Discussiones Mathematicae
Graph Theory 43 (2023) 5–16
https://doi.org/10.7151/dmgt.2353

THE EXISTENCE OF PATH-FACTOR COVERED GRAPHS

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Abstract

A spanning subgraph H of a graph G is called a $P_{\geq k}$-factor of G if every component of H is isomorphic to a path of order at least k, where $k \geq 2$. A graph G is called a $P_{\geq k}$-factor covered graph if there is a $P_{\geq k}$-factor of G covering e for any $e \in E(G)$. In this paper, we obtain two special classes of $P_{\geq 2}$-factor covered graphs. We also obtain two special classes of $P_{\geq 3}$-factor covered graphs. Furthermore, it is shown that these results are all sharp.

Keywords: path-factor, $P_{\geq 2}$-factor covered graph, $P_{\geq 3}$-factor covered graph, claw-free graph, isolated toughness.

2010 Mathematics Subject Classification: 05C70, 05C38.

1. Introduction

We consider only finite simple graph, unless explicitly stated. We refer to [6] for the notation and terminologies not defined here. Let $G = (V(G), E(G))$ be a simple graph, where $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A subgraph H of G is called a spanning subgraph of G if $V(H) = V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is called an induced subgraph of G if every pair of vertices in H which are adjacent in G are also adjacent in H. For $v \in V(G)$, the degree of v in G is denoted by $d_G(v)$. A graph G is said to be claw-free if there is no induced subgraph of G isomorphic to $K_{1,3}$.

For a family of connected graphs $\mathcal{F}$, a spanning subgraph H of a graph G is called an $\mathcal{F}$-factor of G if each component of H is isomorphic to some graph in $\mathcal{F}$. A spanning subgraph H of a graph G is called a $P_{\geq k}$-factor of G if every component of H is isomorphic to a path of order at least k. For example, a $P_{\geq 3}$-factor means a graph factor in which every component is a path of order at least three. A graph G is called a $P_{\geq k}$-factor covered graph if there is a $P_{\geq k}$-factor of G covering e for any $e \in E(G)$.

Since Tutte proposed the well known Tutte 1-factor theorem [15], there are many results on graph factors [2, 3, 8, 9, 16] and $P_{\geq k}$-factors in claw-free graphs and cubic graphs [4, 12, 13]. More results on graph factors can be found in the survey papers and books in [2, 14, 18]. We use $\omega(G)$, $i(G)$ to denote the number of components and isolated vertices of a graph G, respectively. For a subset $X \subseteq V(G)$, $G-X$ denotes the graph obtained from G by deleting all the vertices of X. Akiyama, Avis and Era [1] proved the following theorem, which is a criterion for a graph to have a $P_{\geq 2}$-factor.

Theorem 1 (Akiyama *et al.* [2]). *A graph G has a $P_{\geq 2}$-factor if and only if $i(G-X) \leq 2|X|$ for all $X \subseteq V(G)$.*

Kaneko [10] introduced the concept of a sun and gave a characterization for a graph with a $P_{\geq 3}$-factor. It is perhaps the first characterization of graphs which have a path factor not including P_2. Recently, Kano *et al.* [11] presented a simpler proof for Kaneko's theorem [10].

A graph H is called factor-critical if $H-\{v\}$ has a 1-factor for each $v \in V(H)$. Let H be a factor-critical graph and $V(H) = \{v_1, v_2, \ldots, v_n\}$. By adding new vertices $\{u_1, u_2, \ldots, u_n\}$ together with new edges $\{v_iu_i : 1 \leq i \leq n\}$ to H, the resulting graph is called a sun. Note that, according to Kaneko [10], we regard K_1 and K_2 also as a sun, respectively. Usually, the suns other than K_1 are called big suns. It is called a sun component of $G-X$ if the component of $G-X$ is isomorphic to a sun. We denote by $sun(G-X)$ the number of sun components in $G-X$.

