.* [*A note on equimatchable graphs*, Australas. J. Combin. 46 (2010) 185–190] has also provided a structural characterization for equimatchable graphs with girth at least five, in particular, a characterization for equimatchable bipartite graphs with girth at least six. In this paper, we extend the characterization of Frendrup by eliminating the girth condition. For an equimatchable graph, an edge is said to be a critical-edge if the graph obtained by the removal of this edge is not equimatchable. An equimatchable graph is called edge-critical, denoted by ECE, if every edge is critical. Noting that each ECE-graph can be obtained from some equimatchable graph by recursively removing non-critical edges, each equimatchable graph can also be constructed from some ECE-graph

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by joining some non-adjacent vertices. Our study reduces the characterization of equimatchable bipartite graphs to the characterization of bipartite ECE-graphs.

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1. INTRODUCTION

A graph G is called *equimatchable* if every maximal matching of G is a maximum matching, that is, every maximal matching in G has the same cardinality. The concept of equimatchability was introduced in 1974 independently by Grünbaum [7], Lewin [11], and Meng [13]. In 1984, Lesk *et al.* [10] formally introduced equimatchable graphs and provided a characterization of equimatchable graphs via Gallai-Edmonds decomposition, yielding a polynomial-time recognition algorithm. In 2014, Demange and Ekim [2] gave an alternative characterization of equimatchable graphs yielding a more efficient recognition algorithm using alternating chain-based arguments.

In the literature, the structure of equimatchable graphs are extensively studied by several authors. The first study is the characterization of equimatchable graphs with a perfect matching, that is, randomly matchable graphs. In 1979, Sumner [14] proved that the only connected randomly matchable graphs are the complete graphs K_{2n} and complete bipartite graphs $K_{n,n}$ for $n \geq 1$. On the other hand, the work in [10] provided a characterization for general equimatchable graphs, that is, not only randomly matchable but also equimatchable graphs without a perfect matching. Particularly, [10] gave a characterization of equimatchable bipartite graphs in terms of subsets of neighborhoods of vertices in the smaller partite set. Although it is a complete characterization of equimatchable bipartite graphs, it does not provide much insight about the structure of equimatchable bipartite graphs. Besides, this characterization does not lead to an efficient polynomial-time recognition algorithm, namely $O(n^4)$ time recognition algorithm as in [2], for equimatchable bipartite graphs. Recently, the work in [3] reformulated the characterization of equimatchable bipartite graphs given in [10]. A connected bipartite graph is equimatchable if and only if each of its maximal matchings saturates all vertices in the smaller partite set. Furthermore, Frendrup *et al.* [6] provided a structural characterization of equimatchable graphs with girth at least five. Particularly, they showed that an equimatchable graph with girth at least five is either one of C_5 and C_7 or a member of graph family, which consists of K_2 and all bipartite graphs with partite sets V_1 and V_2 such that all vertices in V_1 are stems and no vertex from V_2 is a stem. However, the work in [6] provides a partial characterization of equimatchable bipartite graphs,

namely, a characterization of equimatchable bipartite graphs with girth at least six.

Motivated by the lack of a structural characterization for the general case of equimatchable bipartite graphs, we study equimatchable bipartite graphs in this paper. By the reformulation given in [3] for the characterization of equimatchable bipartite graphs, any bipartite supergraph of an equimatchable bipartite graph obtained by joining some pair of non-adjacent vertices from different partite sets is also an equimatchable bipartite graph with the same vertex partition. Hence, we intuitively consider equimatchable bipartite subgraphs of an equimatchable bipartite graph with the same vertex partition. For an equimatchable graph, an edge is said to be a *critical-edge* if the graph obtained by the removal of this edge is not equimatchable. An equimatchable graph is called *edge-critical*, denoted by *ECE*, if every edge is critical. Notice here that the smallest equimatchable bipartite subgraph of an equimatchable bipartite graph with the same vertex partition is indeed an edge-critical equimatchable bipartite graph with the same vertex partition. That is, each bipartite ECE-graph can be obtained from some equimatchable bipartite graph having the same vertex partition by recursively removing non-critical edges; moreover, each equimatchable bipartite graph can also be constructed from some bipartite ECE-graph by joining some non-adjacent vertices from different partite sets. Hence, it is sufficient to focus on the structure of edge-critical equimatchable bipartite graphs instead of the structure of the general case of equimatchable bipartite graphs.

Section 2 is devoted to basic definitions, notations, and known results on equimatchable graphs. In Section 3, we provide some structural results for equimatchable bipartite graphs by using Gallai-Edmonds decomposition. Particularly, we extend the partial characterization of Frendrup *et al.* [6] to all equimatchable bipartite graphs by showing that at least one of the following is true: each vertex in the smaller partite set of an equimatchable bipartite graph is a stem or a vertex of an induced $K_{2,2}$. We also show that removing a cut vertex from an equimatchable bipartite graph preserves the smaller and larger partite sets in the partition vertices. In Section 4, we discuss the structure of bipartite ECE-graphs. We first point out that every connected bipartite ECE-graph is 2-connected. We then provide a characterization for the general case of bipartite ECE-graphs. Finally, in Section 5 we conclude the paper and present an open question.

2. PRELIMINARIES

In this section, we first give some graph-theoretical definitions and notations that will be used in the forthcoming sections and then present some results about equimatchable bipartite graphs, which will lay the foundation for our arguments

in Section 3.

