

## ELIMINATION PROPERTIES FOR MINIMAL DOMINATING SETS OF GRAPHS

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### Abstract

A dominating set of a graph is a vertex subset such that every vertex not in the subset is adjacent to at least one in the subset. In this paper we study whenever there exists a new dominating set contained (respectively, containing) the subset obtained by removing a common vertex from the union of two minimal dominating sets. A complete description of the graphs satisfying such elimination properties is provided.

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### 1. INTRODUCTION

A set  $D$  of vertices of a graph  $G$  is a *dominating set* if every vertex of  $G$  belongs to  $D$  or is adjacent to some vertex of  $D$ . A *minimal dominating set* is a dominating set with no proper dominating subsets. Domination in graphs is a widely

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researched branch of graph theory, both from a theoretical and an algorithmic point of view (see [5, 6]).

Since the deletion of a vertex from a minimal dominating set results in a non-dominating set of  $G$ , a natural question that arises at this point is to find minimal dominating sets, not containing a vertex  $u$ , close to the minimal dominating sets that contain  $u$ . This approach can be done in several ways, for example, by considering *exchange type properties* or by considering *elimination type properties*. Roughly speaking, the exchange and the elimination type properties can be understood as follows: the deletion of a vertex can be seen as a node that fails in a network modelled by the graph and so, in this framework, with the exchange type properties we want to change the node that fails to another node; while with the elimination type properties we want to replace the node that fails by nodes belonging to some special sets containing it. These kinds of exchange and elimination type properties appear in some other contexts, such as determining sets and resolving sets of graphs or computational geometry (see for example [2, 3]). As far as the authors know, neither exchange nor elimination type properties have been considered for minimal dominating sets of graphs.

This paper deals with the study of two elimination type properties for minimal dominating sets of graphs. Namely, here we consider the *lower* and the *upper* elimination properties defined as follows.

We say that a graph  $G$  satisfies the *lower elimination property* if for any two different minimal dominating sets  $D_1, D_2$  of  $G$  such that  $D_1 \cap D_2 \neq \emptyset$  and for every vertex  $u \in D_1 \cap D_2$ , there is a minimal dominating set contained in  $(D_1 \cup D_2) \setminus \{u\}$ , (observe that the lower elimination property is the circuit elimination property in Matroid Theory, see [10, 11]). Analogously, we say that  $G$  satisfies the *upper elimination property* if for any two different minimal dominating sets  $D_1, D_2$  of  $G$  such that  $D_1 \cap D_2 \neq \emptyset$  and for every vertex  $u \in D_1 \cap D_2$ , there is a minimal dominating set that contains  $(D_1 \cup D_2) \setminus \{u\}$ .

Notice that a trivial class of graphs satisfying both elimination properties is the class of graphs whose minimal dominating sets are pairwise disjoint (that is, graphs where the family of minimal dominating sets forms a partition of the vertex set or, in other words, graphs where every vertex belongs to exactly one minimal dominating set). Even though the elimination type properties that we study have not been considered previously, several results concerning pairwise disjoint minimal dominating sets have been studied from different points of view (see for instance [1, 7]).

The goal of this paper is to characterize the family of graphs satisfying these elimination properties. We have completely solved this problem obtaining, surprisingly, that both families are exactly the same, that is, a graph satisfies the upper elimination property if and only if it satisfies the lower elimination property.

The structure of the paper is as follows. Some definitions and notations on graphs are recalled in Section 2. In Section 3 we provide a complete description of the graphs such that their family of minimal dominating sets forms a partition (Proposition 3), and we present a complete description of the graphs satisfying the lower elimination property (Theorem 4). The results concerning the upper elimination property are gathered in Section 4. Namely, we present the description of general families of subsets satisfying the upper elimination property (Theorem 10); we demonstrate that the upper elimination property implies the lower elimination property (Corollary 11); and, in the domination case, we prove that both properties are equivalent (Corollary 12). Finally, in Section 5 we summarise our results and we present some open problems.

## 2. PRELIMINARIES

A *graph*  $G$  is an ordered pair  $(V(G), E(G))$  comprising a finite set  $V(G)$  of *vertices* together with a (possibly empty) set  $E(G)$  of *edges* which are two-element subsets of  $V(G)$  (for general references on graph theory see [4, 12]). If  $e = \{x, y\} \in E(G)$ , then  $x$  and  $y$  are said to be *adjacent vertices*. An edge  $\{x, y\}$  will be denoted by  $xy$ . A vertex  $x$  of a graph  $G$  is a *universal* vertex if  $x$  is adjacent to  $y$  for every vertex  $y \in V(G) \setminus \{x\}$ . For every  $W \subseteq V(G)$ , we denote by  $G[W]$  the subgraph of  $G$  induced by  $W$ .

A *dominating set* for a graph  $G = (V(G), E(G))$  is a subset  $D$  of  $V(G)$  such that every vertex not in  $D$  is adjacent to at least one member of  $D$ . Since any superset of a dominating set of  $G$  is also a dominating set of  $G$ , the collection  $D(G)$  of the dominating sets of a graph  $G$  is a monotone increasing family of subsets of  $V(G)$ . Therefore,  $D(G)$  is uniquely determined by the family  $\min(D(G))$  of its inclusion-minimal elements. Let us denote by  $\mathcal{D}(G)$  the family of the *inclusion-minimal dominating sets* of the graph  $G$ . This family has been studied under different points of view, see for instance [8, 9].

