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EFFICIENT (j,k)-DOMINATING FUNCTIONS

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Abstract

For positive integers j and k, an *efficient* (j,k)*-dominating function* of a graph $G = (V,E)$ is a function $f : V \to \{0,1,2,\ldots,j\}$ such that the sum of function values in the closed neighbourhood of every vertex equals k. The relationship between the existence of efficient (j,k)-dominating functions and various kinds of efficient dominating sets is explored. It is shown that if a strongly chordal graph has an efficient (j,k)-dominating function, then it has an efficient dominating set. Further, every efficient (j,k)-dominating function of a strongly chordal graph can be expressed as a sum of characteristic functions of efficient dominating sets. For $j < k$ there are strongly chordal graphs with an efficient dominating set but no efficient (j,k)-dominating function. The problem of deciding whether a given graph has an efficient (j,k)-dominating function is shown to be NP-complete for all positive integers j and k, and solvable in polynomial time for strongly chordal graphs

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when $j = k$. By taking $j = 1$ we obtain NP-completeness of the problem of deciding whether a given graph has an efficient k-tuple dominating set for any fixed positive integer k. Finally, we consider efficient $(2, 2)$-dominating functions of trees. We describe a new constructive characterization of the trees with an efficient dominating set and a constructive characterization of the trees with two different efficient dominating sets. A number of open problems and questions are stated throughout the work.

Keywords: efficient (j, k)-dominating function, efficient dominating set, k-tuple dominating set, strongly chordal graph, tree, complexity.

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1. Introduction

Recall that a *dominating set* of a graph $G = (V, E)$ is a subset $D \subseteq V$ such that, for every vertex $x \in V$, $|N[x] \cap D| \geq 1$, where $N[x] = N(x) \cup \{x\}$ is the closed neighbourhood of x. The *domination number of* G, denoted $\gamma(G)$, is the smallest size of a dominating set of G. If $|N[x] \cap D| = 1$ for all $x \in V$, then D is called an *efficient dominating set* of G.

A wealth of information on dominating sets and efficient dominating sets in graphs can be found in the two volume set by Haynes, Hedetniemi and Slater [16, 17]. Efficient dominating sets are also called *perfect codes*, or *perfect dominating sets* because they first arose in the work of Biggs on error-correcting codes [3]. In the literature the term "perfect dominating set" often means a subset $D \subseteq V$ such that $|N[x] \cap D| \geq 1$ for every vertex $x \in V \setminus D$, for example see [22], so we will not use it. Efficient dominating sets were later independently introduced by Bange, Bartsaukas and Slater [2]. The problem of deciding whether a given graph has an efficient dominating set is NP-complete even for some restricted graph families (see [4] and its references). Linear time algorithms to find an efficient dominating set of a tree, if one exists, are described in [2, 20]. The existence of a linear-programming algorithm that decides if a strongly chordal graph has an efficient dominating set follows from the work of Farber [8]. The trees that have an efficient dominating set have been constructively characterized, as have the trees with two disjoint efficient dominating sets [2]. While a characterization of the trees with a unique dominating set is known [12], no characterization of the trees with a unique efficient dominating set has been given.

For an integer $k \geq 1$, a *k-tuple dominating set* of a graph G is a subset $D \subseteq V$ such that $|N[v] \cap D| \geq k$ for every $v \in V$. If the equality $|N[v] \cap D| = k$ holds for every $v \in V$, then D is an *efficient k-tuple dominating set*. Clearly, D is a 1-tuple dominating set of G if and only if D is a dominating set of G. Only graphs with minimum degree at least $k-1$ have a k-tuple dominating set. Harary and Haynes [14] (also see [16, 17]) were the first to study k-tuple domination in graphs. The case $k = 2$ has received the most attention, for example see [13, 15, 19]. Efficient 2-tuple domination was studied by Chellali, Khelladi and Maffray [6] under the name *exact double domination*.

A function $f : V(G) \to \{0,1\}$ is the characteristic function of a dominating set D of a graph G if and only if $\sum_{x \in N[v]} f(x) \geq 1$ for each vertex $v \in V$. The set D is an efficient dominating set if and only if equality holds for each vertex $v \in V$. Similarly, f is the characteristic function of a k-tuple dominating set of a graph G if and only if $\sum_{x \in N[v]} f(x) \geq k$ for each vertex $v \in V$, and the k-tuple dominating set is efficient if and only if equality holds for each vertex $v \in V$.

A great variety of other dominating functions have received attention. See [16, 17, 19, 21] for details. We mention one example which is related to the present work. An *Italian dominating function* of a graph G is a function $f : V(G) \to \{0,1,2\}$ such that $\sum_{x \in N[v]} f(x) \geq 2$ for each vertex v with $f(v) = 0$. The function f is a *perfect Italian dominating function* if $\sum_{x \in N[v]} f(x) = 2$ for each vertex v with $f(v) = 0$. It is proved in [18], that if f is a perfect Italian dominating function of a tree with n vertices, then $\sum_{v \in V} f(v) \leq \frac{4}{5}n$, and this bound is best possible.

We now state the main definition needed for this work. It first appears in Rubalcaba and Slater [21]. For positive integers j and k, a (j,k)*-dominating function* of a graph G is a function $f : V(G) \to \{0,1,\ldots,j\}$ such that $\sum_{x \in N[v]} f(x) \geq k$ for each vertex $v \in V(G)$. If $\sum_{x \in N[v]} f(x) = k$ for each vertex $v \in V(G)$, then f is an *efficient* (j,k)*-dominating function* of G.

The *weight* of the (j,k)-dominating function f is $\sum_{x \in V(G)} f(x)$. The (j,k)-*domination number of* G, denoted $\gamma_{(j,k)}(G)$, is the minimum weight of a (j,k)-dominating function of G, if such a function exists.

By definition $\gamma_{(1,1)}(G) = \gamma(G)$ because a $(1,1)$-dominating function is the characteristic function of a dominating set. A graph may not have a (j,k)-dominating function for some values of j and k: for example, it follows from considering the closed neighbourhood of a leaf that a tree has a (j,k)-dominating function only when $j \geq k/2$; a graph with isolated vertices has a (j,k)-dominating function if and only if $j \geq k$.