All graphs in this paper are finite, simple, and undirected. For a graph $G = (V(G), E(G))$, $V(G)$ and $E(G)$ denote the set of vertices and edges in G , respectively. An edge joining the vertices u and v in G will be denoted by uv . A *bipartite graph* G is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2 . If $|V_1| = |V_2|$, then we say that G is *balanced*. For a vertex v in G and a subset $X \subseteq V(G)$, $N_G(v)$ denotes the set of neighbors of v in G , while $N_G(X)$ denotes the set of all vertices adjacent to at least one vertex of X in G . We omit the subscript G when it is clear from the context. The *order* of G is denoted by $|V(G)|$ and the *degree* of a vertex v of G is denoted by $d(v)$. A vertex of degree one is called a *leaf* and a vertex adjacent to a leaf is called a *stem*. For a graph G and $U \subseteq V(G)$, the subgraph induced by U is denoted by $G[U]$. The *difference* $G \setminus H$ of two graphs G and H is defined as the subgraph induced by the difference of their vertex sets, that is, $G \setminus H = G[V(G) \setminus V(H)]$. For a graph G and a vertex v of G , the subgraph induced by $V(G) \setminus v$ is denoted by $G - v$ for the sake of brevity. We also denote by $G \setminus e$ the graph $G(V, E \setminus \{e\})$. The cycle and complete graph on n vertices are denoted by C_n and K_n , respectively, while the complete bipartite graph with partite sets of sizes n and m is denoted by $K_{n,m}$. The length of a shortest cycle in G is called the *girth* of G . For a graph G , $c(G)$ denotes the number of components in G . A set of vertices S of a graph G such that $c(G \setminus S) > c(G)$ is called a *cut set*. A vertex v is called a *cut vertex* if $\{v\}$ is a cut set. A graph is called *2-connected* if its cut sets have at least 2 vertices.

A *matching* in a graph G is a set $M \subseteq E(G)$ of pairwise nonadjacent edges of G . A vertex v of G is *saturated* by M if $v \in V(M)$ and *exposed* by M otherwise. A matching M is called *maximal* in G if there is no other matching of G that contains M . A matching is called a *minimum maximal matching* of G if it is a maximal matching of minimum size. A matching is called a *maximum matching* of G if it is a matching of maximum size. The size of a maximum matching of G is denoted by $\nu(G)$. A matching M in G is a *perfect matching* if M saturates all vertices in G , that is, $V(M) = V(G)$. For a vertex v , a matching M is called a *matching isolating* v if $\{v\}$ is a component of $G \setminus V(M)$. A graph G is *equimatchable* if every maximal matching of G is a maximum matching, that is, every maximal matching has the same cardinality. A graph G is *randomly matchable* if it is an equimatchable graph admitting a perfect matching. A graph G is *almost equimatchable* if the difference between the maximum and minimum size of maximal matchings of G is 1. A graph G is *factor-critical* if $G - v$ has a perfect matching for every vertex v of G . Note here that a factor-critical graph cannot be bipartite, since if you choose a vertex from the smaller partite set (or from any partite set if their cardinalities are equal) there cannot be a perfect matching in the rest of the graph.

The characterization of randomly matchable graphs was provided in [14] as follows.

Theorem 1 [14]. *A connected graph is randomly matchable if and only if it is isomorphic to K_{2n} or $K_{n,n}$, $n \geq 1$.*

The following well-known result, which is called *Hall's Theorem*, gives necessary and sufficient condition for the existence of a perfect matching in bipartite graphs as follows.

Theorem 2 [8]. *A bipartite graph $G = (A \cup B, E)$ contains a matching saturating all vertices in A if and only if it satisfies $|N(S)| \geq |S|$ for all subset $S \subseteq A$.*

3. STRUCTURE OF EQUIMATCHABLE BIPARTITE GRAPHS

This section is devoted to investigate the structure of equimatchable bipartite graphs, more simply EB-graphs. Since a graph is equimatchable if and only if all of its components are equimatchable, it suffices to focus on connected EB-graphs. Indeed, some characterizations of EB-graphs are already provided in the literature, see [6] and [10]. For example, the following characterization of equimatchable graphs with girth at least five, not necessarily bipartite, was provided in [6].

Theorem 3 [6]. *Let G be a connected equimatchable graph with girth at least 5. Then $G \in \mathcal{F} \cup \{C_5, C_7\}$, where \mathcal{F} is the family of graphs containing K_2 and all connected bipartite graphs with bipartite sets V_1 and V_2 such that all vertices in V_1 are stems and no vertex from V_2 is a stem.*

Although this characterization provides information about the structure of EB-graphs, it is only a partial characterization. More precisely, Theorem 3 explicitly describes the structure of EB-graphs with girth at least six. On the other hand, the work in [10] provides a general characterization for EB-graphs in the following way.

Theorem 4 [10]. *A connected bipartite graph $G = (U \cup V, E)$ with $|U| \leq |V|$ is equimatchable if and only if for all $u \in U$, there exists a non-empty $X \subseteq N(u)$ such that $|N(X)| \leq |X|$.*

This characterization yields a polynomial-time recognition algorithm for EB-graphs, namely $O(n^4)$ time recognition algorithm [2]. The following result is a more intuitive reformulation of the characterization of EB-graphs in Theorem 4 by using Hall's Theorem.

Theorem 5 [3]. *Let $G = (U \cup V, E)$ be a connected bipartite graph with $|U| \leq |V|$. Then G is equimatchable if and only if every maximal matching of G saturates all vertices in U .*

Although Theorem 4 provides a complete characterization for EB-graphs leading to a polynomial-time recognition algorithm, it lacks an explicit description of the structure. We also note that recognition algorithm generated from the characterization in Theorem 4 does not lead to an efficient algorithm. These observations about the characterizations given in Theorem 3 and Theorem 4 motivated us to reexamine the structure of EB-graphs.