Dominating sets of a graph are closely related to independent sets. An *independent set* of a graph  $G$  is a set of vertices such that no two of them are adjacent. It is clear that an independent set is also a dominating set if and only if it is an inclusion-maximal independent set (see [4]). Therefore, any inclusion-maximal independent set of a graph is necessarily also an inclusion-minimal dominating set.

The *join*  $G_1 \vee G_2$  of two graphs  $G_1$  and  $G_2$  with pairwise disjoint sets of vertices  $V(G_1)$  and  $V(G_2)$  is the graph with vertex set  $V(G_1) \cup V(G_2)$  and edge set  $E(G_1) \cup E(G_2) \cup \{x_1x_2 : x_1 \in V(G_1), x_2 \in V(G_2)\}$ . The following lemma deals with the minimal dominating sets of the join of two graphs. Its proof is a straightforward consequence of the definitions.

**Lemma 1.** *Let  $G_1, G_2$  be two graphs with pairwise disjoint set of vertices. Then  $\mathcal{D}(G_1 \vee G_2) = \mathcal{D}(G_1) \cup \mathcal{D}(G_2) \cup \{\{x_1, x_2\} : x_i \in V(G_i) \text{ and } x_i \text{ is not a universal vertex in } G_i, i \in \{1, 2\}\}$ . In particular,  $\mathcal{D}(G_1 \vee G_2) = \mathcal{D}(G_1) \cup \mathcal{D}(G_2)$  if and only if the graph  $G_1$  or the graph  $G_2$  is isomorphic to a complete graph  $K_n$ .*

### 3. LOWER ELIMINATION PROPERTY

In this section we characterize those graphs  $G$  that satisfy the lower elimination property. Recall that a graph  $G$  satisfies the lower elimination property if for every pair of different sets  $D_1, D_2 \in \mathcal{D}(G)$  such that  $D_1 \cap D_2 \neq \emptyset$  and every  $x \in D_1 \cap D_2$ , there exists  $D_3 \in \mathcal{D}(G)$  such that  $D_3 \subseteq (D_1 \cup D_2) \setminus \{x\}$ .

**Remark 2.** Notice that the case  $D_1 = D_2$  has no sense, because if  $D_1 = D_2$ , then  $D_3 \subseteq (D_1 \cup D_2) \setminus \{x\} = D_1 \setminus \{x\} \subsetneq D_1$ , and so a contradiction is obtained because  $D_1$  and  $D_3$  are two inclusion-minimal dominating sets.

Trivial examples of graphs satisfying the lower elimination property are complete graphs and empty graphs. In general, every graph whose minimal dominating sets are pairwise disjoint satisfies the lower elimination property. The following proposition provides a complete description of these graphs.

**Proposition 3.** *The minimal dominating sets of a graph  $G$  are pairwise disjoint if and only if  $G$  is isomorphic to a complete graph  $K_n$ , or to an empty graph  $\overline{K_n}$ , or to the join  $K_r \vee \overline{K_s}$  of a complete graph and an empty graph.*

**Proof.** Clearly, if  $G$  is isomorphic to  $K_n$  or  $\overline{K_n}$ , then  $\mathcal{D}(G)$  is a partition of  $V(G)$ . Now, by applying Lemma 1 we get that if  $G$  is isomorphic to the join graph  $K_r \vee \overline{K_s}$  then the minimal dominating sets of  $G$  are pairwise disjoint.

Conversely, assume that  $G$  is a graph such that  $\mathcal{D}(G)$  is a pairwise disjoint family of subsets. Suppose that  $G$  is neither a complete graph nor an empty graph. Let  $D$  be a maximal independent set of vertices of  $G$  with  $|D| \geq 2$  (such a set  $D$  exists because  $G$  is not a complete graph), and let  $z \in V(G) \setminus D$  (such a vertex  $z$  exists because  $G$  is not an empty graph). We claim that  $z$  is adjacent to every vertex of  $D$ . Indeed, if  $z$  is not adjacent to some  $x \in D$ , then the independent set  $\{z, x\}$  can be extended to a maximal independent set, so a minimal dominating set, that has a non-empty intersection with the minimal dominating set  $D$ , contradicting the hypothesis. Thus every vertex of  $V(G) \setminus D$  is adjacent to every vertex of  $D$ . Hence  $G = G[D] \vee G[V \setminus D]$ , where  $G[D]$  is an empty graph. We want to prove that  $V \setminus D$  induces a complete subgraph of  $G$ . Suppose on the contrary that there exists a non-universal vertex  $z$  in  $G[V \setminus D]$ . From Lemma 1 it follows that, for every  $x \in D$ , the vertex set  $\{x, z\}$  is a minimal dominating set of  $G$  that has a non-empty intersection with the

minimal dominating set  $D$ . This leads us to a contradiction because  $\mathcal{D}(G)$  is a pairwise disjoint family of subsets. ■

From Proposition 3 we get that, in general, minimal dominating sets do not need to be pairwise disjoint (that is, the family of minimal dominating sets of a graph does not necessarily provide a partition of the vertex set of the graph). Therefore, in order to find a complete description of the graphs satisfying the lower elimination property we must look for graphs  $G$  satisfying this property and with non-disjoint minimal dominating sets. We stress that from our characterization result (see Theorem 4) it follows that there are only two families of graphs  $G$  fulfilling both conditions: the complete multipartite graph  $K_{2,\dots,2}$  and the join  $K_r \vee K_{2,\dots,2}$  of a complete graph  $K_r$  and a complete multipartite graph  $K_{2,\dots,2}$ .