Theorem 2 (Kaneko [10]). *A graph G has a $P_{\geq 3}$-factor if and only if $sun(G-X) \leq 2|X|$ for all $X \subseteq V(G)$.*

Zhang and Zhou [19] proposed the concept of path-factor covered graph, which is a generalization of matching cover. They also obtained a characterization for $P_{\geq 2}$-factor and $P_{\geq 3}$-factor covered graphs, respectively.

Theorem 3 (Zhang *et al.* [19]). *Let G be a connected graph. Then G is a $P_{\geq 2}$-factor covered graph if and only if $i(G-S) \leq 2|S| - \varepsilon(S)$ for all $S \subseteq V(G)$, where $\varepsilon(S)$ is defined by*

$$\varepsilon(S) = \begin{cases} 2 & \text{if } S \neq \emptyset \text{ and } S \text{ is not an independent set,} \\ 1 & \text{if } S \neq \emptyset,\ S \text{ is an independent set and there exists} \\ & \text{a component of } G-S \text{ with at least two vertices,} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4 (Zhang *et al.* [19])**.** *Let G be a connected graph. Then G is a $P_{\geq 3}$-factor covered graph if and only if $sun(G-S) \leq 2|S| - \varepsilon(S)$ for all $S \subseteq V(G)$, where $\varepsilon(S)$ is defined by*

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For a connected graph G, its toughness, denoted by $t(G)$, was first introduced by Chvátal [7] as follows. If G is complete, then $t(G) = +\infty$; otherwise,

$$t(G) = \min\left\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G),\ \omega(G-S) \geq 2\right\}.$$

Bazgan, Benhamdine, Li and Woźniak [5] showed a toughness condition for the existence of a $P_{\geq 3}$-factor in a graph.

Theorem 5 (Bazgan, Benhamdine, Li and Woźniak [5])**.** *Let G be a graph with at least three vertices. If $t(G) \geq 1$, then G includes a $P_{\geq 3}$-factor.*

For a connected graph G, its isolated toughness, denoted by $I(G)$, was first introduced by Yang, Ma and Liu [17] as follows. If G is complete, then $I(G) = +\infty$; otherwise,

$$I(G) = \min\left\{\frac{|S|}{i(G-S)} : S \subseteq V(G),\ i(G-S) \geq 2\right\}.$$

Recently, Zhou and Wu [20] obtained three classes of $P_{\geq 3}$-factor covered graphs.

Theorem 6 (Zhou and Wu [20])**.** *A graph G is a $P_{\geq 3}$-factor covered graph if one the following holds.*

(i) *G is a connected graph with at least three vertices and $t(G) > 2/3$;*

(ii) *G is a connected graph with at least three vertices and $I(G) > 5/3$;*

(iii) *G is a k-regular graph with $k \geq 2$.*

In this paper, we proceed to investigate $P_{\geq k}$-factor covered graphs. We respectively obtain two special classes of $P_{\geq 2}$-factor covered graphs and $P_{\geq 3}$-factor covered graphs. Our main results will be shown in Sections 2 and 3, respectively.

2. $P_{\geq 2}$-Factor Covered Graphs

In this section, we mainly obtain two special classes of $P_{\geq 2}$-factor covered graphs. First, we will give a sufficient condition for a connected claw-free graph to be a $P_{\geq 2}$-factor covered graph as following. Note that the result in Theorem 7 is sharp in the sense that there exists a connected claw-free graph of minimum degree 1, which is not a $P_{\geq 2}$-factor covered graph. An example can be seen in Figure 1.

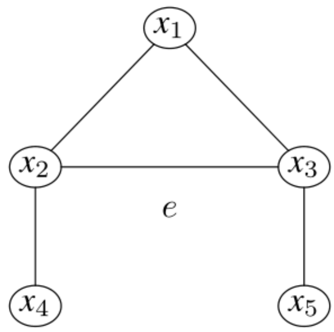

Figure 1. A connected claw-free graph of minimum degree 1 that does not contain any $P_{\geq 2}$-factor covering $e = x_2x_3$.