This paper is organized as follows. In the remainder of this section we make some observations about (j,k)-dominating functions. For example, the ordered pairs (j,k) such that a graph G has a (j,k)-dominating function are determined in Proposition 1.1 below. We also relate (j,k)-dominating functions of a graph G to k-tuple dominating sets of a related graph. The next section contains observations about the relationship between the (j,k)-domination number and the domination number, and about efficient (j,k)-domination. Strongly chordal graphs are considered in Section 3. It is shown that if a strongly chordal graph has an efficient (j,k)-dominating function, then it has an efficient dominating set. The converse holds if and only if $j = k$. It is also shown that every efficient

(j,k)-dominating function of a strongly chordal graph can be written as a sum of efficient $(1,1)$-dominating functions, that is, as a sum of characteristic functions of efficient dominating sets. The main result of Section 4 is that, for all integers j and k, it is NP-complete to decide whether a given graph has an efficient (j,k)-dominating function. When $j = k$ the problem is solvable in polynomial time for strongly chordal graphs. By taking $j = 1$ in the main result of the section we obtain NP-completeness of the problem of deciding whether a given graph has an efficient k-tuple dominating set for all integers k. Efficient $(2,2)$-dominating functions of trees are considered in the final section. We give a constructive characterization of the trees with an efficient $(2,2)$-dominating function, as well as some subclasses where the values the function can take are restricted to 0 and 1, or to be such that the function is onto $\{0,1,2\}$. The results imply a constructive characterization of the trees with an efficient dominating set which is simpler than the one in [2], essentially the same constructive characterization of the trees with two disjoint efficient dominating sets as in [2], and a constructive characterization of the trees with two different efficient dominating sets.

Every graph has a dominating set but, as remarked above, only graphs with minimum degree at least $k-1$ have a k-tuple dominating set. We now determine which graphs have a (j,k)-dominating function.

Proposition 1.1. *For positive integers j and k, a graph G has a (j,k)-dominating function if and only if $(1+\delta)j \geq k$, where δ denotes the minimum degree of a vertex of G.*

Proof. Suppose G has a (j,k)-dominating function. Then the sum of function values in the closed neighbourhood of any vertex of minimum degree must be at least k. Since $f(x) \leq j$ for all $x \in V$, this implies $(1+\delta)j \geq k$.

On the other hand, if $(1+\delta)j \geq k$, then the function f that assigns j to each vertex is a (j,k)-dominating function. ∎

Consider the following array

$$\begin{array}{cccc}
\gamma_{(1,1)} & \gamma_{(1,2)} & \gamma_{(1,3)} & \cdots \\
\gamma_{(2,1)} & \gamma_{(2,2)} & \gamma_{(2,3)} & \cdots \\
\gamma_{(3,1)} & \gamma_{(3,2)} & \gamma_{(3,3)} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}$$

By Proposition 1.1, for a given graph G, the j-th row extends to $\gamma_{(j,j(1+\delta))}$, after which $\gamma_{(j,k)}$ does not exist. It follows from the definition that, in each row, any entry is less than or equal to all of those on its right which exist, and in each column, any entry that exists is greater than or equal to all of those below it. It is easy to see that $\gamma_{(j,k)}(G) = \gamma_{(k,k)}(G)$ for all integers $j \geq k$. This is the reason the condition $j \leq k$, which appears in the definition of a (j,k)-dominating function in [21], has been removed.

Question 1.2. *Is it true that, for each $k \geq 2$, there exists a graph G_k, with minimum degree k, such that all inequalities arising from rows 1 through $k-1$ of the array are strict, and all inequalities arising from columns 2 through k are strict?*

We close this section by establishing a connection between (j, k)-dominating functions and k-tuple dominating sets (i.e., $(1, k)$-dominating functions) in a related graph.

The *wreath product*, or *lexicographic product* of the graphs G and H is the graph $G\,wr\,H$ with vertex set $\{(v, w) : v \in V(G),\, w \in V(H)\}$ such that vertices (u, x) and (v, y) are adjacent in $G\,wr\,H$ if and only if either $uv \in E(G)$, or $u = v$ and $xy \in E(H)$.

Theorem 1.3. *For positive integers j and k, a graph G has a (j, k)-dominating function of weight w if and only if $G\,wr\,K_j$ has a k-tuple dominating set of cardinality w.*

Proof. Suppose $f : V \to \{0, 1, 2, \ldots, j\}$ is (j, k)-dominating function of G with weight w. For each vertex $u \in V(G)$, we have $f(u) \in \{0, 1, \ldots, j\}$. Hence, let D_u be a set of $f(u)$ vertices of $G\,wr\,K_j$ such that $D_u \subseteq \{(u, z) : z \in V(K_j)\}$. Let $D = \bigcup_{u \in V(G)} D_u$. Then $|D| = \sum_{u \in V(G)} |D_u| = \sum_{u \in V(G)} f(u) = w$. Further, for each vertex $v \in V(G)$ we have $\sum_{x \in N[v]} f(x) = \sum_{x \in N[v]} |D_x| \geq k$, so that by the structure of the wreath product $|N[(v, z)] \cap D| \geq k$ for any $z \in V(K_j)$. Therefore D is a k-tuple dominating set of $G\,wr\,K_j$ of cardinality w.

Now suppose D is a k-tuple dominating set of $G\,wr\,K_j$ of cardinality w. For each vertex $u \in V(G)$, let $D_u = \{(u, z) : z \in V(K_j)\} \cap D$ and $f(u) = |D_u|$. Then $f(u) \in \{0, 1, \ldots, j\}$, and also $\sum_{u \in V(G)} f(u) = \sum_{u \in V(G)} |D_u| = |D| = w$. By the structure of the wreath product we have $\sum_{x \in N[v]} |D_x| = \sum_{x \in N[v]} f(x) \geq k$ for each vertex $v \in V(G)$. Therefore f is a (j, k)-dominating function of G with weight w. ■

2. EFFICIENT (j, k)-DOMINATING FUNCTIONS

Let v and w be different vertices of K_{t+2}, where $t \geq 1$. By Proposition 1.1, the graph $K_{t+2} - vw$ has a (j, k)-dominating function if and only if $(t + 1)j \geq k$. We claim that it has an efficient (j, k)-dominating function if and only if $tj \geq k$. Note that an efficient (j, k)-dominating function must assign the value 0 to v and w, otherwise the sum of function values in the neighbourhood of any other vertex is greater than k. If $tj \geq k$, then any function $f : V \to \{0, 1, \ldots, j\}$ that assigns 0 to v and w and has weight k on the complete subgraph $V \setminus \{v, w\}$ is an efficient (j, k)-dominating function. Now suppose $k > tj$. Then any (j, k)-dominating function must assign a positive weight to v and w, and cannot be efficient.

The example in the previous paragraph can be generalized as follows. A *dominating vertex* of a graph G is a vertex adjacent to all other vertices of G.