**Theorem 4.** *A graph  $G$  satisfies the lower elimination property if and only if  $G$  is isomorphic to one of the following graphs.*

1. *The complete graph  $K_n$ .*
2. *The empty graph  $\overline{K_n}$ .*
3. *The complete multipartite graph  $K_{2,\dots,2}$ .*
4. *The join of a complete graph and an empty graph  $K_r \vee \overline{K_s}$ .*
5. *The join of a complete graph and a complete multipartite graph  $K_r \vee K_{2,\dots,2}$ .*

**Proof.** First, suppose that  $G$  is isomorphic to one of the graphs of the list above. If  $G$  is isomorphic to  $K_n$ , or to  $\overline{K_n}$ , or to  $K_r \vee \overline{K_s}$ , then the minimal dominating sets of  $G$  are pairwise disjoint and, so,  $G$  satisfies the lower elimination property. In addition, the complete multipartite graph  $K_{2,\dots,2}$  satisfies the lower elimination property because its minimal dominating sets are all the vertex subsets of cardinality 2. Moreover, from Lemma 1 it follows that: if  $G'$  is a graph satisfying the lower elimination property, then the join graph  $K_r \vee G'$  of a complete graph  $K_r$  and  $G'$  also satisfies the lower elimination property. In particular, we get that the join graph  $K_r \vee K_{2,\dots,2}$  satisfies the lower elimination property. So we conclude that if  $G$  is a graph isomorphic to one of the graphs of the list above, then  $G$  satisfies the lower elimination property.

Now we are going to prove that if a graph  $G$  satisfies the lower elimination property then  $G$  is isomorphic either to  $K_n$ , or to  $\overline{K_n}$ , or to  $K_{2,\dots,2}$ , or to  $K_r \vee \overline{K_s}$ , or to  $K_r \vee K_{2,\dots,2}$ . So, let  $G$  be a graph satisfying the lower elimination property. We claim that, in such a case, the graph  $G$  is a complete multipartite graph. Let us prove our claim. Suppose that  $G$  satisfies the lower elimination property and that  $G$  is not a complete multipartite graph. It is well known that the class of complete multipartite graphs is equivalent to the class of  $\overline{P_3}$ -free graphs. Therefore, since we are assuming that  $G$  is not a complete multipartite graph,

we conclude that  $G$  contains  $\overline{P_3}$  as an induced subgraph. So, there exist vertices  $u, v, w \in V(G)$  such that  $uv \notin E(G)$ ,  $uw \notin E(G)$  and  $vw \in E(G)$ . Consider a maximal independent set  $D_1$  containing  $u$  and  $v$ , and a maximal independent set  $D_2$  containing  $u$  and  $w$ . Both sets  $D_1$  and  $D_2$  are minimal dominating sets with no vertex adjacent to  $u$ . Therefore, the set  $(D_1 \cup D_2) \setminus \{u\}$  has no vertex adjacent to  $u$ , implying that  $(D_1 \cup D_2) \setminus \{u\}$  does not contain a minimal dominating set, which is not possible for a graph that satisfies the lower elimination property. This completes the proof of our claim.

From our claim it follows there are integers  $n_1 \geq \dots \geq n_k \geq 1$  such that the graph  $G$  is isomorphic to the complete multipartite graph  $K_{n_1, \dots, n_k}$ . Let  $V(G) = V_1 \cup \dots \cup V_k$  be the corresponding  $k$ -partition of the vertex set of  $G$  (so  $|V_i| = n_i$  for  $1 \leq i \leq k$ ). Observe that if  $k = 1$  then  $G$  is isomorphic to the empty graph  $\overline{K_n}$ . Therefore, we may assume that  $k \geq 2$ . Clearly, if  $n_1 = 1$  then  $G$  is isomorphic to the complete graph  $K_n$ , while if  $n_1 = 2$  then  $G$  is isomorphic either to a complete multipartite graph  $K_{2, \dots, 2}$  or to a join of a complete graph and a complete multipartite graph  $K_r \vee K_{2, \dots, 2}$ . To complete the proof of the theorem it is enough to show that  $n_2 = \dots = n_k = 1$  whenever  $k \geq 2$  and  $n_1 \geq 3$  (because then we conclude that  $G$  is isomorphic to a join  $K_r \vee \overline{K_s}$ ). Suppose on the contrary that  $k \geq 2$ ,  $n_1 \geq 3$  and  $n_2 \geq 2$ . Let  $x_1, x_2, x_3$  be three different vertices of  $V_1$  and  $y_1, y_2$  two different vertices of  $V_2$ . Clearly, the sets  $D_1 = \{x_1, y_1\}$  and  $D_2 = \{x_2, y_1\}$  are minimal dominating sets of  $G$ . However, the set  $D_1 \cup D_2 \setminus \{y_1\} = \{x_1, x_2\}$  does not contain any minimal dominating set, because  $x_3$  is not dominated by  $\{x_1, x_2\}$ . Thus a contradiction follows because we are assuming that the graph  $G$  satisfies the lower elimination property. ■

#### 4. UPPER ELIMINATION PROPERTY

In this section we characterize the graphs  $G$  that satisfy the upper elimination property. Recall that  $G$  satisfies the upper elimination property if for every pair of different sets  $D_1, D_2 \in \mathcal{D}(G)$  such that  $D_1 \cap D_2 \neq \emptyset$  and every  $x \in D_1 \cap D_2$ , there exists  $D_3 \in \mathcal{D}(G)$  such that  $(D_1 \cup D_2) \setminus \{x\} \subseteq D_3$ .