Theorem 7. *Let G be a connected claw-free graph of minimum degree at least 2. Then G is a $P_{\geq 2}$-factor covered graph.*

Proof. Suppose G is not a $P_{\geq 2}$-factor covered graph. Then by Theorem 3, there exists a subset $S \subseteq V(G)$ such that $i(G-S) > 2|S| - \varepsilon(S)$. In terms of the integrality of $i(G-S)$, we obtain that $i(G-S) \geq 2|S| - \varepsilon(S) + 1$. We will distinguish two cases below to show that G is a $P_{\geq 2}$-factor covered graph.

Case 1. $|S| \leq 1$. If $S = \emptyset$, then $\varepsilon(S) = |S| = 0$ by the definition of $\varepsilon(S)$. It follows easily that

$$i(G) = i(G-S) \geq 2|S| - \varepsilon(S) + 1 = 1.$$

On the other hand, $i(G) \leq \omega(G) = 1$ since G is a connected graph. Combining the results above, we obtain $i(G) = 1$, which contradicts the connectivity of G.

If $|S| = 1$, let $S = \{s\}$. We obtain $\varepsilon(S) \leq 1$ by the definition of $\varepsilon(S)$. If $\varepsilon(S) = 0$, then

$$\omega(G-S) \geq i(G-S) \geq 2|S| - \varepsilon(S) + 1 = 3.$$

Therefore, if either $\varepsilon(S)$ is 0 or 1, then there are at least three components of $G - \{s\}$. It follows easily that there exists a claw with center vertex s in G, a contradiction.

Case 2. $|S| \geq 2$. Let $|S| = k \geq 2$ and $S = \{s_1, s_2, \ldots, s_k\}$. By the definition of $\varepsilon(S)$, we have $\varepsilon(S) \leq 2$. It follows easily that

$$i(G-S) \geq 2|S| - \varepsilon(S) + 1 = 2|S| - 1.$$

Let $i(G-S) = m \geq 2k-1$ and $\{x_1, x_2, \ldots, x_m\}$ be the set of isolated vertices of $G-S$. Since the minimum degree of G is at least two, we immediately obtain the number of edges incident with the vertices in $\{x_1, x_2, \ldots, x_m\}$ is at least $2m$. Because G does not have multiple edges and

$$\frac{2m}{|S|} = \frac{2m}{k} \geq \frac{2(2k-1)}{k} = 4 - \frac{2}{k} \geq 3,$$

there must exist a vertex $s_i \in S$ adjacent to at least three vertices in $\{x_1, x_2, \ldots, x_m\}$ by pigeonhole principle. It follows easily that there exists a claw with center vertex s_i in G, a contradiction.

Combining Case 1 and Case 2, Theorem 7 is proved. ■

Next, we study the relationship between isolated toughness and $P_{\geq 2}$-factor covered graphs, and obtain an isolated toughness condition for the existence of $P_{\geq 2}$-factor covered graphs. The example in Figure 2 shows the sharpness of the results in Theorem 8 in the sense that there exists a connected graph with $I(G) = 2/3$, which is not a $P_{\geq 2}$-factor covered graph.

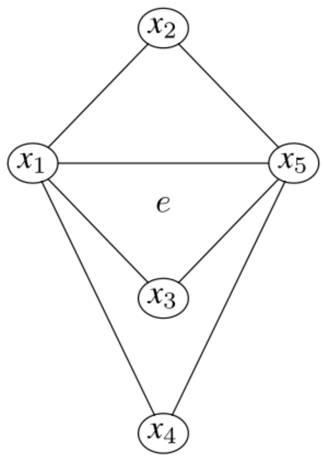

Figure 2. A connected graph with $I(G) = 2/3$ that does not contain any $P_{\geq 2}$-factor covering $e = x_1x_5$.

Theorem 8. *Let G be a connected graph with at least two vertices. If $I(G) > 2/3$, then G is a $P_{\geq 2}$-factor covered graph.*

Proof. If G is a complete graph with at least two vertices, obviously G is a $P_{\geq 2}$-factor covered graph. Thus we may assume that G is a connected graph with at least two vertices and not complete. Suppose G is not a $P_{\geq 2}$-factor covered graph. Then by Theorem 3, there exists a subset $S \subseteq V(G)$ such that $i(G-S) > 2|S| - \varepsilon(S)$. Then, by the integrality of $i(G-S)$, we obtain that $i(G-S) \geq 2|S| - \varepsilon(S) + 1$.