Proposition 2.1. *Let G be a graph with a dominating vertex. Then G has an efficient (j,k)-dominating function if and only if $j|D| \geq k$, where D is the set of dominating vertices. Furthermore, any efficient (j,k)-dominating function of G must assign the value 0 to all vertices in $V \setminus D$.*

Proof. Suppose G has an efficient (j,k)-dominating function, f. We claim f must assign the value 0 to all vertices in $V \setminus D$. Suppose $v \in V \setminus D$ is such that $f(v) > 0$. Let u be a vertex such that $uv \notin E$. Then the sum of the function values in the closed neighbourhood of u does not equal the sum of the function values in the closed neighbourhood of any vertex in D (which equals the weight of f). This proves the claim. Since $f(x) > 0$ only if $x \in D$, it follows that $j|D| \geq k$.

Now suppose $j|D| \geq k$. Then there exists a (j,k)-dominating function which has weight exactly k on the complete subgraph induced by D, and assigns the value 0 to all vertices of $V \setminus D$. Such a function is an efficient (j,k)-dominating function of G. ■

Question 2.2. *Is it true that, for every pair of positive integers (j,k) with $\frac{k}{1+\delta} < j \leq k$, there exists a graph with an efficient (j,k)-dominating function but no efficient $(j-1,k)$-dominating function?*

The following theorem implies that there is only one possible weight an efficient (j,k)-dominating function of a graph G can have $\gamma_{(j,k)}(G)$.

Theorem 2.3 [21]**.** *Let j and k be positive integers. If f is an efficient (j,k)-dominating function of the graph G, the weight of f is $\gamma_{(j,k)}(G)$.*

A natural question is whether there is a bound of the form $\gamma_{(j,k)}(G) \leq c_{j,k} \cdot \gamma(G)$ for all graphs G, where $c_{j,k}$ is a constant. We now show that the answer is yes if $j = k$ and no if $j < k$.

Proposition 2.4. *Let G be a graph. Then $\gamma_{(k,k)}(G) \leq k\gamma(G)$. Further, if G has an efficient dominating set, then $\gamma_{(k,k)}(G) = k\gamma(G)$.*

Proof. Let D be a minimum dominating set of G, and $f : V \to \{0, 1, \ldots, k\}$ be defined by

$$f(x) = \begin{cases} k, & x \in D, \\ 0, & x \notin D. \end{cases}$$

Then f is a (k,k)-dominating function of G with weight $k\gamma(G)$, from which it follows that $\gamma_{(k,k)}(G) \leq k\gamma(G)$.

If D is an efficient dominating set, then $|D| = \gamma(G)$. The function f is then an efficient (k,k)-dominating function of G. It follows from Theorem 2.3 that the weight of f is $\gamma_{(k,k)}(G)$. ■

Proposition 2.5. *Let j and k be positive integers such that $j < k$. For any constant $c_{j,k}$ there exists a graph G such that $\gamma_{(j,k)}(G) > c_{j,k}\gamma(G)$.*

Proof. Let $c_{j,k}$ be a given constant. Let H be a graph with $n \geq 2$ vertices. Let $p > c_{j,k}$ and let G be constructed from H by attaching p vertices of degree 1 at each vertex of H. Since $\gamma(G) = n$, we have

$$\gamma_{(j,k)}(G) \geq \gamma_{(k-1,k)}(G) = (k-1)n + np > np > c_{j,k}\gamma(G).$$

■

We now relate the existence of certain efficient (k,k)-dominating functions to the existence of efficient dominating sets.

Proposition 2.6. *Let $k \geq 2$ be an integer. A graph G has an efficient (k,k)-dominating function that takes only the values 0 and k if and only if it has an efficient dominating set. Further, if every efficient (k,k)-dominating function of G takes only the values 0 and k, then G has a unique efficient dominating set.*

Proof. Suppose f is an efficient (k,k)-dominating function of G that takes only the values 0 and k. Then $f = kg$, where $g : V(G) \to \{0,1\}$ is the characteristic function of an efficient dominating set. Thus G has efficient dominating set.

Conversely, suppose G has an efficient dominating set D. If g is the characteristic function of D, then $f = kg$ is an efficient (k,k)-dominating function of G.

Finally, suppose D_1 and D_2 are different efficient dominating sets of G with characteristic functions f_1 and f_2, respectively. Since $D_1 \neq D_2$ there is a vertex x such that $f_1(x) = 1$ and $f_2(x) = 0$. Then $(k-1)f_1 + f_2$ is an efficient (k,k)-dominating function of G that does not take only the values 0 and k (since $k \geq 2$). Therefore, if every efficient (k,k)-dominating function of G takes only the values 0 and k, then G has a unique efficient dominating set. ■

The condition that the efficient (k,k)-dominating function takes only the values 0 and k cannot be removed for all graphs. The function that assigns the value 1 to each vertex of the cycle C_n is an efficient $(3,3)$-dominating function of C_n, but C_n has an efficient dominating set if and only if $n \equiv 0 \pmod 3$. The condition can be removed for strongly chordal graphs; see Corollary 3.5.

The following results are similar to those in the literature for various types of dominating sets. See [6], for example. Recall that the maximum degree of a vertex of the graph G is denoted by Δ.

Proposition 2.7. *Let G be a graph on n vertices. If G has an efficient (j,k)-dominating function, then*

$$\frac{kn}{1+\Delta} \leq \gamma_{(j,k)}(G) \leq \frac{kn}{1+\delta}.$$

Proof. Suppose f is an efficient (j,k)-dominating function of G. Since, for each vertex w, the quantity $f(w)$ appears $1+deg(w)$ times in $\sum_{v\in V}\sum_{x\in N[v]} f(x)$, and this sum equals kn, we have

$$(1+\delta)\gamma_{(j,k)} = (1+\delta)\sum_{x\in V} f(x) \le \sum_{v\in V}\sum_{x\in N[v]} f(x) \le (1+\Delta)\sum_{x\in V} f(x) = (1+\Delta)\gamma_{(j,k)}$$

from which the result follows. ■

Corollary 2.8. *If G is an r-regular graph with n vertices and an efficient (j,k)-dominating function f, then $\gamma_{(j,k)}(G) = \frac{kn}{1+r}$.*

Corollary 2.9. *If G is an r-regular graph with n vertices which has an efficient (j,k)-dominating function, then $(1+r)$ divides kn.*

The fact that the necessary condition in the above corollary is not in general sufficient is suggested by the fact that it in no way depends on j. By Proposition 1.1, the condition $(1+\delta)j \ge k$ is also necessary. These two conditions together are also not sufficient. For example, it is easy to see that, for $t \ge 1$, the complete bipartite graph $K_{2t+1,2t+1}$ has no efficient $(t+1,t+1)$-dominating function even though both conditions are satisfied.