**Remark 5.** Notice that if  $x \in D_3$ , then  $D_1, D_2 \subseteq D_3$ , and by the minimality of  $D_3$ , we obtain that  $D_3 = D_1$  and  $D_3 = D_2$ , that is not possible because  $D_1 \neq D_2$ . Hence  $x \notin D_3$ . On the other hand, the case  $D_1 = D_2$  would give a trivial property, taking  $D_3 = D_1$ .

As a consequence of Theorem 4, we obtain the following result relating both elimination properties in a graph.

**Proposition 6.** *If a graph  $G$  satisfies the lower elimination property, then it also satisfies the upper elimination property.*

**Proof.** If  $G$  satisfies the lower elimination property, then the graph  $G$  is isomorphic to a graph in the list of Theorem 4. It is easy to check that these graphs also satisfy the upper elimination property. ■

To characterize the graphs satisfying the lower elimination property (Corollary 12), we will use several results concerning elimination properties for general families of subsets. First we need to introduce some terminology.

Let  $\Omega$  be a non-empty finite set. A *clutter*  $\mathcal{H}$  on  $\Omega$ , also known as *Sperner family*, is a family of pairwise non-comparable subsets of  $\Omega$ . Observe that, for a graph  $G$ , the family  $\mathcal{D}(G)$  is a clutter on  $V(G)$ .

The elimination properties defined for the family of minimal dominating sets of a graph can be extended to the more general frame of clutters. Concretely, we say that a clutter  $\mathcal{H}$  satisfies the *lower elimination property* if for every  $A, B \in \mathcal{H}$  with  $A \neq B$  such that  $A \cap B \neq \emptyset$ , and for every  $x \in A \cap B$ , there exists  $C \in \mathcal{H}$  such that  $C \subseteq (A \cup B) \setminus \{x\}$ . Analogously, we say that the clutter  $\mathcal{H}$  satisfies the *upper elimination property* if for every  $A, B \in \mathcal{H}$  with  $A \neq B$  such that  $A \cap B \neq \emptyset$ , and for every  $x \in A \cap B$ , there exists  $C \in \mathcal{H}$  such that  $(A \cup B) \setminus \{x\} \subseteq C$ . Clearly, every clutter whose elements are pairwise disjoint satisfies both the lower and upper elimination property.

Notice that, from the definition, the clutters  $\mathcal{H}$  satisfying the lower elimination property are exactly the set of *circuits of a matroid* (the reader is referred to [10, 11] for general references on matroid theory). However, as far as we know the upper elimination property has not been considered in the literature and we now focus on it.

**Remark 7.** From Proposition 6, if  $\mathcal{H} = \mathcal{D}(G)$  for some graph  $G$  and  $\mathcal{H}$  satisfies the lower elimination property, then it also satisfies the upper one. However, this relationship between both elimination properties does not hold for a general clutter. Let us give an example. On the finite set  $\Omega = \{1, 2, 3, 4\}$  we consider the clutter  $\mathcal{H} = \{\{1, 2, 3\}, \{1, 2, 4\}, \{3, 4\}\}$ . It is easy to check that  $\mathcal{H}$  satisfies the lower elimination property. However, if  $A = \{1, 2, 3\}$ ,  $B = \{1, 2, 4\}$ , then  $1 \in A \cap B$  and  $(A \cup B) \setminus \{1\} = \{2, 3, 4\}$ . This last set is not contained in any element of  $\mathcal{H}$ . So  $\mathcal{H}$  does not satisfy the upper elimination property.

From now on, our aim is to provide a complete characterization of clutters satisfying the upper elimination property (Theorem 10). To this end we study first the elimination properties for uniform clutters. Recall that the *uniform clutter*  $\mathcal{U}_{k,\Omega}$  is the family of all subsets of cardinality  $k \geq 1$  of a finite set  $\Omega$ .

**Proposition 8.** *For every  $1 \leq k \leq |\Omega|$ , the uniform clutter  $\mathcal{U}_{k,\Omega}$  satisfies the lower elimination property. Moreover,  $\mathcal{U}_{k,\Omega}$  satisfies the upper elimination property if and only if  $k \in \{1, 2, |\Omega| - 1, |\Omega|\}$ .*

**Proof.** If  $k = 1$ , then the elements of  $\mathcal{U}_{k,\Omega}$  are pairwise disjoint, and so  $\mathcal{U}_{k,\Omega}$  satisfies the lower elimination property. Next let us prove that  $\mathcal{U}_{k,\Omega}$  satisfies the lower elimination property whenever  $2 \leq k \leq |\Omega|$ . For it, take  $A_1, A_2 \in \mathcal{U}_{k,\Omega}$  with  $A_1 \neq A_2$ , and let  $x \in A_1 \cap A_2$ . Since  $|A_1| = |A_2| = k$ , we have  $|(A_1 \cup A_2) \setminus \{x\}| \geq k$ , and so  $(A_1 \cup A_2) \setminus \{x\}$  contains a subset with cardinality  $k$ . Therefore, there exists  $A_3 \in \mathcal{U}_{k,\Omega}$  such that  $A_3 \subseteq (A_1 \cup A_2) \setminus \{x\}$ .