The class of equimatchable graphs is separated into two complementary subclasses, namely factor-critical and non-factor-critical equimatchable graphs. Since a factor-critical (equimatchable) graph cannot be bipartite, all EB-graphs are non-factor-critical. However, the converse is not necessarily true; that is, not all non-factor-critical equimatchable graphs are bipartite. For instance, the graph consisting of two K_4 with one common vertex is a non-factor-critical equimatchable graph that is not bipartite. Therefore, we concentrate only on non-factor-critical equimatchable graphs in this paper.

The following well-known structural result, which is called *Gallai-Edmonds decomposition*, provides an important characterization for general graphs based on maximum matchings as follows.

Theorem 6 [12]. *For any graph G , let us denote by $D(G)$ the set of vertices which are exposed by at least one maximum matching of G and by $A(G)$ the vertices of $V(G) \setminus D(G)$ which are neighbors of at least one vertex of $D(G)$. Let $C(G) = V(G) \setminus (D(G) \cup A(G))$. Then:*

1. *Every component of the graph $G[D(G)]$ is factor-critical,*
2. *$G[C(G)]$ has a perfect matching,*
3. *Every maximum matching of G matches every vertex of $A(G)$ to a vertex of a distinct component of $G[D(G)]$.*

By definition, for an equimatchable graph G , if G admits a perfect matching then $C(G) = V(G)$. Remark that by Theorem 1, a connected EB-graph G admitting a perfect matching is $K_{n,n}$ where $n \geq 1$. If G is a connected equimatchable graph G without a perfect matching, one can observe the following result.

Lemma 7 [10]. *Let G be a connected equimatchable graph with no perfect matching. Then $C(G) = \emptyset$ and $A(G)$ is an independent set in G .*

By the Gallai-Edmonds decomposition, if G is a connected equimatchable graph without a perfect matching and $A(G) = \emptyset$, then G is factor-critical. That is, the class of EB-graphs does not contain such a graph G since each EB-graph

is non-factor-critical. Therefore, it suffices to focus on connected equimatchable graphs without a perfect matching and $A(G) \neq \emptyset$, or equivalently connected non-factor-critical equimatchable graphs without a perfect matching. The following result is not explicitly given in [10], but it is an immediate consequence of Theorems 3 and 4 in [10].

Lemma 8 [10]. *Let G be a connected equimatchable non-factor-critical graph with no perfect matching. Then G is bipartite if and only if each component of $G[D(G)]$ is a singleton.*

Corollary 9. *Let G be a connected EB-graph with no perfect matching. Then, $C(G) = \emptyset$, and each of $D(G)$ and $A(G)$ is a nonempty independent set.*

For the rest of the paper, $G = (U \cup V, E)$ denotes a connected EB-graph with $|U| < |V|$. By Theorems 1 and 5, the only connected EB-graph with equal partite sets is $K_{n,n}$ where $n \geq 1$. In the next lemma, we show that the parts U and V of G correspond to the sets $A(G)$ and $D(G)$, respectively, where $C(G)$ is empty.

Lemma 10. *Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph and (D, A, C) be its Gallai-Edmonds decomposition. Then we have $C = \emptyset$, $A = U$ and $D = V$.*

Proof. Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph and (D, A, C) be its Gallai-Edmonds decomposition. By Corollary 9, we have $C = \emptyset$ and each of D and A is a nonempty independent set since G has no perfect matching. By Theorem 5, every maximal matching of G saturates all vertices in U . The Gallai-Edmonds decomposition implies that D does not have any vertex from U and then all vertices of U are in A . By Corollary 9, we observe that $A = U$ and $D = V$ since A is a nonempty independent set. ■

Corollary 11. *Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph. Then there exists an isolating matching for each $v \in V$ and there is no isolating matching for any $u \in U$.*

The following result, which provides a characterization for EB-graphs by using the Gallai-Edmonds decomposition, can be easily verified by combining Theorem 1 and Lemma 10.

Corollary 12. *Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected EB-graph with Gallai-Edmonds decomposition (D, A, C) .*

- (i) *If G admits a perfect matching, that is, $|U| = |V|$, then $C = V(G)$, $D = \emptyset$ and $A = \emptyset$. In particular, G is $K_{n,n}$ where $n \geq 1$.*

- (ii) If G admits no perfect matching, that is, $|U| < |V|$, then $C = \emptyset$, $A = U$ and $D = V$.

In the next result, we extend the characterization of Frendrup *et al.* [6] stated in Theorem 3 to all EB-graphs as follows.

Lemma 13. *Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph. Then each vertex $u \in U$ satisfies at least one of the followings.*

- (i) u is a stem in G ,
- (ii) u is included in a subgraph $K_{2,2}$ in G .

Proof. Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph. Let $u \in U$ and $N(u) = \{v_1, v_2, \dots, v_n\}$ be the set of neighbors of u in V . If one vertex in $N(u)$ is a leaf in G , then we are done. Assume to the contrary that none of the vertices in $N(u)$ is a leaf in G . If u is not included in a subgraph $K_{2,2}$ in G , then there is no pair of vertices in $N(u)$ having a common neighbor except u . It implies that there exists a matching isolating u in G . Since it contradicts with Corollary 11, we deduce that there exists at least one pair, say $\{v_1, v_2\}$, of vertices in $N(u)$ having a common neighbor except u , say u^* . It follows that the vertices $\{u, v_1, u^*, v_2\}$ induce a $K_{2,2}$ in G , as desired. ■

Corollary 14. *For a connected EB-graph $G = (U \cup V, E)$ with $|U| < |V|$, there is no leaf in U and there is no stem in V .*

Thus, each vertex $u \in U$ has at least two neighbors in V . Unless $|U| = 1$, since G is connected, each vertex $u \in U$ has a neighbor $v \in V$ which is not a leaf. Remark that Theorem 3 states that in an EB-graph with girth at least 5, all vertices in either U or V are stems, whereas Lemma 13 provides that in such a graph, all vertices in U are indeed stems. Hence, Lemma 13 together with Theorem 3 lead to the following corollary.