A graph can have an efficient (j,k)-dominating function even though it has no efficient dominating set, as was noted above for any cycle C_n where $n \not\equiv 0 \pmod 3$. More generally, if k is a positive integer then a $(k-1)$-regular graph G has an efficient (j,k)-dominating function for all integers j with $1 \le j \le k$: define $f : V \to \{0,1,2,\ldots,j\}$ by $f(x) = 1$ for all $x \in V$. Such a graph can have an efficient dominating set only if k divides $|V|$.

We now give further examples of graphs with an efficient (k,k)-dominating function and no efficient dominating set.

Suppose first that k is even. Let ℓ be a positive integer, and G_ℓ be the cubic graph obtained from a cycle $v_1, v_2, \ldots, v_{8\ell}$ by joining vertex v_i to vertex $v_{i+4\ell}$ for $i = 1,2,\ldots,4\ell$. Then G_ℓ has no efficient dominating set. The function that assigns $k/2$ to each vertex with an odd subscript and 0 to each vertex with an even subscript is an efficient $(k/2,k)$-dominating function, and hence an efficient (k,k)-dominating function.

Now suppose k is odd. Let $\ell \ge k \ge 3$ be an integer, and H_ℓ be any graph obtained from a cycle $v_1, v_2, \ldots, v_{3\ell}$ by adding a new vertex v and joining it to k vertices in the set $\{v_i : i \equiv 1 \pmod 3\}$. Then H_ℓ has no efficient dominating set. The function f defined by $f(x) = 0$ and

$$f(v_i) = \begin{cases} 0, & i \equiv 0 \pmod 3, \\ 1, & i \equiv 1 \pmod 3, \\ k-1, & i \equiv 2 \pmod 3, \end{cases}$$

is an efficient $(k-1,k)$-dominating function, and hence an efficient (k,k)-dominating function. Figure 1 shows H_3.

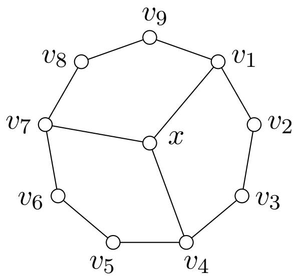

Figure 1. The graph H_3.

3. STRONGLY CHORDAL GRAPHS

In this section we show that the class of strongly chordal graphs has the property that the existence of an efficient (j,k)-dominating function is equivalent to the existence of an efficient dominating set. We show further that every (j,k)-dominating function of a strongly chordal graph is a sum of characteristic functions of efficient dominating sets.

A graph G is *strongly chordal* if its vertices can be ordered $v_1, v_2, \ldots, v_n$ such that for each i, j, k and l the following conditions are satisfied.

(a) If $i < j < k$ and $v_iv_j, v_iv_k \in E$, then $v_jv_k \in E$;

(b) If $i < j$, $k < l$ and $v_iv_k, v_iv_l, v_jv_k \in E$, then $v_jv_l \in E$.

Since any vertex ordering that satisfies (a) is a perfect elimination ordering, every strongly chordal graph is chordal. A forbidden subgraph characterization of strongly chordal graphs was independently given by Farber [8, 9], and Chang and Nemhauser [5].

A $(0,1)$-matrix is called *balanced*, if it does not have a submatrix which is the incidence matrix of an odd cycle, and *totally balanced* if it does not have a submatrix which is the incidence matrix of any cycle of length at least 3.

Totally balanced matrices, strongly chordal graphs, and integer / linear programming are connected in the following theorems. Note that a graph G has an efficient dominating set if and only if there is a $(0,1)$-vector $\mathbf{x}$ such that $(A+I)\mathbf{x} = 1$, where A is the adjacency matrix of G.

Theorem 3.1 [8, 9]**.** *A graph G is strongly chordal if and only if the matrix $A(G)+I$ is totally balanced, where A is the adjacency matrix of G.*

Theorem 3.2 [10]**.** *If the matrix B is balanced, then the polyhedron $Bx = 1$, $x \geq 0$ is either empty, or every vertex has $(0,1)$-coordinates.*

If a strongly chordal graph G has an efficient (j,k)-dominating function, then the polyhedron $\mathcal{P}$ determined by $(A+I)\mathbf{x}=1$, $x \geq 0$ is non-empty. By Theorem 3.2, each vertex of $\mathcal{P}$ corresponds to an efficient dominating set of G. The characteristic function of an efficient dominating set is an efficient $(1,1)$-dominating function. We now show that the efficient $(1,1)$-dominating functions corresponding to vertices of $\mathcal{P}$ are the building blocks of the efficient (j,k)-dominating functions of G.

Theorem 3.3. *For every integer $k \geq 1$, if a strongly chordal graph G has an efficient (j,k)-dominating function f, then there exist efficient $(1,1)$-dominating functions $g_1, g_2, \ldots, g_k$ such that $f = g_1 + g_2 + \cdots + g_k$.*

Proof. The proof is by induction on k. The statement is clearly true when $k=1$ since an efficient $(1,1)$-dominating function is the characteristic function of an efficient dominating set. Suppose it is true when $k = t-1$, where $t \geq 2$.

Let G be a strongly chordal graph with an efficient (s,t)-dominating function f. Then, setting $x_v = \frac{1}{t}f(v)$ for each $v \in V$ gives a solution $\mathbf{x}$ to $(A+I)\mathbf{x}=1$, $x \geq 0$. Since the polyhedron $\mathcal{P}$ determined by the equation is non-empty, by Theorem 3.2 its vertices have $(0,1)$-coordinates. Note that if $\mathbf{x} \in \mathcal{P}$, then $0 \leq x_v \leq 1$ for all vertices $v \in V(G)$, so $||\mathbf{x}||_2 \leq \sqrt{|V(G)|}$. In particular, $\mathcal{P}$ is bounded and thus is the convex hull of its extreme points.

Hence $\mathbf{x}$ is a (possibly trivial) convex combination of vertices of $\mathcal{P}$. Recall that each vertex of $\mathcal{P}$ corresponds to an efficient dominating set of T. Therefore there exists an efficient $(1,1)$-dominating function g_t such that $g_t(v) \leq f(v)$ for all $v \in V$.

Since f is an efficient (s,t)-dominating function of G and $t \geq 2$, the function $f - g_t$ is an efficient $(s, t-1)$-dominating function of G. By the induction hypothesis there exist efficient $(1,1)$-dominating functions $g_1, g_2, \ldots, g_{t-1}$ such that $f - g_t = g_1 + g_2 + \cdots + g_{t-1}$. Hence f can be expressed as the sum of t efficient $(1,1)$-dominating functions of G. ■

The path P_5 is strongly chordal and has an efficient $(4,4)$-dominating function f that assigns 0 to the middle vertex of the path and 2 to all other vertices. The function g is the sum of two copies of the characteristic function of the efficient dominating set consisting of the first and fourth vertices of the path, and two copies of the characteristic function of the efficient dominating set consisting of the second and fifth vertices of the path. In particular, note that if g_4 is any one of these characteristic functions, then $g_4(v) < f(v)$ for all vertices $v \in V$.