Now, we must demonstrate that  $\mathcal{U}_{k,\Omega}$  satisfies the upper elimination property if and only if  $k \in \{1, 2, |\Omega| - 1, |\Omega|\}$ . It is straightforward to check that the uniform clutters  $\mathcal{U}_{1,\Omega}$ ,  $\mathcal{U}_{2,\Omega}$ ,  $\mathcal{U}_{|\Omega|-1,\Omega}$  and  $\mathcal{U}_{|\Omega|,\Omega}$  satisfy the upper elimination property. Thus, the proof will be completed by showing that if  $|\Omega| \geq 5$  and  $3 \leq k \leq |\Omega| - 2$ , then the uniform clutter  $\mathcal{U}_{k,\Omega}$  does not satisfy the upper elimination property. So let  $|\Omega| \geq 5$  and  $3 \leq k \leq |\Omega| - 2$ . Set  $\Omega = \{x_1, \dots, x_{|\Omega|}\}$  and let  $A_1 = \{x_1, \dots, x_k\}$  and  $A_2 = (A_1 \cup \{x_{k+1}, x_{k+2}\}) \setminus \{x_1, x_2\}$ . Then  $A_1, A_2 \in \mathcal{U}_{k,\Omega}$  and  $x_k \in A_1 \cap A_2$ . However, there is no  $A \in \mathcal{U}_{k,\Omega}$  containing  $(A_1 \cup A_2) \setminus \{x_k\}$  because  $|(A_1 \cup A_2) \setminus \{x_k\}| = k + 1$ . ■

**Lemma 9.** *Let  $\mathcal{H}$  be a clutter satisfying the upper elimination property. Let  $A_1 \in \mathcal{H}$  be a set of maximum cardinality  $r$  among all elements of  $\mathcal{H}$  with non-empty intersection with some element of  $\mathcal{H}$ . Then,  $r \geq 2$  and, if  $A_2 \in \mathcal{H}$  with  $A_1 \neq A_2$  and  $A_1 \cap A_2 \neq \emptyset$ , then the following statements hold.*

- (i)  $|A_1| = |A_2| = r$ ,  $|A_1 \cup A_2| = r + 1$  and  $|A_1 \cap A_2| = r - 1$ .
- (ii)  $\mathcal{U}_{r, A_1 \cup A_2} \subseteq \mathcal{H}$ .
- (iii) If  $r \geq 3$  and  $A \in \mathcal{H}$  with  $(A_1 \cup A_2) \cap A \neq \emptyset$ , then  $A \in \mathcal{U}_{r, A_1 \cup A_2}$ .

**Proof.** First of all observe that  $r \geq 2$  because otherwise, if  $r = 1$ , then  $A_1 = \{x_1\}$ , and so  $A_1 \cap A = \emptyset$  for all  $A \in \mathcal{H}$  different from  $A_1$  (recall that a clutter is a family of pairwise non-comparable subsets).

Let us prove statement (i).

By assumption we have  $A_1, A_2 \in \mathcal{H}$  with  $A_1 \neq A_2$  and  $A_1 \cap A_2 \neq \emptyset$ . Let  $x \in A_1 \cap A_2$  and let  $A_3 \in \mathcal{H}$  such that  $(A_1 \cup A_2) \setminus \{x\} \subseteq A_3$  (such a set  $A_3 \in \mathcal{H}$  exists because  $\mathcal{H}$  satisfies the upper elimination property).

We claim that  $A_3 = (A_1 \cup A_2) \setminus \{x\}$  and that  $|(A_1 \cup A_2) \setminus \{x\}| = r$ . Let us prove our claim. As  $\mathcal{H}$  is a clutter,  $A_1 \setminus \{x\} \notin \mathcal{H}$ . Hence  $A_1 \setminus \{x\} \subsetneq A_3$ , and thus  $|A_3| \geq r$ . But  $A_1 \cap A_3 \neq \emptyset$ . So by the choice of  $A_1$  we have that  $|A_3| \leq |A_1| = r$ . Therefore we conclude that  $|A_3| = r$ . Moreover, the sets  $A_1$  and  $A_2$  are non-comparable (because  $A_1, A_2 \in \mathcal{H}$  are different). So  $A_1 \setminus A_2 \neq \emptyset$  and  $A_2 \setminus A_1 \neq \emptyset$ . Hence  $A_1 \setminus \{x\} \subsetneq (A_1 \cup A_2) \setminus \{x\} \subseteq A_3$ , so  $r - 1 = |A_1 \setminus \{x\}| < |(A_1 \cup A_2) \setminus \{x\}| \leq |A_3| = r$ . Therefore  $|(A_1 \cup A_2) \setminus \{x\}| = r$ , and thus  $A_3 = (A_1 \cup A_2) \setminus \{x\}$ .