Corollary 15. *Let G be a connected EB-graph with girth at least 6. Then $G \in \mathcal{F}$, where \mathcal{F} is the family of graphs containing K_2 and all connected bipartite graphs with bipartite sets V_1 and V_2 with $|V_1| \leq |V_2|$ such that all vertices in V_1 are stems and no vertex from V_2 is a stem.*

Note here that the EB-graph family \mathcal{F} described in Theorem 3 and Corollary 15 contains not only all EB-graphs with girth at least six but also some EB-graphs with girth less than six, that is, EB-graphs with girth exactly four. On the other hand, it is obvious that \mathcal{F} contains some but definitely not all EB-graphs with girth four. For instance, complete bipartite graphs which are not included in the graph family \mathcal{F} are trivially EB-graphs with girth four. So, in this paper we deal with EB-graphs with girth exactly four, whereas some of these graphs are contained in the graph family \mathcal{F} .

We conclude this section with some observations providing an insight about the structure of connected EB-graphs.

Lemma 16 [5]. *Let $G \neq K_2$ be a connected equimatchable graph. Then $\nu(G) = \nu(G \setminus e)$ for any edge $e \in E(G)$.*

Proof. Let $G \neq K_2$ be a connected equimatchable graph and $e = uv \in E(G)$. For a maximal matching M of G containing uv , $M \setminus uv$ is a matching of $G \setminus uv$ with size $\nu(G) - 1$. It follows that $\nu(G \setminus uv) \geq \nu(G) - 1$; that is, $\nu(G \setminus uv)$ equals either $\nu(G) - 1$ or $\nu(G)$. Assume to the contrary that $\nu(G \setminus uv) = \nu(G) - 1$. Since $G \neq K_2$ and is connected, there exists a vertex $w \in N(u) \cup N(v)$. Without loss of generality, we say $w \in N(u)$; that is, $wu \in E(G)$. Hence, there also exists another maximal matching M' in G which can be obtained by extending the edge wu . Since G is equimatchable, $|M'| = |M|$. Note here that M' is also a maximal matching in $G \setminus uv$ with size $\nu(G)$, contradicting with the assumption that $\nu(G \setminus uv) = \nu(G) - 1$. Therefore, $\nu(G \setminus uv) = \nu(G)$ for every edge uv . ■

In Lemma 18, we extend the following known result about the cut vertices in equimatchable graphs to EB-graphs as in the following way.

Lemma 17 [1]. *Let G be a connected equimatchable graph with a cut vertex c . Then each component of $G - c$ is also equimatchable.*

Lemma 18. *Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph with a cut vertex c . Then each component of $G - c$ is also a connected EB-graph $H = (U_H \cup V_H, E_H)$ with $|U_H| < |V_H|$ such that $U_H \subseteq U$ and $V_H \subseteq V$.*

Proof. Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph with a cut vertex c . Let H_1, H_2, \dots, H_k ($k \geq 2$) be the components of $G - c$. By Lemma 17, each H_i is an EB-graph for $i \in 1, 2, \dots, k$. Assume that a connected EB-graph $H = (U_H \cup V_H, E_H)$ with $|U_H| < |V_H|$ is a component of $G - c$ such that $U_H \subseteq U$ and $V_H \subseteq V$ or $U_H \subseteq V$ and $V_H \subseteq U$. Since G is connected, c is adjacent to some vertices of H . Since G is bipartite, c is adjacent to some vertices of only U_H or V_H . Let x be a neighbor of c in H and $e = cx$ be an edge joining c and x in G . We then examine the following complementary cases.

- In the case $c \in V$ in G , $x \in U$ in G . Assume to the contrary that $x \in V_H$ in H . Then, by Corollary 11, there exists a matching M_H isolating x in H . Note here that the only neighbor of x in G which is not contained in H is the vertex c . Since c is a cut vertex of G , that is, $k \geq 2$, there exists another component H' of $G - c$. Let x' be a neighbor of c in H' and $e' = cx'$ be an edge joining c and x' in G . Hence, it is easy to see that the matching $M_H \cup \{e'\}$ is a matching isolating x in G . By Corollary 11, it contradicts with $x \in U$ in G . Therefore, we obtain that $x \in U_H$ in the case $c \in V$.

• In the case $c \in U$ in G , $x \in V$ in G . Assume to the contrary that $x \in U_H$ in H . Then, by Corollary 11, there exists a matching M isolating x in G . Note that the only neighbor of x in G which is not contained in H is the vertex c . Besides, the vertices c and x have no common neighbor in G , since G is bipartite. Let $e' \in M$ be such that $e' = cx'$ is an edge joining c and $x' \notin N(x)$ in G . Hence, it is easy to see that the matching $M \setminus \{e'\}$ isolates x in H . By Corollary 11, it contradicts with the assumption $x \in U_H$ in G . Therefore, we obtain that $x \in V_H$ in the case $c \in U$.