Corollary 3.4. *Let G be a strongly chordal graph. If G has an efficient (j,k)-dominating function, then G has an efficient dominating set.*

The converse of Corollary 3.4 does not hold when $j < k$. The star $K_{1,n}$, $n \geq 2$, is a strongly chordal graph with an efficient dominating set. It has efficient (j, k)-dominating function only if $j = k$. The converse of the corollary holds when $j = k$. This statement strengthens Theorem 2.6 for strongly chordal graphs.

Corollary 3.5. *Let G be a strongly chordal graph. Then G has an efficient (k, k)-dominating function if and only if it has an efficient dominating set.*

Proof. The forward implication of the first statement follows from Corollary 3.4, and the backwards implication follows from Proposition 2.6. ∎

Problem 3.6. Given a graph G with an efficient (k, k)-dominating function, find bounds for the smallest positive integer j such that it has an efficient (j, k)-dominating function.

Theorem 3.3 makes it possible to connect efficient dominating sets in strongly chordal graphs and efficient (k, k)-dominating functions in a stronger way than in the two corollaries above.

Theorem 3.7. *Let G be a strongly chordal graph. For any integer $k \geq 2$.*

1. *If G has an efficient (k, k)-dominating function, then every efficient (k, k)-dominating function of G takes only the values 0 and k if and only if G has a unique efficient dominating set.*
2. *There exists an efficient $(1, k)$-dominating function of G if and only if G has k pairwise disjoint efficient dominating sets.*
3. *There exists an efficient (k, k)-dominating function of G that takes a value t such that $1 < t < k$ at some vertex if and only if G has at least two different efficient dominating sets.*

Proof. Let G be a strongly chordal graph that has an efficient (k, k)-dominating function. We prove each statement in turn.

1. If G has an efficient (k, k)-dominating function, then, by Corollary 3.5, G has an efficient dominating set (irrespective of the values the function takes). To prove the converse, we prove the contrapositive. Suppose G has two different efficient dominating sets D_1 and D_2 with characteristic functions f_1 and f_2, respectively. Since $D_1 \neq D_2$, there exists a vertex x such that $f_1(x) = 0$ and $f_2(x) = 1$. Since $k \geq 2$, the function $(k-1)f_1 + f_2$ is an efficient (k, k)-dominating function of G that takes the value $1 < k$.

2. Suppose f is an efficient $(1, k)$-dominating function of G. By Theorem 3.3 there exist efficient $(1, 1)$-dominating functions $g_1, g_2, \ldots, g_k$ such that $f = g_1 + g_2 + \cdots + g_k$. Let D_i be the efficient dominating set whose characteristic function is g_i, $1 \leq i \leq k$. Since $f(x) \in \{0, 1\}$ for all $x \in V$, the sets $D_1, D_2, \ldots, D_k$ are pairwise disjoint. Conversely, if G has k pairwise disjoint efficient dominating

sets, then the sum of their respective characteristic functions is an efficient $(1, k)$-dominating function.

3. Suppose f is an efficient (k, k)-dominating function of G that takes a value t such that $1 < t < k$ at the vertex x. Then, as in the proof of statement 2, f is the sum of characteristic functions of (not necessarily different) efficient dominating sets $D_1, D_2, \ldots, D_k$. Since $f(x) = t$ and $t < k$, the vertex x belongs to $t < k$ of these sets. Therefore G has at least two different efficient dominating sets. Conversely, suppose G has at least two different efficient dominating sets with characteristic functions f_1 and f_2, respectively. Then $tf_1 + (k - t)f_2$ is an efficient (k, k)-dominating function of G that takes the value t at some vertex. ■

The efficient dominating sets in statement 3 of the above theorem need not have non-empty intersection, as is demonstrated by the example immediately following Theorem 3.3. More generally, if f_1 and f_2 are the characteristic functions of the two efficient dominating sets of P_5, then for $a, b \in \mathbb{Z}$ with $1 \leq a \leq b$ and $a + b \geq 3$, the function $af_1 + bf_2$ is an efficient $(a + b, a + b)$-dominating function of P_5 that takes a value strictly between 1 and $a + b$ at some vertex.

4. Complexity

The problem of deciding whether a given graph has an efficient dominating set (i.e., an efficient $(1, 1)$-dominating function) is NP-complete, even when restricted to chordal graphs [22]. Chellali and others proved that the problem of deciding whether a given graph has an efficient $(1, 2)$-dominating function is NP-complete [6]. We show that the problem of deciding whether a given graph has an efficient (j, k)-dominating function is NP-complete for all j and k, where $k \geq 2$. It follows that the problem of deciding whether a given graph has an efficient (j, k)-dominating function is NP-complete for all positive integers j and k.

Let G be a graph. For $p \geq 1$, the p-th power of G is the graph G^p with the same vertex set as G in which vertices u and v are adjacent if and only if the distance between u and v in G is less than or equal to p.

Lemma 4.1. *Let j and k be integers such that $k \geq 2$. Let X_k be the $(k - 1)$-st power of the cycle $v_0, v_1, \ldots, v_{2k^2-k-1}, v_0$ of length $2k^2 - k$. Then X_k has an efficient (j, k)-dominating function and, if f is an efficient (j, k)-dominating function of X_k, then $f(v_0) = f(v_{2k-1}) = f\big(v_{2(2k-1)}\big) = \cdots = f\left(v_{(k-1)(2k-1)}\right)$.*

Proof. To see the first statement, let $f : V \to \{0, 1\}$ be

$$f(v_i) = \begin{cases} 1, & i \equiv 0, 1, \ldots, k - 1 \pmod{2k - 1}, \\ 0, & i \equiv k, k + 1, \ldots, 2k - 2 \pmod{2k - 1}. \end{cases}$$

Then the closed neighbourhood of every vertex v_i contains exactly k vertices to which f assigns the value 1. Thus f is an efficient $(1,k)$-dominating function of X_k, and hence an efficient (j,k)-dominating function of X_k.