Case 1. $|S| \leq 1$. If $|S| = 0$, by the definition of $\varepsilon(S)$, we have $S = \emptyset$ and $\varepsilon(S) = 0$. It follows immediately that

$$i(G) = i(G-S) \geq 2|S| - \varepsilon(S) + 1 = 1,$$

which contradicts the connectivity of G.

Thus we may assume $|S| = 1$, we have $\varepsilon(S) \leq 1$ by the definition of $\varepsilon(S)$. It follows easily that

$$i(G-S) \geq 2|S| - \varepsilon(S) + 1 \geq 2|S|.$$

By the definition of $I(G)$, we have that

$$I(G) \leq \frac{|S|}{i(G-S)} \leq \frac{1}{2},$$

which contradicts $I(G) > 2/3$.

Case 2. $|S| \geq 2$. In this case, it follows from the definition of $\varepsilon(S)$ that $\varepsilon(S) \leq 2$, which implies that

$$i(G-S) \geq 2|S| - \varepsilon(S) + 1 \geq 2|S| - 1 \geq 3.$$

Thus we immediately obtain

$$|S| \leq \frac{i(G-S)+1}{2}.$$

By the definition of $I(G)$, we have

$$I(G) \leq \frac{|S|}{i(G-S)} \leq \frac{i(G-S)+1}{2i(G-S)} \leq \frac{1}{2} + \frac{1}{2i(G-S)} \leq \frac{1}{2} + \frac{1}{6} = \frac{2}{3},$$

which contradicts $I(G) > 2/3$.

This completes the proof of Theorem 8. ■

3. $P_{\geq 3}$-Factor Covered Graphs

In this section, we mainly obtain two special classes of $P_{\geq 3}$-factor covered graphs. First, we give a minimum degree condition for a connected claw-free graph to be

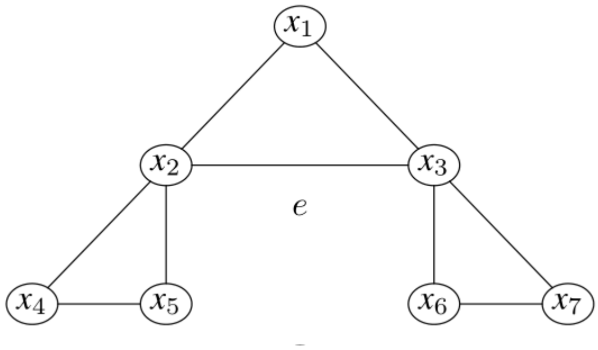

Figure 3. A connected claw-free graph of minimum degree 2 that does not contain any $P_{\geq 3}$-factor covering $e = x_2x_3$.

a $P_{\geq 3}$-factor covered graph as following. Note that the results in Theorem 9 is also sharp in the sense that there exists a connected claw-free graph of minimum degree 2, which is not a $P_{\geq 3}$-factor covered graph. It is shown by the example in Figure 3.

Theorem 9. *Let G be a connected claw-free graph of minimum degree at least* 3. *Then G is a $P_{\geq 3}$-factor covered graph.*

Proof. Suppose G is not a $P_{\geq 3}$-factor covered graph. Then by Theorem 4, there exists a subset $S \subseteq V(G)$ such that $sun(G-S) > 2|S| - \varepsilon(S)$. In terms of the integrality of $sun(G-S)$, we obtain that $sun(G-S) \geq 2|S| - \varepsilon(S) + 1$. We will distinguish two cases below to show that G is a $P_{\geq 3}$-factor covered graph.

Case 1. $|S| \leq 1$. If $S = \emptyset$, then $\varepsilon(S) = |S| = 0$ by the definition of $\varepsilon(S)$. It follows easily that

$$sun(G) = sun(G-S) \geq 2|S| - \varepsilon(S) + 1 = 1.$$

On the other hand, $sun(G) \leq \omega(G) = 1$ since G is a connected graph. Combining the results above, we obtain that G is a big sun, which contradicts the minimum degree of G.