Therefore, we conclude that $U_H \subseteq U$ and $V_H \subseteq V$ irrespective of whether $c \in U$ or $c \in V$. ■

In short, Lemma 18 implies that for each cut vertex c of G , the components of $G - c$ are induced EB-subgraphs preserving (U, V) -partitions of G .

4. EDGE-CRITICAL EQUIMATCHABLE BIPARTITE GRAPHS

Recall that for an equimatchable graph, an edge is a *critical-edge* if the graph obtained by removal of this edge is not equimatchable, and an equimatchable graph is *edge-critical* if every edge is critical. For the sake of brevity, we will call edge-critical equimatchable graphs as *ECE-graphs*. The goal of this section is to describe the structure of a generating subclass of EB-graphs, namely *edge-critical EB-graphs*.

Notice that the complete graph K_2 (or equivalently the complete bipartite graph $K_{1,1}$) is equimatchable but not edge-critical. In fact, its edge is not critical since the remaining graph obtained by removal of the edge contains only isolated vertices, which is trivially equimatchable. The following results about ECE-graphs, not necessarily bipartite, are frequently used in our arguments.

Lemma 19 [5]. *Let $G \neq K_2$ be a connected equimatchable graph. Then $uv \in E(G)$ is critical if and only if there is a matching of G containing uv and saturating $N_{G \setminus uv}(\{u, v\})$.*

Proof. Let $G \neq K_2$ be a connected equimatchable graph with $uv \in E(G)$.

(\Rightarrow) Assume that $uv \in E(G)$ is critical; that is, G is equimatchable but $G \setminus uv$ is not equimatchable. Then $G \setminus uv$ admits two maximal matchings M_1 and M_2 with different sizes, say $|M_1| < |M_2|$. It follows that $|M_1| = \nu(G) - 1$ and $|M_2| = \nu(G)$, since $G \setminus uv$ is obtained by the removal of only uv from G . That is, M_2 is a maximal matching of G whereas M_1 is not maximal in G . It implies that M_1 leaves both u and v exposed in $G \setminus uv$ and $M_1 \cup \{uv\}$ is a maximal matching of G . Hence, it follows that M_1 saturates all vertices in $N_{G \setminus uv}(\{u, v\})$, because otherwise M_1 cannot be maximal in $G \setminus uv$. Therefore, the maximal matching $M_1 \cup \{uv\}$ of G is the desired matching.

(\Leftarrow) Conversely, assume that there is a matching of G containing uv and saturating $N_{G \setminus uv}(\{u, v\})$. Without loss of generality, we extend this matching to a maximal matching M of G containing uv and saturating $N_{G \setminus uv}(\{u, v\})$. It follows that $M \setminus uv$ is a maximal matching in $G \setminus uv$. There also exist another maximal matching M' in $G \setminus uv$ which can be obtained by extending the edge vw for some $w \in N(v)$ with $w \neq u$ or uz for some $z \in N(u)$ with $z \neq v$. Notice that the matching M' has size $\nu(G)$ since it is also a maximal matching of G , which is equimatchable. Hence, $G \setminus uv$ is not equimatchable since the maximal matchings M and M' of G have different sizes $\nu(G)$ and $\nu(G) - 1$. It implies that the edge uv is critical. ■

Corollary 20. *A connected equimatchable graph $G \neq K_2$ is edge-critical if and only if there is a matching containing uv and saturating $N(\{u, v\})$ for every $uv \in E(G)$.*

Corollary 21. *All randomly matchable connected graphs except K_2 are edge-critical.*

Recall that $G = (U \cup V, E)$ is a connected EB-graph with $|U| < |V|$. We consider a bipartite supergraph G' of G obtained by joining a pair of non-adjacent vertices of $u \in U$ and $v \in V$ by the edge uv . By Theorem 5, a maximal matching of G cannot be extended to a larger matching in G' since maximal matchings of G saturate all vertices in U . It follows that $\nu(G) = \nu(G') = |U|$. Let us consider a minimum maximal matching M of G' which is obtained by extending the edge uv . Note that all edges of M except uv are indeed the edges of G . Then, by Theorem 5, the matching $M - uv$ of G is not maximal since it does not saturate the vertex $u \in U$. We extend $M - uv$ to a maximal matching M' in G . Since G is an EB-graph, by Theorem 5, we have $\nu(G) = |M'| = |U|$. Since M is a maximal matching of G' containing the edge $uv \notin E(G)$, we need to add at least one and at most two edges to $M \setminus uv$ in order to extend it to M' ; namely, the edges saturating u and v . Note here that M' has to saturate the vertex u but may not saturate the vertex v . It follows that $|U| \geq |M| \geq |M'| - 1 = |U| - 1$. Thus, $|M|$ is equal to either $|U|$ or $|U| - 1$; equivalently, it is equal to either $\nu(G)$ or $\nu(G) - 1$.

In the case where $|M| = \nu(G)$, by definition of M , all maximal matchings of G' have the same size, namely $|U|$. Thus, the supergraph G' of G is also a connected EB-graph with the same vertex set $U \cup V$. In such a case, uv is a non-critical edge of G' . In the case where $|M| = \nu(G) - 1$, we observe that maximal matchings of G' have size either $|U|$ or $|U| - 1$. It follows that the supergraph G' of G is a connected almost equimatchable bipartite graph with the vertex set $U \cup V$. Besides, the work in [4] provides a characterization for almost equimatchable graphs. All these leads us to the following observation.