To see the second statement, observe that since

$$\sum_{x\in N[v_{k-1}]} f(x) = \sum_{i=0}^{2k-2} f(v_i) = k$$

and

$$\sum_{x\in N[v_k]} f(x) = \sum_{i=1}^{2k-1} f(v_i) = k,$$

it follows that $f(v_0) = f(v_{2k-1})$. Similarly, $f(v_{2k-1}) = f(v_{2(2k-1)}) = \cdots = f(v_{(k-1)(2k-1)})$. ■

We prove the main result in this section using a transformation from monotone 1-in-3-SAT, the variant of SAT which is described at the start of the proof (also, see [11], problem L04). NP-completeness of monotone 1-in-3-SAT follows from Schaefer's Dichotomy Theorem [7].

Theorem 4.2. *Let j and k be integers such that $k \geq 2$. The problem of deciding whether a given graph has an efficient (j,k)-dominating function is NP-complete.*

Proof. The transformation is from monotone 1-in-3-SAT. Suppose we are given an instance of monotone 1-in-3-SAT with variables $x_1, x_2, \ldots, x_n$ and 3-variable clauses $c_1, c_2, \ldots, c_m$ such that no clause contains a negated variable. Construct a graph G as follows.

Corresponding to each variable x_i there is a copy X_i of the $(k-1)$-st power of the cycle $v_{i,0}, v_{i,1}, \ldots, v_{i,2k^2-k-1}, v_{i,0}$ of length $2k^2-k$. Corresponding to each clause c_t there is a vertex ℓ_t. If clause $c_t = x_{t_1} \vee x_{t_2} \vee x_{t_3}$, then for $1 \leq q \leq 3$ add edges joining ℓ_t to vertices $v_{t_q,0}, v_{t_q,2k-1}, \ldots, v_{t_q,(k-1)(2k-1)}$. This completes the construction of G. The transformation can clearly be carried out in polynomial time.

We claim that there is a satisfying truth assignment in which each clause contains exactly one true variable if and only if G has an efficient (j,k)-dominating function.

Suppose there is a satisfying truth assignment in which each clause contains exactly one true variable. Define the function $f : V(G) \to \{0,1\}$ as follows. If x_i is true, then the restriction of f to X_i is such that $f(v_{i,0}) = 1$, and if x_i is false, then the restriction of f to X_i is such that $f(v_{i,0}) = 0$. All other f-values for X_i are assigned according to Lemma 4.1. Set $f(\ell_t) = 0$, $1 \leq t \leq m$. Since each clause contains exactly one true variable, it follows from Lemma 4.1, that the sum of function values in the closed neighbourhood of each vertex ℓ_t equals k. Since the same holds for all other vertices by the construction of G and the definition of f, the function f is an efficient $(1,k)$-dominating function of G. Therefore, f is an efficient (j,k)-dominating function of G.

Now suppose G has an efficient (j,k)-dominating function f. We claim that $f(\ell_t)=0$, $1\le t\le m$. Without loss of generality assume $f(\ell_1)>0$, and ℓ_1 is adjacent to $v_{1,0}, v_{1,2k-1}, \ldots, v_{1,(k-1)(2k-1)}$. We will obtain a contradiction. Since $f(\ell_1)>0$, we have

$$\sum_{x\in N_{X_1}[v_{1,t(2k-1)}]} f(x) < k, \quad 0\le t\le k-1.$$

But the set $\{v_{1,t(2k-1)} : 0\le t\le k-1\}$ is an efficient dominating set of X_1, so that

$$\sum_{i=0}^{2k^2-k-1} f(v_{1,i}) < k^2.$$

On the other hand, the set $\{v_{1,t(2k-1)+1} : 0\le t\le k-1\}$ is an efficient dominating set of X_1, and none of the vertices in this set are adjacent to ℓ_1, so that

$$\sum_{x\in N_{X_1}[v_{1,t(2k-1)+1}]} f(x) = k, \quad 0\le t\le k-1.$$

But this implies

$$\sum_{i=0}^{2k^2-k-1} f(v_{1,i}) = k^2,$$

a contradiction. This proves the claim.

Define a truth assignment for $x_1, x_2, \ldots, x_n$ by setting x_i to be true if $f(v_{i,0}) > 0$, and false if $f(v_{i,0})=0$. Since

$$\sum_{x\in N[\ell_t]} f(x) = k$$

for $1\le t\le m$, it follows from Lemma 4.1 that each clause contains exactly one true variable. Therefore the given instance of monotone 1-in-3-SAT has a satisfying truth assignment in which each clause contains exactly one true variable.

This completes the proof. ■

One can observe that, in the above proof, the function f must be $(0,1)$-valued.

Corollary 4.3. *Let j and k be integers. The problem of deciding whether a given graph has an efficient (j,k)-dominating function is NP-complete.*

Corollary 4.4. *For every integer $k \geq 1$ the problem of deciding whether a given graph has an efficient k-tuple dominating set is NP-complete.*

By Theorem 3.2 there is a polynomial-time algorithm to decide whether a strongly chordal graph has an efficient dominating set (also see [4, 8]). The following then becomes an immediate consequence of Corollary 3.5.

Corollary 4.5. *There is a polynomial-time algorithm to determine whether a strongly chordal graph has an efficient (k, k)-dominating function.*

Proof. By Corollary 3.5 this is equivalent to the existence of an efficient dominating set in G, which is in turm equivalent to the existence of a feasible solution to $(A + I)\mathbf{x} = 1$, $\mathbf{x} \geq 0$. ■

5. Efficient (2, 2)-Dominating Functions of Trees

Since trees are strongly chordal graphs, by Corollary 3.5 a tree has an efficient $(2, 2)$-dominating function if and only if it has an efficient dominating set. One such function is obtained by assigning the value 2 to the vertices of an efficient dominating set, and the value 0 to all other vertices. Theorem 3.7 describes the circumstances under which these are the only efficient $(2, 2)$-dominating functions, or when a tree has an efficient $(2, 2)$-dominating function that takes only the values 0 and 1, or that takes all of the values 0, 1, and 2.

Let us introduce the following notations.

- $\mathcal{T}_{02}$ be the set of trees which are such that every efficient $(2, 2)$-dominating function uses only the values 0 and 2, i.e., the trees with a unique efficient dominating set.
- $\mathcal{T}_{01}$ be the set of trees which have an efficient $(2, 2)$-dominating function using only 0 and 1, that is, the set of trees which have an efficient double dominating set or, equivalently, two disjoint efficient dominating sets.
- $\mathcal{T}_{012}$ be the set of trees that admit a surjective efficient $(2, 2)$-dominating function. These are the trees with two different efficient dominating sets.

No tree has an efficient $(2, 2)$-dominating function that takes only the values 1 and 2 (consider the sum of function values in closed neighbourhood of a leaf versus the closed neighbourhood of its neighbour). Thus the set of trees which have an efficient $(2, 2)$-dominating function is $\mathcal{T} = \mathcal{T}_{01} \cup \mathcal{T}_{012} \cup \mathcal{T}_{02}$. Note that $\mathcal{T}_{02}$ is disjoint from $\mathcal{T}_{01} \cup \mathcal{T}_{012}$.