If $|S| = 1$, let $S = \{s\}$. We obtain $\varepsilon(S) \leq 1$ by the definition of $\varepsilon(S)$. If $\varepsilon(S) = 0$, then

$$\omega(G-S) \geq sun(G-S) \geq 2|S| - \varepsilon(S) + 1 = 3.$$

Otherwise $\varepsilon(S) = 1$, then there exists a non-sun component of $G-S$ and thus

$$\omega(G-S) \geq sun(G-S) + 1 \geq 2|S| - \varepsilon(S) + 1 + 1 = 3.$$

Therefore, if either $\varepsilon(S)$ is 0 or 1, then there are at least three components of $G - \{s\}$. It follows easily that there exists a claw with center vertex s in G, a contradiction.

This completes the proof of Case 1.

Case 2. $|S| \geq 2$. Let $|S| = k \geq 2$ and $S = \{s_1, s_2, \ldots, s_k\}$. By the definition of $\varepsilon(S)$, we have $\varepsilon(S) \leq 2$. It follows easily that

$$sun(G - S) \geq 2|S| - \varepsilon(S) + 1 \geq 2|S| - 1.$$

Let $sun(G-S) = m \geq 2k-1$ and $\{C_1, C_2, \ldots, C_m\}$ be the set of sun components of $G - S$. For any sun component C_i of $G - S$, let $L(C_i) \subseteq V(C_i)$ be the set of vertices with exactly one neighbour vertex in C_i, and $L(C_i) = V(C_i)$ if $C_i \cong K_1$, where $1 \leq i \leq m$. Let $E(S, V(C_i) \setminus L(C_i))$ be the set of edges in graph G between vertex a and b for any $a \in S, b \in V(C_i) \setminus L(C_i)$ for $1 \leq i \leq m$. Then we construct a bipartite multigraph H from G by deleting all edges of

$$E(G[S]) \cup \left(\bigcup_{i=1}^{m} E(S, V(C_i) \setminus L(C_i)) \right)$$

and all vertices of

$$V(G) \setminus S \cup \left(\bigcup_{i=1}^{m} V(C_i) \right)$$

and contracting each C_i to a vertex c_i for $1 \leq i \leq m$.

Claim 1. *For any vertex $u, v \in V(H)$, there are at most two edges between u and v in H.*

Proof. Without loss of generality, we assume $u = s_1$ and $v = c_1$. Suppose there are three edges between u and v in H. Then there are three vertices in $L(C_1)$ corresponding to the vertex c_1, denoted by $\left\{c_1^1, c_1^2, c_1^3\right\}$. By the definition of big sun, $c_1^i c_1^j \notin E(G)$ for any $1 \leq i < j \leq 3$, which implies a claw with center vertex u in G. This is a contradiction. □

Since the minimum degree of G is at least three, it is clear that $d_H(c_i) \geq 3$ for $1 \leq i \leq m$. Trivially,

$$|E(H)| \geq 3m \geq 3(2k-1) = 6k - 3.$$

By pigeonhole principle and

$$\frac{|E(H)|}{|S|} \geq \frac{3m}{k} \geq \frac{6k-3}{k} = 6 - \frac{3}{k} > 4,$$

there must exist a vertex $s_i \in S$ incident with at least five edges in $E(H)$. According to Claim 1 and pigeonhole principle, there exists at least three vertices, denoted by $\{c_1, c_2, c_3\}$, adjacent to s_i. Since $\{s_i, c_1, c_2, c_3\}$ induces a claw in H, it follows easily that there exists a claw with center vertex s_i in G, a contradiction. This completes the proof of Case 2.

Combining Case 1 and Case 2, Theorem 9 is proved. ■

Next, we investigate the relationship between planar graphs and $P_{\geq 3}$-factor covered graphs, and obtain a connectivity condition for a planar graph to be a $P_{\geq 3}$-factor covered graphs as following. The example in Figure 4 shows the sharpness of the results in Theorem 11 in the sense that there exists a 2-connected planar graph, which is not a $P_{\geq 3}$-factor covered graph.