Observation 22. *For a connected EB-graph $G = (U \cup V, E)$ with $|U| < |V|$ except $K_{n,m}$ for some n and m , and a connected bipartite supergraph G' of G obtained by joining a pair of non-adjacent vertices of U and V by an edge, one of the followings holds.*

- (i) G' is equimatchable such that all maximal matchings of G' have size $|U|$.
- (ii) G' is almost equimatchable such that a maximal matching of G' has size $|U|$ or $|U| - 1$.

The next result is a characterization for EB-supergraphs of EB-graphs as follows.

Theorem 23. *Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph except $K_{n,m}$ for some n and m , and let G' be a supergraph of G obtained by joining a pair of non-adjacent vertices of $u \in U$ and $v \in V$ by the edge uv . G' is a connected EB-graph if and only if every maximal matching of G' obtained by extending the edge uv saturates $N_G(v)$.*

Proof. Let $G = (U \cup V, E)$ with $|U| < |V|$ be a connected EB-graph except $K_{n,m}$ for some n and m and G' be a supergraph of G obtained by joining a pair of non-adjacent vertices of $u \in U$ and $v \in V$ by the edge uv . It is clear that G' is connected. By Observation 22, G' is either equimatchable or almost equimatchable.

(\Rightarrow) Suppose that G' is an EB-supergraph of G . Then, by Theorem 5, every maximal matching of G' saturates all vertices in U . Hence, every maximal matching of G' obtained by extending the edge uv saturates $N_G(v)$.

(\Leftarrow) Suppose that every maximal matching of G' obtained by extending the edge uv saturates $N_G(v)$. We will show that G' is an EB-graph. Assume to the contrary that G' is almost equimatchable. Then, by Observation 22, G' admits maximal matchings of two different sizes, namely $|U| - 1$ and $|U|$. For maximal matchings of G' with $|M| = |U|$, the theorem holds and we are done. Let M be a maximal matching of G' with $|M| = |U| - 1$. Then, M is a maximal matching obtained by extending the edge uv ; otherwise M is a maximal matching of G , which is equimatchable with $\nu(G) = |U|$. Let $u^* \in U$ be the vertex exposed by M . Since $\nu(G) = |U|$, it is obvious that $M \setminus uv$ is not maximal in G ; particularly, $M \setminus uv$ does not saturate the vertices u and u^* in U . Since G is equimatchable with $\nu(G) = |U|$, we can extend the matching $M \setminus uv$ to a maximal matching M^* of G by adding two edges saturating the vertices u and u^* . Since u^* is saturated by all maximal matchings in G and exposed by M in G' , it follows that there exists an edge u^*v in G ; that is, u^* is adjacent to v . Hence, $u^* \in N(v)$ is exposed by M . Therefore, we conclude that every maximal matching of G' with size $|U| - 1$ exposes a vertex of $N(v)$, a contradiction. ■

Observation 22 and Theorem 23 imply that a supergraph G' of G obtained by joining a pair of non-adjacent vertices of $u \in U$ and $v \in V$ by the edge uv is a connected bipartite almost equimatchable graph if and only if there exists a maximal matching of G' obtained by extending the edge uv such that the matching leaves exposed a vertex of $N_G(v)$.

Note here that the largest such EB-supergraph of G is trivially the complete bipartite graph. We intuitively consider EB-subgraphs of G with the same vertex set $U \cup V$. Note that the smallest such EB-subgraph of G , say H , is indeed an edge-critical EB-graph with the same vertex set $U \cup V$; while the other EB-subgraphs with the same vertex set $U \cup V$ are exactly bipartite supergraphs of H . That is, each edge-critical EB-graph can be obtained from some EB-graph having the same vertex set by recursively removing non-critical edges. Therefore, in order to characterize all EB-graphs it is sufficient to characterize all edge-critical EB-graphs; that is, the class of edge-critical EB-graphs form a generating subclass of EB-graphs. Notice here that an edge-critical EB-graph H may not be connected. Since a graph is equimatchable if and only if each of its components is equimatchable, it suffices to focus on connected edge-critical EB-graphs.

From here onwards, we focus on edge-critical EB-graphs. In the following lemma, we show that edge-critical EB-graphs cannot have a cut vertex; that is, all edge-critical EB-graphs are 2-connected.

Lemma 24. *Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected edge-critical EB-graph. Then G is 2-connected; that is, G has no cut vertex.*

Proof. Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected edge-critical EB-graph. In the case where $|U| = |V|$, by Theorem 5, G has a perfect matching. Then by Theorem 1, G is $K_{n,n}$ for some $n \geq 1$ and by Corollary 21, we omit $K_2 = K_{1,1}$ and we have $n \geq 2$. That is, G is 2-connected and we are done.

We now suppose that $|U| < |V|$. Assume to the contrary that G has a cut vertex c . Let H_1, H_2, \dots, H_k ($k \geq 2$) be connected components of $G - c$ such that $d_i \in H_i$ where $d_1, d_2, \dots, d_k \in N(c)$ and $i \in [k]$. By Lemma 18, each H_i is an EB-subgraph preserving (U, V) -partitions of G where $i \in [k]$. By Corollary 20, for each edge $e_i = cd_i$, there exists a matching M^i containing the edge cd_i and saturating all vertices in $N(\{c, d_i\})$ for $i \in [k]$. For $j \in [k]$ with $j \neq i$, we define $M_j^i = M^i \cap E(H_j)$ which is a matching in H_j saturating all vertices in $N_{H_j}(c)$. Note that $M_1^i \cup M_2^i \cup \dots \cup M_{i-1}^i \cup M_{i+1}^i \cup \dots \cup M_k^i$ is a matching saturating all neighbors of c except the neighbors in H_i . Without loss of generality, since $k \geq 2$, let us consider the cases $i = 1$ and $i = 2$. For $i = 1$, $M_2^1 \cup M_3^1 \cup \dots \cup M_k^1$ is a matching saturating all neighbors of c except the neighbors in H_1 ; and for $i = 2$, $M_1^2 \cup M_3^2 \cup \dots \cup M_k^2$ is a matching saturating all neighbors of c except the neighbors in H_2 . It follows that $M = M_1^2 \cup M_2^1 \cup M_3^1 \cup \dots \cup M_k^1$ is a matching saturating all neighbors of c in G . Hence, by Corollary 11, we conclude that

$c \in V$ and $d_1, d_2, \dots, d_k \in U$. By Lemma 18, it follows that $d_i \in U_{H_i}$ for $i \in [k]$.