Stars with at least three vertices have an efficient dominating set, and hence an efficient $(2, 2)$-dominating function. These belong to $\mathcal{T}_{02}$, and do not belong to $\mathcal{T}_{01} \cup \mathcal{T}_{012}$. The path P_5 belongs to $\mathcal{T}_{01}$ and does not belong to $\mathcal{T}_{012}$. The tree obtained by subdividing each edge of a star with at least four vertices once belongs to $\mathcal{T}_{012}$ and does not belong to $\mathcal{T}_{01}$. The tree obtained from two copies of P_5 by adding an edge joining their central vertices belongs to $\mathcal{T}_{01} \cap \mathcal{T}_{012}$.

The proof of the following is straightforward, and omitted.

Proposition 5.1. 1. *The path $P_n \in \mathcal{T}_{01}$ if and only if $n \equiv 2 \pmod 3$.*

2. *No path belongs to $\mathcal{T}_{012}$.*
3. *The path $P_n \in \mathcal{T}_{02}$ if and only if $n \not\equiv 2 \pmod 3$.*

No tree has a $(1,k)$-dominating function when $k \geq 3$ (consider the closed neighbourhood of a leaf). Further, no tree has more than two disjoint efficient dominating sets (consider how a leaf is dominated). Thus, statement (2) in the theorem above has no direct analogy for $k \geq 3$.

Problem 5.2. For $k > 2$, characterize the trees that have k different efficient dominating sets.

We now construct a set $\mathcal{T}_c$ of vertex-coloured trees in such a way that, for each tree $T \in \mathcal{T} = \mathcal{T}_{01} \cup \mathcal{T}_{012} \cup \mathcal{T}_{02}$, all efficient $(2,2)$-dominating functions of T are also described. The vertices of each tree $T_f \in \mathcal{T}_c$ are painted one of the three colours red, blue and white. These colours represent the vertices that are assigned the values 2, 1, 0, respectively, by the efficient $(2,2)$-dominating function f of T. The set $\mathcal{T}$ is obtained from $\mathcal{T}_c$ by ignoring the vertex colours.

Initially $P_1, P_2 \in \mathcal{T}_c$. The vertex of P_1 is painted red, and the vertices of P_2 are both painted blue. In the sequel we refer to these a *red* P_1 and a *blue* P_2, respectively. Suppose $T' \in \mathcal{T}_c$. Add to $\mathcal{T}_c$ any tree that can be obtained from T' by a sequence of the following operations.

- **O1.** Add a new vertex w by adding an edge from w to a red vertex of T'. Paint the vertex w white.
- **O2.** Add a path x_1x_2 consisting of two new vertices by joining x_1 to a white vertex of T'. Paint x_1 white and x_2 red.
- **O3.** Add a path $y_1y_2y_3$ consisting of three new vertices by joining y_1 to a blue vertex of T'. Paint y_1 white, and both y_2 and y_3 blue.
- **O4.** Add a path $P_5 = z_1z_2z_3z_4z_5$ consisting of five new vertices by joining z_3 to a white vertex of T'. Paint z_3 white, and z_1, z_2, z_4 and z_5 blue.

Theorem 5.3. *Let T be a tree. A function $f : V \to \{0,1,2\}$ is an efficient $(2,2)$-dominating function of T if and only if a vertex-coloured copy of T can be constructed, starting from a red P_1 or a blue P_2 using a sequence of operations* O1, O2, O3, O4.

Proof. By construction, if $T \in \mathcal{T}_c$, then T has an efficient $(2,2)$-dominating function in which the vertices in $f^{-1}(2), f^{-1}(1), f^{-1}(0)$ are painted red, blue and white, respectively.

For the converse, suppose the tree T has an efficient $(2,2)$-dominating function f. We construct a vertex-coloured tree $T_f \in \mathcal{T}_c$ such that the red, blue, and white vertices correspond to $f^{-1}(2), f^{-1}(1), f^{-1}(0)$, respectively.

If $|V(T)| \leq 2$, then T_f is either a red P_1, a blue P_2, or a P_2 with a red vertex and a white vertex. Each of these vertex-coloured trees belongs to $\mathcal{T}_c$.

For $n \geq 3$, suppose that for any tree with fewer than n vertices and any efficient $(2,2)$-dominating function g of T, there exists a vertex-coloured tree $T_g \in \mathcal{T}_c$ such that the red, blue, and white vertices correspond to $g^{-1}(2), g^{-1}(1), g^{-1}(0)$, respectively.

Let T be a tree with n vertices and an efficient $(2,2)$-dominating function f. Let ℓ be an end of a longest path of T. Then ℓ is a leaf. Let s be the unique neighbour of ℓ. We consider cases depending on $f(\ell)$.

Case 1. $f(\ell) = 0$. Then $f(s) = 2$, and the restriction, g, of f to $V(T-\ell)$ is an efficient $(2,2)$-dominating function of $T-\ell$. By the induction hypothesis, there exists a vertex-coloured tree $(T-\ell)_g \in \mathcal{T}_c$ such that the red, blue, and white vertices correspond to $g^{-1}(2), g^{-1}(1), g^{-1}(0)$, respectively. The vertex-coloured tree T_f can be constructed from $(T-\ell)_g$ by operation O1.

Case 2. $f(\ell) = 2$. Then $f(s) = 0$. We claim that the vertex s must have degree 2. The claim is true if T is P_3, so assume $T \neq P_3$. If s is adjacent to a leaf $\ell' \neq \ell$, then $f(\ell') = 2$. But then the sum of function values in the closed neighbourhood of s is greater than 2. If s is adjacent to two internal vertices of T, then there is longer path than the one under consideration. Both possibilities lead to a contradiction, which proves the claim. Let $t \neq \ell$ be adjacent to s. Then $f(t) = 0$. Thus the restriction, g, of f to $V(T - \{\ell, s\})$ is an efficient $(2,2)$-dominating function of $T - \{\ell, s\}$. By the induction hypothesis, there exists a vertex-coloured tree $(T - \{\ell, s\})_g \in \mathcal{T}_c$ such that the red, blue, and white vertices correspond to $g^{-1}(2), g^{-1}(1), g^{-1}(0)$, respectively. The vertex-coloured tree T_f can be constructed from $(T - \{\ell, s\})_g$ by operation O2.

Case 3. $f(\ell) = 1$. Then $f(s) = 1$ and, similar to the previous case, the vertex s must have degree 2. Let $t \neq \ell$ be adjacent to s. Then $f(t) = 0$. We consider subcases depending on the degree of t.