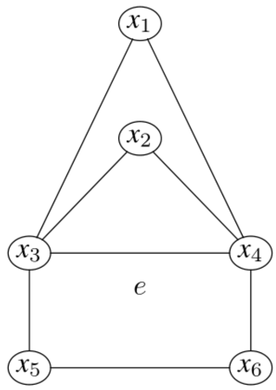

Figure 4. A 2-connected planar graph that does not contain any $P_{\geq 3}$-factor covering $e = x_3x_4$.

Lemma 10 [6]**.** *Let G be a connected planar graph with at least three vertices. If G does not contain triangles, then $|E(G)| \leq 2|G| - 4$.*

Theorem 11. *Let G be a 3-connected planar graph. Then G is a $P_{\geq 3}$-factor covered graph.*

Proof. Suppose G is not a $P_{\geq 3}$-factor covered graph. By Theorem 4, there exists a subset $S \subseteq V(G)$ such that $sun(G - S) > 2|S| - \varepsilon(S)$. According to the integrality of $sun(G - S)$, we obtain that $sun(G - S) \geq 2|S| - \varepsilon(S) + 1$. We distinguish three cases below to show that G is a $P_{\geq 3}$-factor covered graph.

Case 1. $|S| = 0$. In this case, by the definition of $\varepsilon(S)$, we have $S = \emptyset$ and $\varepsilon(S) = 0$. Since G is a connected graph, $sun(G) \leq \omega(G) = 1$. On the other hand, we obtain that

$$sun(G) = sun(G - S) \geq 2|S| - \varepsilon(S) + 1 = 1.$$

It follows easily that $sun(G) = 1$, which is to say G is a big sun. By the definition of sun, it contradicts the fact that G is 3-connected. This completes the proof of Case 1.

Case 2. $|S| = 1$. In this case, we obtain $\varepsilon(S) \leq 1$ by the definition of $\varepsilon(S)$. It follows immediately that

$$sun(G - S) \geq 2|S| - \varepsilon(S) + 1 \geq 2.$$

Let $S = \{x\} \subseteq V(G)$. Since $\omega(G-S) \geq sun(G-S) \geq 2$, x is a cut-vertex of G, which contradicts the fact that G is 3-connected. This completes the proof of Case 2.

Case 3. $|S| \geq 2$. In this case, we obtain $\varepsilon(S) \leq 2$ by the definition of $\varepsilon(S)$. It follows immediately that

$$sun(G-S) \geq 2|S| - \varepsilon(S) + 1 \geq 2|S| - 1.$$

Set $|S| = s$. We denote by $Sun(G-S)$ the set of sun components in $G-S$. Since $sun(G-S) \geq 2|S|-1$, let $C_1, C_2, \ldots, C_{2s-1}$ be $2s-1$ distinct sun components where $C_i \in Sun(G-S)$ for any $1 \leq i \leq 2s-1$. Then we construct a bipartite graph H from G by contracting each C_i to a vertex c_i for $1 \leq i \leq 2s-1$ and deleting all edges of $E(G[S])$ and all vertices of

$$V(G) \setminus (S \cup \left(\bigcup_{i=1}^{2s-1} V(C_i)) \right).$$

Since G is 3-connected, it is clear that $d_H(c_i) \geq 3$ for $1 \leq i \leq 2s-1$. Trivially,

$$|H| = s + (2s-1) = 3s - 1 \geq 5,$$

and

$$|E(H)| \geq 3\,(2s-1) = 6s - 3.$$

As G is a 3-connected planar graph, it is easy to see that H is also a connected planar graph. According to the fact that a bipartite graph does not contain any odd cycles, Lemma 10 implies that

$$6s - 3 \leq |E(H)| \leq 2|H| - 4 = 2\,(3s-1) - 4 = 6s - 6,$$

which is a contradiction. This completes the proof of Case 3.

Combining Cases 1–3, Theorem 11 is proved. ■

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Received 1 December 2019
Revised 2 July 2020
Accepted 6 July 2020