Remember that for each edge $e_i = cd_i$, M^i is a matching containing the edge cd_i and saturating all vertices in $N(\{c, d_i\})$ for $i \in [k]$. By defining $T_i = M^i \cap E(H_i)$ for $i \in [k]$, for each edge $e_i = cd_i$, there also exists a matching T_i in H_i saturating all vertices in $N_{H_i}(d_i)$ for $i \in [k]$. It is easy to see that each T_i is indeed a matching isolating d_i in H_i for $i \in [k]$. Hence, by Corollary 11, we conclude that $d_i \in V_{H_i}$, which is a contradiction. ■

The next theorem provides a characterization for edge-critical EB-graphs as follows.

Theorem 25. *A connected bipartite graph $G = (U \cup V, E)$ with $|U| \leq |V|$ except K_2 is an edge-critical EB-graph if and only if for every $u \in U$, $|N(S)| \geq |S|$ holds for any subset $S \subseteq N(u)$ and the equality holds only for $S = N(u)$.*

Proof. Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected bipartite graph except K_2 .

(\Rightarrow) Suppose that G is an edge-critical EB-graph. We will first show that for every $u \in U$, $|N(S)| > |S|$ holds for any subset $S \subset N(u)$. Assume to the contrary that there exists $u \in U$ such that $|N(S)| \leq |S|$ for some $S \subset N(u)$. Then there exists $w \in N(u) \setminus S$. Since $|N(S)| \leq |S|$, there is no matching of G saturating $N_{G \setminus uw}(\{u, w\})$ and containing uw . By Lemma 19, G is not an ECE-graph which is a contradiction. Thus, for all $u \in U$, $|N(S)| > |S|$ holds for any subset $S \subset N(u)$. We will now show that $|N(S)| \geq |S|$ for $S = N(u)$. Since G is an EB-graph, by Theorem 4, for all $u \in U$, there exists a non-empty $S \subseteq N(u)$ such that $|N(S)| \leq |S|$. Since for all $u \in U$, $|N(S)| > |S|$ holds for any subset $S \subset N(u)$, it follows that for all $u \in U$, $S = N(u)$ satisfies $|N(S)| \leq |S|$. Assume that there exists $u \in U$, $|N(S)| < |S|$ holds for $S = N(u)$. Then, we observe that there is no matching saturating all vertices in $N(u)$. Hence, for any $v \in N(u)$, there is no matching of G saturating $N_{G \setminus uv}(\{u, v\})$ and containing uv . By Lemma 19, the edge uv is not critical, contradicting with G being an ECE-graph. That is, for all $u \in U$, $|N(S)| = |S|$ holds for $S = N(u)$. Therefore, for every $u \in U$, $|N(S)| \geq |S|$ for any subset $S \subseteq N(u)$ and the equality holds only for $S = N(u)$, as desired.

(\Leftarrow) Suppose that for every $u \in U$, $|N(S)| \geq |S|$ holds for any subset $S \subseteq N(u)$ and the equality holds only for $S = N(u)$. Since $S = N(u)$ satisfies $|N(S)| = |S|$, G is an equimatchable graph by Theorem 4. Assume to the contrary that there exists a vertex v such that $u \in U$, $v \in N(u)$ and $uv \in E(G)$ is not critical; that is, $G \setminus uv$ is equimatchable. Then, by Theorem 4, there exists $X \subseteq N_{G \setminus uv}(u)$ such that $|N_{G \setminus uv}(X)| \leq |X|$. Note also that $X \subseteq N_{G \setminus uv}(u) \subset N(u)$. It follows that $|N(X)| = |N_{G \setminus uv}(X)| \leq |X|$. Since $X \subset N(u)$ and $|N(S)| \geq |S|$ holds for any subset $S \subseteq N(u)$, we also have $|N(X)| \geq |X|$, implying that $|N(X)| = |X|$. However, $X \neq N(u)$ and this contradicts with the fact that

$|N(S)| = |S|$ holds only for $S = N(u)$. Therefore, $uv \in E(G)$ is critical for all $uv \in E(G)$ and hence, G is an edge-critical EB-graph. ■

If the equality $|U| = |V|$ holds, we can give an explicit characterization for edge-critical EB-graphs as follows.

Proposition 26. *A connected bipartite graph $G = (U \cup V, E)$ with $|U| = |V|$ is an edge-critical EB-graph if and only if it is isomorphic to $K_{n,n}$ for some $n \geq 2$.*

Proof. Let $G = (U \cup V, E)$ with $|U| = |V|$ be a connected bipartite graph except K_2 . Suppose that G is an edge-critical EB-graph. By Theorem 5, G has a perfect matching. Then by Theorem 1 and Corollary 21, G is $K_{n,n}$ for $n \geq 2$. Conversely, by Theorem 1 and Corollary 21, $K_{n,n}$ is an edge-critical EB-graph for some $n \geq 2$. ■

The next theorem provides another characterization for edge-critical EB-graphs in terms of induced subgraphs as follows.