Suppose the vertex t has degree 2. Then, the restriction, g, of f to $V(T - \{\ell, s, t\})$ is an efficient $(2,2)$-dominating function of $T - \{\ell, s, t\}$. By the induction hypothesis, there exists a vertex-coloured tree $(T - \{\ell, s, t\})_g \in \mathcal{T}_c$ such that the red, blue, and white vertices correspond to $g^{-1}(2), g^{-1}(1), g^{-1}(0)$, respectively. The vertex-coloured tree T_f can be constructed from $(T - \{\ell, s, t\})_g$ by operation O3.

Suppose the vertex t has degree 3. Since $f(t) = 0$ and $f(s) = 1$, then the vertex t cannot be adjacent to a leaf. By the choice of ℓ, the vertex t must have a neighbour $s' \neq s$ which is a adjacent to a leaf. Since t has a neighbour assigned the value 1, we must have $f(s') = 1$. This, in turn, implies that s' has degree 2. It follows that the leaf ℓ' adjacent to s' has $f(\ell') = 1$. Since t has two neighbours assigned the value 1, its third neighbour must be assigned the value 0. Then, the restriction, g_1, of f to $V(T - \{\ell, s, t, s', \ell'\})$ is an efficient $(2,2)$-dominating function of $T - \{\ell, s, t, s', \ell'\}$. By the induction hypothesis, there exists a vertex-coloured tree $(T - \{\ell, s, t, s', \ell'\})_{g_1} \in \mathcal{T}_c$ such that the red, blue, and white vertices correspond to $g^{-1}(2), g^{-1}(1), g^{-1}(0)$, respectively. The vertex-coloured tree T_f can be constructed from $(T - \{\ell, s, t, s', \ell'\})_{g_1}$ by operation O4.

Finally, suppose t has degree at least 4. Then, as above, t is not adjacent to a leaf and all neighbours of t, except possibly one, are adjacent to leaves. Since t has a neighbour assigned the value 1, it has one other neighbour assigned the value 1, and all other neighbours are assigned the value 0. Hence t has a neighbour s' which is adjacent to a leaf and is such that $f(s'') = 0$. The vertex s'' can be adjacent to only one leaf, say ℓ'', and $f(\ell'') = 2$. Then, the restriction, g_2 of f to $V(T - \{\ell'', s''\})$ is an efficient $(2,2)$-dominating function of $T - \{\ell'', s''\}$. By the induction hypothesis, there exists a vertex-coloured tree $(T - \{\ell'', s''\})_{g_2} \in \mathcal{T}_c$ such that the red, blue, and white vertices correspond to $g_2^{-1}(2), g_2^{-1}(1), g_2^{-1}(0)$, respectively. The vertex-coloured tree T_f can be constructed from $T - \{\ell'', s''\})_{g_2}$ by operation O2.

The result now follows by induction. ∎

Suppose the tree T has an efficient $(2,2)$-dominating function f. We use T_f to denote the vertex-coloured tree in $\mathcal{T}_c$ such that the red, blue, and white vertices correspond to $f^{-1}(2), f^{-1}(1), f^{-1}(0)$, respectively. If, in the construction of T_f, operations O1, O2, O3, and O4 have been applied a total of k_1, k_2, k_3, and k_4 times, respectively, then the weight of f is $2 + 2k_2 + 2k_3 + 4k_4$.

A constructive characterization of the trees that have an efficient dominating set was given by Bange, Barkauskas and Slater [2]. A simpler constructive characterization is an immediate consequence of the previous theorem. Recall that an efficient dominating set of a tree T corresponds to an efficient $(2,2)$-dominating function of T that takes only the values 0 and 2.

Corollary 5.4. *Let T be a tree. Then T has an efficient $(2,2)$-dominating function f, that takes only the values 0 and 2 if and only if T_f can be constructed from a red P_1 using operations O1 and O2.*

Proof. Suppose T has an efficient $(2,2)$-dominating function f, that takes only the values 0 and 2. Then, the tree $T_f \in \mathcal{T}_c$ has no blue vertices. Hence T_f is constructed from a red P_1 using operations O1 and O2.

On the other hand, if the vertex-coloured tree T_f can be constructed from a red P_1 using operations O1 and O2, then T_f has no blue vertices. Therefore the efficient $(2,2)$-dominating function f takes only the values 0 and 2. ∎

Bange, Barkauskas and Slater [2] also gave a constructive characterization of the trees with two disjoint efficient dominating sets. Essentially the same characterization can be derived from Theorem 5.3 similarly to Corollary 5.4.

Corollary 5.5 [2]**.** *Let T be a tree. Then T has an efficient $(2, 2)$-dominating function f, that takes only the values 0 and 1 (i.e., $T \in \mathcal{T}_{01}$) if and only if T_f can be constructed from a blue P_2 using only* O3 *and* O4*.*

It is possible to modify our construction to use "light blue" and "dark blue" vertices so that if $T \in \mathcal{T}_{01}$, then the set of light blue vertices and the set of dark blue vertices form two disjoint efficient dominating sets of T. To do this, O3 needs to be replaced by O3$'$ and O3$''$ in a way that takes into account whether the new vertices are being joined to a light blue vertex or a dark blue vertex. Operation O4 also needs to be modified so that the two efficienct dominating sets of P_5 are distinguished by the two shades of blue.

The trees in $\mathcal{T}_{012}$ can be similarly characterized.

Corollary 5.6. *Let T be a tree. Then T has a surjective efficient $(2, 2)$-dominating function f (i.e., $T \in \mathcal{T}_{012}$) if and only T_f can be constructed from a red P_1 or a blue P_2 such that at least one of* O1, O2 *is used, and at least one of* O3, O4 *is used.*

The trees with two different efficient dominating sets can also be constructively characterized.

Corollary 5.7. *Let T be a tree. Then T has an efficient $(2, 2)$-dominating function f, which assigns the value 1 to at least one vertex (i.e., $T \in \mathcal{T}_{01} \cup \mathcal{T}_{012}$) if and only if T_f can be constructed from a red P_1 or a blue P_2 such that at least one of* O3 *and* O4 *is used.*

Problem 5.8. Characterize the trees that have a unique efficient dominating set.

We close with a proposition which suggests that the problem above may be difficult.

Proposition 5.9. *Any tree T is an induced subtree of a tree T' which has a unique efficient dominating set.*

Proof. Let T be a tree. Let T' be the corona of T with respect to K_1, that is, for each vertex $x \in V(T)$ add a new vertex x' and join it to x. Then T is a subtree of T' and the set $\{x' : x \in V(T)$ is the unique efficient dominating set of T'. ∎

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