Now we will prove that  $|A_1 \setminus A_2| = 1$  and that  $|A_2 \setminus A_1| = 1$ . Clearly  $|A_2 \setminus A_1| = 1$  because  $r = |(A_1 \cup A_2) \setminus \{x\}| = |A_1 \setminus \{x\}| + |A_2 \setminus A_1| = (r - 1) + |A_2 \setminus A_1|$ . Let us demonstrate that  $|A_1 \setminus A_2| = 1$ . Suppose on the contrary that  $|A_1 \setminus A_2| \geq 2$



(recall that  $A_1 \setminus A_2 \neq \emptyset$  because  $A_1, A_2 \in \mathcal{H}$  are different and so they are non-comparable). Let  $\{y_1, y_2\} \subseteq A_1 \setminus A_2$ , with  $y_1 \neq y_2$ . Since  $y_1 \in A_1 \cap A_3$ , by the upper elimination property, there exists  $A_4 \in \mathcal{H}$  such that  $(A_1 \cup A_3) \setminus \{y_1\} \subseteq A_4$ . On one hand,  $y_1 \notin A_2$ . On the other hand,  $(A_1 \cup A_3) \setminus \{y_1\} = (A_1 \cup A_2) \setminus \{y_1\}$ . Hence it follows that  $A_2 \subseteq (A_1 \cup A_3) \setminus \{y_1\} \subseteq A_4$ . Therefore,  $A_2 = A_4$  because  $A_2, A_4 \in \mathcal{H}$  are non-comparable. However,  $y_2 \in A_1 \setminus A_2 = A_1 \setminus A_4$  and  $y_2 \in (A_1 \cup A_3) \setminus \{y_1\} \subseteq A_4$ , which is a contradiction.

At this point we are going to complete the proof of statement (i). From  $A_3 = (A_1 \cup A_2) \setminus \{x\}$  and  $|A_1| = |A_3| = r$ , we get that  $|A_1 \cup A_2| = r + 1$ . Then, since  $A_1 \cup A_2 = (A_1 \setminus A_2) \cup (A_2 \setminus A_1) \cup (A_1 \cap A_2)$  and since  $|A_1 \setminus A_2| = |A_2 \setminus A_1| = 1$ , we deduce  $|A_1 \cap A_2| = r - 1$ . Finally, from  $A_2 = (A_2 \setminus A_1) \cup (A_1 \cap A_2)$  we get that  $|A_2| = r$ .

Now let us prove statement (ii).

From statement (i) we have that  $|A_1| = |A_2| = r$ , that  $|A_1 \cup A_2| = r + 1$  and that  $|A_1 \cap A_2| = r - 1$ . Set  $A_1 = \{x_1, \dots, x_r\}$  and  $A_2 = \{x_2, \dots, x_{r+1}\}$  (recall that  $r \geq 2$ ). Let  $A \subseteq A_1 \cup A_2 = \{x_1, x_2, \dots, x_{r+1}\}$  with  $|A| = r$ . We must demonstrate that if  $A \neq A_1, A_2$  then  $A \in \mathcal{H}$ . Without loss of generality we may assume that  $A = \{x_1, x_2, \dots, x_{r+1}\} \setminus \{x_2\}$ . As  $x_2 \in A_1 \cap A_2$ , from the upper elimination property it follows that there exists  $A'_2 \in \mathcal{H}$  such that  $A = (A_1 \cup A_2) \setminus \{x_2\} \subseteq A'_2$ . But  $A'_2 \in \mathcal{H}$  and  $A_1 \cap A'_2 \neq \emptyset$ . Therefore we can apply statement (i) to  $A'_2$ . In this way we deduce that  $|A'_2| = r$ . Therefore  $A = A'_2$  and, in particular,  $A = A'_2 \in \mathcal{H}$ .

Finally, let us demonstrate statement (iii). So, from now on let us assume that  $r \geq 3$ .

Let  $A \in \mathcal{H}$  with  $(A_1 \cup A_2) \cap A \neq \emptyset$  and  $A \neq A_1, A_2$ . We must demonstrate that  $|A| = r$  and that  $A \subseteq A_1 \cup A_2$ .

From statement (i) it follows that the subsets  $A_1$  and  $A_2$  play the same role. Therefore, without loss of generality we may assume that  $A_1 \cap A \neq \emptyset$  and hence, by applying statement (i) to  $A_1$  and  $A$  we conclude that  $|A| = r$ , that  $|A_1 \cup A| = r + 1$  and that  $|A_1 \cap A| = r - 1$ . To finish the proof we must demonstrate that  $A \subseteq A_1 \cup A_2$ . Suppose, for contradiction, that  $A \not\subseteq A_1 \cup A_2$ . Let  $y \in A \setminus (A_1 \cup A_2)$  and, as before, set  $A_1 = \{x_1, \dots, x_r\}$  and  $A_2 = \{x_2, \dots, x_{r+1}\}$ . Since  $r \geq 3$  and  $|A_1 \cap A| = r - 1$ , hence  $A_2 \cap A \neq \emptyset$  and so, by applying statement (i) to  $A_2$  and  $A$  we get that  $|A_2 \cup A| = r + 1$  and that  $|A_2 \cap A| = r - 1$ . Consequently,  $A = \{x_2, \dots, x_r, y\}$ . Choose  $A'_1 \in \mathcal{U}_{r, A_1 \cup A_2}$  so that  $\{x_1, x_{r+1}\} \subseteq A'_1$ . Then, by statement (ii),  $A'_1 \in \mathcal{H}$  and now, by applying statement (i) to  $A'_1$  and  $A$  we get that  $|A'_1 \cap A| = r - 1$ . Therefore, a contradiction is obtained because  $A'_1 \cap A \subsetneq \{x_2, \dots, x_r\}$ . This completes the proof of the lemma. ■