Theorem 27. *A connected bipartite graph $G = (U \cup V, E)$ with $|U| \leq |V|$ is an edge-critical EB-graph if and only if for any $u \in U$, the subgraph $H = (U_H \cup V_H, E_H)$ of G induced by the vertices $N(u)$ and $N(N(u))$ is a 2-connected balanced bipartite subgraph of G such that for all $v \in V_H$, there exists a perfect matching containing uv in H .*

Proof. Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected bipartite graph.

(\Rightarrow) Suppose that G is an edge-critical EB-graph. Let $u \in U$ and H be the subgraph of G induced by the vertices $N(u)$ and $N(N(u))$. By Theorem 25, we have $|N(u)| = |N(N(u))|$, implying that $|U_H| = |V_H|$. That is, H is a balanced bipartite subgraph of G . Then by Corollary 20, for any neighbor v of u , there exists a matching containing uv and saturating all other neighbors of u in G . It implies that for all $v \in V_H$, there exists a perfect matching containing uv in H since H is a balanced bipartite graph. Notice that by definition of H , u is adjacent to all vertices in V_H , implying that there is no cut vertex in V_H . Suppose that $u^* \in U_H$ is a cut vertex in H . Then, we have $u^* = u$ since u is adjacent to all vertices in V_H . If u is a stem in H , then it is also a stem in G . However, G is 2-connected by Theorem 24, contradiction. Therefore, u is not a stem in H . Since H is a balanced bipartite graph, there exists a component H^* of $H - u$ such that $|U_{H^*}| < |V_{H^*}|$. Then, for any $v \in V_H \setminus V_{H^*}$, there is no perfect matching containing uv in H since $|U_{H^*}| < |V_{H^*}|$, contradicting with Corollary 20. Hence, we conclude that u cannot be a cut vertex, which is a contradiction. Therefore, H is a 2-connected balanced bipartite subgraph of G such that for all $v \in V_H$, there exists a perfect matching containing uv in H .

(\Leftarrow) Suppose that for any $u \in U$, the subgraph $H = (U_H \cup V_H, E_H)$ of G induced by the vertices $V_H = N(u)$ and $U_H = N(N(u))$ is a 2-connected

balanced bipartite subgraph of G such that for all $v \in V_H$, there exists a perfect matching containing uv in H . Since for all $v \in V_H$, there exists a perfect matching containing uv in H , $|N(S)| \geq |S|$ holds for any subset $S \subseteq N(u)$. We assume that there exists a $X \subset N(u)$ such that $|N(X)| = |X|$. By definition of H , it is clear that $u \in N(X)$. Then, there exists a vertex w in $V_H \setminus X$, and there exists a perfect matching M containing uw in H . It implies that the vertices in X is saturated by the vertices in $N(X) - u$ in M . It gives a contradiction since $|N(X) - u| < |N(X)| = |X|$. Hence, the equality $|N(S)| = |S|$ holds only for $S = N(u)$. Therefore, by Theorem 25, we conclude that G is an edge-critical EB-graph. ■

The next result can be verified by Theorem 27. Particularly, the subgraph $H = (U_H \cup V_H, E_H)$ of G induced by the vertices $N(u)$ and $N(N(u))$ is 2-connected, implying that each vertex in $U_H \setminus \{u\}$ has two neighbors in V_H and u is adjacent to all vertices in V_H .

Corollary 28. *Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected edge-critical EB-graph. If $H = (U_H \cup V_H, E_H)$ is a subgraph of G induced by the vertices $N(u)$ and $N(N(u))$ for any $u \in U$, then all vertices in $U_H \setminus \{u\}$ form a C_4 with u .*

We conclude this section with an interesting result.

Lemma 29. *Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected edge-critical EB-graph and $u_1, u_2 \in U$. Then either $N(u_1) = N(u_2)$ or both $N(u_1) \setminus N(u_2)$ and $N(u_2) \setminus N(u_1)$ are nonempty.*

Proof. Let $G = (U \cup V, E)$ with $|U| \leq |V|$ be a connected edge-critical EB-graph and $u_1, u_2 \in U$. Without loss of generality, we assume to the contrary that $N(u_1) \subset N(u_2)$. Let $X = N(u_1) \subset N(u_2)$. By Theorem 27, we have $|N(u_1)| = |N(N(u_1))|$, implying that $|N(X)| = |X|$. However, by Theorem 25 since for $u_2 \in U$, $|N(S)| \geq |S|$ holds for any subset $S \subseteq N(u_2)$ and the equality holds only for $S = N(u_2)$, contradiction. ■

5. CONCLUDING REMARKS

Two characterizations for EB-graphs have been provided by Lesk *et al.* [10] and Frendrup *et al.* [6]. The characterization given in [6] is only a partial characterization for EB-graphs and the characterization given in [10] provides limited information about the structure of EB-graphs.

In this paper, we initially present some observations and preliminary results about the structure of general EB-graphs by using Gallai-Edmonds decomposition. We then discuss bipartite ECE-graphs. An ECE-graph can be obtained

from equimatchable graphs by recursively removing non-critical edges. In this paper, we show that each EB-graph can be obtained by adding an arbitrary number of edges to an edge-critical EB-graph. Thus, we reduce the characterization of EB-graphs to the characterization of edge-critical EB-graphs. Particularly, we provide some characterizations for bipartite ECE-graphs.

An interesting open question is to obtain an efficient algorithm that recognizes whether a given bipartite graph is equimatchable by generating ECE-graphs.

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