**Theorem 10.** *A clutter  $\mathcal{H}$  on  $\Omega$  satisfies the upper elimination property if and only if  $\mathcal{H} = \mathcal{U}_{k_1, \Omega_1} \cup \dots \cup \mathcal{U}_{k_q, \Omega_q}$  where  $\{\Omega_1, \dots, \Omega_q\}$  is a family of pairwise disjoint sets of  $\Omega$  and  $k_i \in \{1, 2, |\Omega_i| - 1, |\Omega_i|\}$  for every  $i \in \{1, \dots, q\}$ .*

**Proof.** Let  $\mathcal{H}$  be a clutter on  $\Omega$ . From Proposition 8 it follows that, if there exists a family  $\{\Omega_1, \dots, \Omega_q\}$  of pairwise disjoint subsets of  $\Omega$  such that  $\mathcal{H} = \mathcal{U}_{k_1, \Omega_1} \cup \dots \cup \mathcal{U}_{k_q, \Omega_q}$  and  $k_i \in \{1, 2, |\Omega_i| - 1, |\Omega_i|\}$  for  $1 \leq i \leq q$ , then the clutter  $\mathcal{H}$  satisfies the upper elimination property. Therefore we only must demonstrate the converse.

So, let  $\mathcal{H}$  be a clutter on  $\Omega$  satisfying the upper elimination property.

Suppose first that all the elements of  $\mathcal{H}$  are pairwise disjoint. Let  $\mathcal{H} = \{A_1, \dots, A_m\}$ . In this case set  $q = m$  and  $\Omega_i = A_i$ . Then,  $\{\Omega_1, \dots, \Omega_q\}$  is a family of pairwise disjoint sets of  $\Omega$  and  $\mathcal{H} = \{A_1\} \cup \dots \cup \{A_m\} = \mathcal{U}_{k_1, \Omega_1} \cup \dots \cup \mathcal{U}_{k_q, \Omega_q}$  where  $k_i = |\Omega_i|$ .

Now suppose that not all elements of  $\mathcal{H}$  are pairwise disjoint.

Let  $A_1 \in \mathcal{H}$  be a set of maximum cardinality  $r \geq 2$  among all elements of  $\mathcal{H}$  with non-empty intersection with some element of  $\mathcal{H}$ .

We claim that there exists  $\Omega_1 \subseteq \Omega$  such that  $A_1 \in \mathcal{U}_{r, \Omega_1} \subseteq \mathcal{H}$  and such that for every  $A \in \mathcal{H}$ , either  $A \cap \Omega_1 = \emptyset$  or  $A \in \mathcal{U}_{r, \Omega_1}$ . To proof our claim, we distinguish two cases,  $r \geq 3$  and  $r = 2$ .

First assume that  $r \geq 3$ . In this case, let  $A_2 \in \mathcal{H}$  be such that  $A_1 \cap A_2 \neq \emptyset$  and set  $\Omega_1 = A_1 \cup A_2$ . By statement (ii) of Lemma 9, we know that  $A_1 \in \mathcal{U}_{r, \Omega_1} \subseteq \mathcal{H}$ . Moreover, if  $A \in \mathcal{H}$  with  $A \neq A_1$  and  $A \cap \Omega_1 \neq \emptyset$  then, from statement (iii) of Lemma 9 we get that  $A \in \mathcal{U}_{r, \Omega_1}$ . Therefore  $\Omega_1$  fulfils the required conditions.

Now we must demonstrate our claim in the case  $r = 2$ . To do this, let us consider the binary relation  $\sim$  on  $\Omega$  defined as follows. If  $a, b \in \Omega$ , then  $a \sim b$  if and only if  $a = b$  or  $\{a, b\} \in \mathcal{H}$ . Clearly, the binary relation  $\sim$  is reflexive and symmetric. Moreover, if  $a, b, c$  are distinct elements of  $\Omega$  with  $a \sim b$  and  $b \sim c$ , then by the upper elimination property and the fact that  $r = 2$ , we have  $\{a, c\} \in \mathcal{H}$ . Therefore the binary relation  $\sim$  is transitive and so  $\sim$  is an equivalence relation. Since  $r = 2$ , we can consider the equivalence class containing  $A_1$ . Let  $\Omega_1$  be this equivalence class. Then, by applying statements (ii) and (iii) of Lemma 9 we get that  $\Omega_1$  satisfies the required conditions.

From our claim we get that there exists a subset  $\Omega_1 \subseteq \Omega$  and an integer  $k_1 \geq 2$  such that  $A_1 \in \mathcal{U}_{k_1, \Omega_1} \subseteq \mathcal{H}$  and such that the following property is fulfilled: every element of  $\mathcal{H}$  intersecting  $\Omega_1$  is in  $\mathcal{U}_{k_1, \Omega_1}$ . At this point let us consider the clutter  $\mathcal{H}_1 = \mathcal{H} \setminus \mathcal{U}_{k_1, \Omega_1}$ . Clearly  $\mathcal{H}_1$  is a clutter on  $\Omega \setminus \Omega_1$  satisfying the upper elimination property. Therefore we can repeat the reasoning for  $\mathcal{H}_1$  and so on. By means of this recursive procedure we get that there exists a family  $\{\Omega_1, \Omega_2, \dots, \Omega_q\}$  of pairwise disjoint subsets of  $\Omega$  such that  $\mathcal{H} = \mathcal{U}_{k_1, \Omega_1} \cup \dots \cup \mathcal{U}_{k_q, \Omega_q}$ . Finally, notice that since the clutter  $\mathcal{H}$  satisfies the upper elimination property, hence the uniform clutters  $\mathcal{U}_{k_i, \Omega_i}$  also satisfy the upper elimination property and, therefore, by Proposition 8,  $k_i \in \{1, 2, |\Omega_i| - 1, |\Omega_i|\}$ . This completes the proof of the theorem. ■

**Corollary 11.** *If a clutter  $\mathcal{H}$  satisfies the upper elimination property, then  $\mathcal{H}$  satisfies the lower elimination property.*

**Proof.** By Theorem 10, if a clutter  $\mathcal{H}$  satisfies the upper elimination property, then  $\mathcal{H} = \mathcal{U}_{k_1, \Omega_1} \cup \dots \cup \mathcal{U}_{k_q, \Omega_q}$  where  $\{\Omega_1, \dots, \Omega_q\}$  is a family of pairwise disjoint sets of  $\Omega$  and  $k_i \in \{1, 2, |\Omega_i| - 1, |\Omega_i|\}$  for every  $i \in \{1, \dots, q\}$ . Let  $A_1$  and  $A_2$  be two different elements of  $\mathcal{H}$ , and let  $x \in A_1 \cap A_2$ . In such a case,  $A_1 \cap A_2 \neq \emptyset$  and, therefore,  $A_1, A_2 \in \mathcal{U}_{k_i, \Omega_i}$  for some  $i \in \{1, \dots, q\}$ . By Proposition 8, the uniform clutter  $\mathcal{U}_{k_i, \Omega_i}$  satisfies the lower elimination property. So, there exists  $A \in \mathcal{U}_{k_i, \Omega_i} \subseteq \mathcal{H}$  such that  $A \subseteq (A_1 \cup A_2) \setminus \{x\}$ . ■

Observe that we can apply this corollary whenever  $\mathcal{H}$  is a domination clutter, that is, whenever  $\mathcal{H} = \mathcal{D}(G)$  for some graph  $G$ . So, by combining the above corollary and Proposition 6 we conclude that even though for general clutters both elimination properties are not equivalent, these conditions are the same for domination clutters. This result is stated in the following corollary.

**Corollary 12.** *A graph  $G$  satisfies the upper elimination property if and only if  $G$  satisfies the lower elimination property.*

## 5. CONCLUSION AND OPEN PROBLEMS

In this paper we have studied the lower and the upper elimination properties in general clutters and in domination clutters. For a general clutter, having the lower elimination property is equivalent to being the set of circuits of a matroid. One of the main goals of this paper is to present a complete characterization and description of the domination clutters satisfying this property. In addition, in regard to the upper elimination property, we provide a complete characterization and description of those clutters satisfying the upper elimination property both for general clutters and for domination clutters. Finally, we demonstrate that, while for a general clutter the upper elimination property and the lower one are not equivalent, in the particular case of the domination clutter of a graph both properties are the same.

The problems we have addressed suggest two future research lines. On one hand, to relate matroids with different families of vertex subsets of graphs and, on the other hand, to generalize the elimination properties from pairs of subsets to a large number of them.

Concretely, given a graph  $G$  and a family  $\Lambda(G)$  of vertex subsets of  $G$ , the question is to decide when  $\Lambda(G)$  defines a matroid  $\mathcal{M}$ . In this paper we answer this question in the case  $\Lambda(G) = \mathcal{D}(G)$ . Other interesting families could be the independent vertex sets or the vertex cover sets.

Regarding the second research line we could define the *r-upper* and the *r-lower* elimination properties as follows. We say that a graph  $G$  satisfies the *r-lower elimination property* if for every  $r$  different minimal dominating sets  $D_1, \dots, D_r$  of  $G$  and for every vertex  $x \in D_1 \cap \dots \cap D_r$ , there is a minimal dominating set contained in  $(D_1 \cup \dots \cup D_r) \setminus \{x\}$ . Analogously, we say that  $G$  satisfies the *r-upper elimination property* if for every  $r$  different minimal dominating sets  $D_1, \dots, D_r$  of  $G$  and for every vertex  $x \in D_1 \cap \dots \cap D_r$ , there is a minimal dominating set that contains  $(D_1 \cup \dots \cup D_r) \setminus \{x\}$ . Therefore, now it could be interesting to provide a complete characterization and description of graphs satisfying either the *r-upper* or the *r-lower* elimination properties (this paper deals with this issue whenever  $r = 2$ ). Observe that, for  $r = 1$ , the 1-lower elimination property has no sense while the 1-upper elimination property would yield *exchange type properties* (as far as we know, the exchange type properties have not been studied in the context of dominating sets of graphs). Finally we stress that, in general, if  $r \geq 2$ , different values of  $r$  will provide different properties. For instance, consider the graph  $G$  with vertex set  $V(G) = \{1, 2, 3, 4, 5\}$  and edge set  $E(G) = \{12, 13, 14, 15, 34, 45\}$ . Then  $\mathcal{D}(G) = \{\{1\}, \{2, 4\}, \{2, 3, 5\}\}$ . Clearly  $G$  satisfies the 3-upper and the 3-lower elimination properties, however it does not satisfy neither the 2-upper nor the 2-lower ones.

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